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NEAREST LINEAR SYSTEMS WITH HIGHLY DEFICIENT REACHABLE SUBSPACES*

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Abstract. We consider the 2-norm distance $\tau_r(A,B)$ from a linear time-invariant dynamical system (A,B) of order n to the nearest system $(A+\Delta A_*, B+\Delta B_*)$ whose reachable subspace is of dimension $r<n$. We first present a characterization to test whether the reachable subspace of the system has dimension r, which resembles and can be considered as a generalization of the Popov–Belevitch–Hautus test for controllability. Then, by exploiting this generalized Popov–Belevitch–Hautus characterization, we derive the main result of this paper, which is a singular value optimization characterization for $\tau_r(A,B)$. A numerical technique to solve the derived singular value optimization problems is described. The numerical results on a few examples illustrate the significance of the derived singular value characterization for computational purposes.

Key words. optimization of singular values, reachable subspace, distance to uncontrollability, distance to rank deficiency, minimal system, Lipschitz continuous

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1. Introduction.

Given a linear time-invariant dynamical system of order n,

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

with zero initial conditions, $A \in \mathbb{C}^{n\times n}$, and $B \in \mathbb{C}^{n\times m}$. This work concerns the distance

$$\tau_r(A,B) = \inf\left\{ \left\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \right\| : \operatorname{rank}(\mathcal{C}(A+\Delta A, B+\Delta B)) \le r \right\} \tag{1.2}$$

with

$$\mathcal{C}(A,B) = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

denoting the controllability matrix, whose rank gives the dimension of the reachable subspace of (1.1). In (1.2) and elsewhere $\|\cdot\|$ denotes the matrix 2-norm unless otherwise stated.

The quantity $\tau_r(A,B)$ can be considered as a generalization of the distance to uncontrollability [20, 1, 19]

$$\tau(A,B) = \inf\left\{ \left\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \right\| : \operatorname{rank}\left(\mathcal{C}(A+\Delta A, B+\Delta B)\right) \le n-1 \right\}$$

for which the singular value characterization

$$\tau(A,B) = \inf_{\lambda\in\mathbb{C}} \sigma_n\left(\begin{bmatrix} A-\lambda I & B \end{bmatrix} \right)$$

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is derived by Eising [6]. Above and throughout this paper, $\sigma_j(C)$ denotes the jth largest singular value of C. Eising's characterization is a simple consequence of the Popov–Belevitch–Hautus (PBH) test, which establishes the equivalence of the rank-deficiency of $\mathcal{C}(A,B)$ and the condition

$$\exists \lambda \in \mathbb{C} \text{ s.t. } \operatorname{rank}\left(\begin{bmatrix} A-\lambda I & B \end{bmatrix}\right) \leq n-1. \tag{1.3}$$

Computational techniques based on Eising's singular value characterization are developed for low precision [2, 13] as well as for high precision [10, 11].

The quantity $\tau_r(A,B)$ provides an upper bound for the distance to nonminimality. The linear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1.4}$$

with zero initial conditions, and $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{p\times m}$ is called *nonminimal* if there exists a system of smaller order $r < n$ which maps the input $u(t)$ to the output $y(t)$ in exactly the same manner as the original system. Let us use the shorthand notation (A,B,C,D) for the system above and pose the problem

$$\begin{aligned} \tau_r(A,B,C) = \inf \Bigg\{ \left\| \begin{bmatrix} \Delta A & \Delta B \\ \Delta C & 0 \end{bmatrix} \right\| : \exists A_r \in \mathbb{C}^{r\times r}, B_r \in \mathbb{C}^{r\times m}, C_r \in \mathbb{C}^{p\times r} \\ (A+\Delta A, B+\Delta B, C+\Delta C, D) \equiv (A_r, B_r, C_r, D) \Bigg\}, \end{aligned} \tag{1.5}$$

where we use $(A+\Delta A, B+\Delta B, C+\Delta C, D) \equiv (A_r, B_r, C_r, D)$ to denote that the left-hand and right-hand arguments are equivalent systems. It is well known that when the reachable subspace of (1.4) is of dimension r, there is an equivalent system of order r (see Proposition 2.27 and its proof in [5]) implying

$$\tau_r(A,B,C) \leq \tau_r(A,B).$$

The distance under consideration $\tau_r(A,B)$ is also related to the sensitivity of the Krylov subspaces. In particular, for the special case when B is a column vector, the range of $\mathcal{C}(A,B)$ reduces to a Krylov subspace. The distance $\tau_r(A,B)$ can be viewed as the backward error so that the perturbed Krylov subspace has dimension r. Consequently, it is inherently connected to the condition numbers for Krylov subspaces, which are formalized and studied by Godunov and his co-authors; see, for instance, [3]. This work, however, concerns the more general case when B is not necessarily a column vector, and the quantity under consideration can be viewed as a backward error for the simultaneous subspace iteration.

The outline of this paper is as follows. In section 2 we derive a generalization of the PBH test (1.3) for the condition $\operatorname{rank}(\mathcal{C}(A,B)) \leq r$. (See Theorem 2.1 for the generalized PBH test and Theorem 2.3 for a linearized version.) In section 3 by exploiting this generalized PBH test we obtain the main result of this paper, which is a singular value optimization characterization for the distance $\tau_r(A,B)$ as summarized in Theorem 3.7. Finally, in section 4 we outline a technique to estimate $\tau_r(A,B)$ numerically to a low precision based on the property that the singular values are Lipschitz continuous over the space of matrices. The numerical results in section 5 display that at least for r close to n the singular value optimization characterization facilitate the computation of $\tau_r(A,B)$.

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Notation. Throughout this paper $A \otimes B$ denotes the Kronecker product of the matrices A and B, i.e., if $A \in \mathbb{C}^{\ell \times p}$, then

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \dots & a_{1p}B \\ a_{21}B & a_{22}B & \dots & a_{2p}B \\ \vdots & & & \vdots \\ a_{\ell 1}B & a_{\ell 2}B & \dots & a_{\ell p}B \end{bmatrix}.$$

We use the notation $\mathrm{vec}(X) = [x_1^T \;\; x_2^T \;\; \dots \;\; x_p^T]^T \in \mathbb{C}^{\ell p}$ for the vectoral form of $X = [x_1 \;\; x_2 \;\; \dots \;\; x_p] \in \mathbb{C}^{\ell \times p}$ with $x_1, \dots, x_p \in \mathbb{C}^{\ell}$. In our derivation we frequently benefit from the identity

$$\mathrm{vec}(AXB) = \left(B^T \otimes A\right) \mathrm{vec}(X)$$

for matrices A, X, B of appropriate sizes. Occasionally the notation $\bar{A}$ is used to represent the conjugation of the matrix A entrywise. We reserve I_j for the identity matrix of size $j \times j$.

2. Generalizations of the Popov–Belevitch–Hautus test.

2.1. A generalized PBH test.

Here we derive an equivalent characterization for the condition $\mathrm{rank}(\mathcal{C}(A,B)) \leq r$ that facilitates the conversion of the problem (1.2) into a singular value optimization problem in the next section. The characterization is in terms of the existence of a monic matrix polynomial of A, of degree $n-r$ of the form

$$P_{n-r}(A) = (A - \lambda_1 I)(A - \lambda_2 I)(A - \lambda_3 I) \dots (A - \lambda_{n-r} I)$$

with roots $\lambda_1, \lambda_2, \dots, \lambda_{n-r} \in \mathbb{C}$, satisfying a particular rank condition. The monic polynomial characterization will be derived under the assumption that the matrix A has simple eigenvalues, i.e., the algebraic multiplicities of all eigenvalues of A are one.[1] In the next subsection we will remove the simple eigenvalue assumption.

Suppose $\mathrm{rank}(\mathcal{C}(A,B)) = k$. Then there exists a state transformation $T \in \mathbb{C}^{n \times n}$ such that

$$\tilde{A} = TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \quad \text{and} \quad \tilde{B} = TB = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \tag{2.1}$$

where $\tilde{A}_{11} \in \mathbb{C}^{k \times k}$, $\tilde{A}_{12} \in \mathbb{C}^{k \times (n-k)}$, $\tilde{A}_{22} \in \mathbb{C}^{(n-k) \times (n-k)}$ is upper triangular, $\tilde{B}_1 \in \mathbb{C}^{k \times m}$, and the pair $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable [5, Theorem 2.12, p. 70]. The pair $(\tilde{A}, \tilde{B})$ is called the *controllability canonical form* for (A,B). Since T has full rank and $\mathcal{C}(\tilde{A}, \tilde{B}) = T\,\mathcal{C}(A,B)$, the state transformation preserves the rank, that is,

$$\mathrm{rank}(\mathcal{C}(A,B)) = \mathrm{rank}(\mathcal{C}(\tilde{A}, \tilde{B})).$$

The uncontrollability of the system is equivalent to the existence of a left eigenvector of A that lies in the left null space of B. This condition can be neatly expressed by the existence of a λ so that $\mathrm{rank}([A - \lambda I \;\; B]) \leq n-1$, which is the PBH test. More generally when $\mathrm{rank}(\mathcal{C}(A,B)) = k$, there exists a left eigenvector of A corresponding

[1] This assumption could be relaxed; it is possible to modify the derivation in this subsection for a matrix A with its controllable eigenvalues different from its uncontrollable eigenvalues.

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to each left eigenvector of $\tilde{A}_{22}$ in the controllability canonical form that is contained in the left null space of B. The PBH test can be generalized based on this geometric observation.

First let us suppose $\operatorname{rank}(\mathcal{C}(A,B)) = k \le r$, and partition $\tilde{A}$ and $\tilde{B}$ into

$$\tilde{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} \quad \text{and} \quad \tilde{B} = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix},$$

where $\hat{A}_{11} \in \mathbb{C}^{r\times r}$, $\hat{A}_{12} \in \mathbb{C}^{r\times(n-r)}$, $\hat{A}_{22} \in \mathbb{C}^{(n-r)\times(n-r)}$, and $\hat{B}_1 \in \mathbb{C}^r$. Note that $\hat{A}_{22}$ is the lower rightmost $(n-r)\times(n-r)$ portion of $\tilde{A}_{22}$ in the controllability canonical form (2.1). Consider the polynomial $P_{n-r}(A)$ with roots equal to the distinct eigenvalues of $\hat{A}_{22}$. Clearly

$$T \begin{bmatrix} P_{n-r}(A) & B \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} P_{n-r}(\tilde{A}) & \tilde{B} \end{bmatrix},$$

meaning

$$\operatorname{rank}\left(\begin{bmatrix} P_{n-r}(A) & B \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} P_{n-r}(\tilde{A}) & \tilde{B} \end{bmatrix}\right).$$

Furthermore, by the Hamilton–Cayley theorem we have

(2.2)
$$\begin{bmatrix} P_{n-r}(\tilde{A}) & \tilde{B} \end{bmatrix} = \begin{bmatrix} P_{n-r}(\hat{A}_{11}) & \hat{\hat{A}}_{12} & \hat{B}_1 \\ 0 & P_{n-r}(\hat{A}_{22}) & 0 \end{bmatrix} = \begin{bmatrix} P_{n-r}(\hat{A}_{11}) & \hat{\hat{A}}_{12} & \hat{B}_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the eigenvalues of $\hat{A}_{11}$ and $\hat{A}_{22}$ are disjoint, the matrices $P_{n-r}(\hat{A}_{11})$ as well as $[P_{n-r}(\hat{A}_{11}) \;\; \hat{\hat{A}}_{12} \;\; \hat{B}_1]$ have full row-rank. Consequently, we deduce that

$$\operatorname{rank}\left(\begin{bmatrix} P_{n-r}(A) & B \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} P_{n-r}(\tilde{A}) & \tilde{B} \end{bmatrix}\right) = r. \tag{2.3}$$

On the contrary, suppose $\operatorname{rank}(\mathcal{C}(A,B)) = k > r$. Consider again

$$\begin{bmatrix} P_{n-r}(\tilde{A}) & \tilde{B} \end{bmatrix} = \begin{bmatrix} P_{n-r}(\tilde{A}_{11}) & \tilde{\hat{A}}_{12} & \tilde{B}_1 \\ 0 & P_{n-r}(\tilde{A}_{22}) & 0 \end{bmatrix}$$

for any polynomial $P_{n-r}(A)$ of degree $n-r$, where upper, lower blocks have k and $n-k$ rows, respectively. Let c_1, c_2 be the number of roots of $P_{n-r}(A)$ (a multiple root, if there is any, counted only once) coinciding with the eigenvalues of $\tilde{A}_{11}$ and $\tilde{A}_{22}$, respectively.

It is straightforward to verify that the left null space of $[P_{n-r}(\tilde{A}_{11}) \;\; \tilde{B}_1]$ is of dimension $\max(c_1 - 1, 0)$ at most due to the fact that $(\tilde{A}_{11}, \tilde{B}_1)$ is controllable. To see this, suppose $\mu_1, \ldots, \mu_{c_1}$ are the roots of $P_{n-r}(A)$ coinciding with the eigenvalues of $\tilde{A}_{11}$. Denote also the left eigenvectors of $\tilde{A}_{11}$ associated with $\mu_1, \ldots, \mu_{c_1}$ by $v_1, \ldots, v_{c_1}$. Then the left null space of $P_{n-r}(\tilde{A}_{11})$ is $\operatorname{span}\{v_1, \ldots, v_{c_1}\}$, whereas the left null space of $[P_{n-r}(\tilde{A}_{11}) \;\; \tilde{B}_1]$ is a subspace of $\operatorname{span}\{v_1, \ldots, v_{c_1}\}$. If the left null space of $[P_{n-r}(\tilde{A}_{11}) \;\; \tilde{B}_1]$ were precisely $\operatorname{span}\{v_1, \ldots, v_{c_1}\}$, this would imply

$$v_j^* \begin{bmatrix} P_{n-r}(\tilde{A}_{11}) & \tilde{B}_1 \end{bmatrix} = 0 \quad \Longrightarrow \quad v_j^* \begin{bmatrix} (\tilde{A}_{11} - \mu_j I) & \tilde{B}_1 \end{bmatrix} = 0$$

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for $j = 1, \ldots, c_1$ contradicting the controllability of $(\tilde{A}_{11}, \tilde{B}_1)$. Consequently, the left null space of $[P_{n-r}(\tilde{A}_{11}) \;\; \tilde{B}_1]$ is a strict subspace of $\text{span}\{v_1, \ldots, v_{c_1}\}$.

On the other hand the left null space of $P_{n-r}(\tilde{A}_{22})$ has dimension $c_2 \leq n-k < n-r$. Therefore, the dimension of the left null space of $[P_{n-r}(\tilde{A}) \;\; \tilde{B}]$ cannot exceed $\max(c_1 + c_2 - 1, c_2) \leq n-r-1$. Consequently

$$\text{rank}\left(\begin{bmatrix} P_{n-r}(A) & B \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} P_{n-r}(\tilde{A}) & \tilde{B} \end{bmatrix}\right) \geq r+1.$$

We conclude with the following generalization of the Popov–Belevitch–Hautus test.

THEOREM 2.1 (generalized PBH test). *Suppose that the matrix A has simple eigenvalues. Then the condition* $\text{rank}(\mathcal{C}(A,B)) \leq r$ *holds if and only if there exists a monic matrix polynomial $P_{n-r}(A)$ of degree $n-r$ such that*

$$\text{rank}\left(\begin{bmatrix} P_{n-r}(A) & B \end{bmatrix}\right) = r.$$

When the system is uncontrollable, that is when $\text{rank}(\mathcal{C}(A,B)) \leq n-1$, Theorem 2.1 implies that $\text{rank}([A - \lambda I \;\; B]) = n-1$ for some λ. This is consistent with the PBH test. However, due to the simple eigenvalue assumption it is possible to conclude that the rank is *precisely* equal to $n-1$ rather than *at most* equal to $n-1$ as suggested by the PBH test.

2.2. Linearization. There are two downsides of Theorem 2.1 which make it difficult to use in our derivation. First of all it is based on the hypothesis that A has simple eigenvalues. Second, in the next section we will attempt to identify the nearest system $(A + \Delta A, B + \Delta B)$ whose reachable subspace has dimension r. The fact that $[P_{n-r}(A + \Delta A) \;\; B + \Delta B]$ depends on the perturbation ΔA nonlinearly is problematic for the derivation. Here we simultaneously linearize $[P_{n-r}(A) \;\; B]$ and remove the simple eigenvalue assumption.

Our starting point is yet again the controllability canonical form (2.1), in particular the uncontrollable portion $\tilde{A}_{22} \in \mathbb{C}^{(n-k)\times(n-k)}$ of $\tilde{A}$ in (2.1). Consider the Sylvester equation

$$\tilde{A}_{22}X - XC(\Lambda, \Gamma) = 0, \tag{2.4}$$

where $X \in \mathbb{C}^{(n-k)\times(n-r)}$ and

$$C(\Lambda,\Gamma) := \begin{bmatrix} \lambda_1 & -\gamma_{21} & \cdots & -\gamma_{(n-r)1} \\ 0 & \lambda_2 & \cdots & -\gamma_{(n-r)2} \\ \vdots & & \ddots & \vdots \\ 0 & & 0 & \lambda_{n-r} \end{bmatrix} \in \mathbb{C}^{(n-r)\times(n-r)} \tag{2.5}$$

with

$$\Lambda = \begin{bmatrix} \lambda_1 & \ldots & \lambda_{n-r} \end{bmatrix}^T \in \mathbb{C}^{n-r},$$
$$\Gamma = \begin{bmatrix} \gamma_{21} & \ldots & \gamma_{(n-r)(n-r-1)} \end{bmatrix}^T \in \mathbb{C}^{(n-r-1)(n-r)/2}.$$

The dimension of the solution space of the Sylvester equation

$$\left\{ X \in \mathbb{C}^{(n-k)\times(n-r)} \;:\; \tilde{A}_{22}X - XC(\Lambda,\Gamma) = 0 \right\}$$

depends on the common eigenvalues of $\tilde{A}_{22}$ and $C(\Lambda,\Gamma)$, and the Jordan structures of these matrices associated with these common eigenvalues.

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Now consider any arbitrary upper triangular matrix $C(\Lambda,\Gamma) \in \mathbb{C}^{(n-r)\times(n-r)}$. Suppose ℓ of the scalars $\lambda_1,\ldots,\lambda_{n-r}$ are distinct, and denote them by $\mu_1,\ldots,\mu_\ell$ appearing $m_1,\ldots,m_\ell$ times in the list $\lambda_1,\ldots,\lambda_{n-r}$, respectively. For generic values of Γ each eigenvalue of $C(\Lambda,\Gamma)$ has geometric multiplicity equal to one [4]. We reserve the notation $\mathcal{G}(\Lambda)$ for this set of generic Γ values, depending on Λ.

Furthermore, let us denote the common eigenvalues of $\tilde{A}_{22}$ and $C(\Lambda,\Gamma)$ by $\mu_{k_1},\ldots,\mu_{k_q}$, and denote the sizes of the Jordan blocks of $\tilde{A}_{22}$ associated with the eigenvalue μ_{k_j} by $c_{k_j,1},\ldots,c_{k_j,\ell_j}$. Then for each $\Gamma \in \mathcal{G}(\Lambda)$ the dimension of the solution space of the Sylvester equation (2.4) is

$$\sum_{j=1}^{q}\sum_{i=1}^{\ell_j}\min(c_{k_j,i},m_{k_j}) \le \sum_{j=1}^{q}\sum_{i=1}^{\ell_j} c_{k_j,i} \le n-k \tag{2.6}$$

(see [8, Theorem 1, page 219]).

First, suppose that $\operatorname{rank}(\mathcal{C}(A,B)) = k \le r$. In particular assume that the set of common eigenvalues $\{\mu_{k_1},\ldots,\mu_{k_q}\}$ is precisely the set of eigenvalues of $C(\Lambda,\Gamma)$, and

$$\sum_{i=1}^{\ell_j} c_{k_j,i} \ge m_{k_j}$$

for each $j=1,\ldots,q$. Then for each $\Gamma \in \mathcal{G}(\Lambda)$ we have

$$\sum_{j=1}^{q}\sum_{i=1}^{\ell_j}\min(c_{k_j,i},m_{k_j}) \ge \sum_{j=1}^{q} m_{k_j} = n-r,$$

so the dimension of the solution space of the Sylvester equation is at least $n-r$.

Now on the contrary suppose that $\operatorname{rank}(\mathcal{C}(A,B)) = k > r$. Then by (2.6) for all $C(\Lambda,\Gamma)$ such that $\Gamma \in \mathcal{G}(\Lambda)$ the dimension of the solution space of the Sylvester equation (2.4) cannot exceed $(n-k)$ and is strictly less than $n-r$. What we deduced so far in terms of the Sylvester equations is summarized below.

THEOREM 2.2. *Consider a linear time-invariant system (A,B) with the controllability canonical form given by* (2.1)*. The following two statements are equivalent:*

(i) *The inequality* $\operatorname{rank}(\mathcal{C}(A,B)) \le r$ *holds.*

(ii) *There exists a* Λ *such that for all* $\Gamma \in \mathcal{G}(\Lambda)$ *we have*

$$\dim\Big\{X \;:\; \tilde{A}_{22}X - XC(\Lambda,\Gamma) = 0\Big\} \ge n-r,$$

where $\tilde{A}_{22}$ *and* $\mathcal{C}(\Lambda,\Gamma)$ *are as defined in* (2.1) *and* (2.5)*, respectively.*

Next we convert the Sylvester equation (2.4) into a linear system. A matrix X is a solution of the Sylvester equation if and only if

$$\Big(I\otimes\tilde{A}_{22} - C(\Lambda,\Gamma)^T\otimes I\Big)\operatorname{vec}(X) = \mathcal{A}(\Lambda,\Gamma)\operatorname{vec}(X) = 0,$$

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where

$$\mathcal{A}(\Lambda,\Gamma) := \begin{bmatrix} \tilde{A}_{22}-\lambda_1 I & 0 & 0 & 0 & & 0 \\ \gamma_{21} I & \tilde{A}_{22}-\lambda_2 I & 0 & 0 & & 0 \\ \gamma_{31} I & \gamma_{32} I & \tilde{A}_{22}-\lambda_3 I & 0 & & \\ & & & \ddots & & \\ & & & & \ddots & \\ \gamma_{(n-r)1} I & \gamma_{(n-r)2} I & & & & \tilde{A}_{22}-\lambda_{n-r} I \end{bmatrix}. \tag{2.7}$$

Consequently, the solution space of the Sylvester equation (2.4) and the null space of $\mathcal{A}(\Lambda,\Gamma)$ are of same dimension.

In the controllability canonical form (2.1) the pair $(\tilde{A}_{11},\tilde{B}_1)$ is controllable meaning $[\tilde{A}_{11}-\lambda_k I \quad \tilde{B}_1]$ is full rank for all λ_k. Therefore the dimension of the left null space of

$$\mathcal{L}(\Lambda,\Gamma,\tilde{A},\tilde{B}) := \begin{bmatrix} \tilde{A}-\lambda_1 I & \tilde{B} & 0 & 0 & 0 & 0 & & 0 & 0 \\ \gamma_{21} I & 0 & \tilde{A}-\lambda_2 I & \tilde{B} & 0 & 0 & & 0 & 0 \\ \gamma_{31} I & 0 & \gamma_{32} I & 0 & \tilde{A}-\lambda_3 I & \tilde{B} & & 0 & 0 \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ \gamma_{(n-r)1} I & 0 & \gamma_{(n-r)2} I & 0 & & & & \tilde{A}-\lambda_{n-r} I & \tilde{B} \end{bmatrix}$$

is the same as the dimension of the null space of $\mathcal{A}(\Lambda,\Gamma)$. Finally, the dimensions of the left null spaces of $\mathcal{L}(\Lambda,\Gamma,\tilde{A},\tilde{B})$ and

$$\begin{aligned} \mathcal{L}(\Lambda,\Gamma,A,B) &= (I_{n-r}\otimes T^{-1})\mathcal{L}(\Lambda,\Gamma,\tilde{A},\tilde{B})\left(I_{n-r}\otimes \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix}\right) \\ &= \begin{bmatrix} A-\lambda_1 I & B & 0 & 0 & & 0 & 0 \\ \gamma_{21} I & 0 & A-\lambda_2 I & B & & 0 & 0 \\ \gamma_{31} I & 0 & \gamma_{32} I & 0 & & 0 & 0 \\ & & & & \ddots & & \\ \gamma_{(n-r)1} I & 0 & \gamma_{(n-r)2} I & 0 & & A-\lambda_{n-r} I & B \end{bmatrix} \end{aligned} \tag{2.8}$$

are also the same where $T\in\mathbb{C}^{n\times n}$ is as defined in the controllability canonical form (2.1). Consequently, Theorem 2.2 can be restated as follows.

THEOREM 2.3 (linearized PBH test). *The following statements are equivalent:*

(i) *The inequality* $\operatorname{rank}(\mathcal{C}(A,B)) \le r$ *holds.*

(ii) *There exists a* Λ *such that for all* $\Gamma\in\mathcal{G}(\Lambda)$ *the inequality* $\operatorname{rank}(\mathcal{L}(\Lambda,\Gamma,A,B)) \le (n-r)(n-1)$ *holds.*

The above result could be considered as a linearized version of Theorem 2.1.

3. A singular value characterization for $\tau_r(A,B)$. For the derivation of the singular value characterization we will depend on the following elementary result [9, Theorem 2.5.3, p. 72].

THEOREM 3.1 (minimal rank). *Let* $A\in\mathbb{C}^{n\times m}$ *and let* $k<\min(m,n)$ *be a positive integer. Then*

$$\inf\{\|\Delta A\| : \operatorname{rank}(A+\Delta A)\le k\} = \sigma_{k+1}(A).$$

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We deduce the following from Theorem 2.3:

$$\begin{aligned}\tau_r(A,B) &= \inf\left\{\left\|\begin{bmatrix}\Delta A & \Delta B\end{bmatrix}\right\| : \operatorname{rank}(\mathcal{C}(A+\Delta A, B+\Delta B)) \le r\right\}\\ &= \inf\left\{\left\|\begin{bmatrix}\Delta A & \Delta B\end{bmatrix}\right\| : \exists \Lambda\ \forall \Gamma \in \mathcal{G}(\Lambda)\right.\\ &\qquad\qquad \left.\operatorname{rank}\left(\mathcal{L}(\Lambda,\Gamma,A+\Delta A,B+\Delta B)\right) \le (n-r)(n-1)\right\}.\end{aligned}$$

Let us define

$$\begin{aligned}&\tau_r(\Lambda,A,B)\\ &\quad := \inf\left\{\left\|\begin{bmatrix}\Delta A & \Delta B\end{bmatrix}\right\| : \operatorname{rank}\left(\mathcal{L}(\Lambda,\Gamma,A+\Delta A,B+\Delta B)\right) \le (n-r)(n-1)\right\}\end{aligned}$$

for any $\Gamma \in \mathcal{G}(\Lambda)$ so that

$$\tau_r(A,B) = \inf_{\Lambda} \tau_r(\Lambda,A,B),$$

where the minimization is over $\mathbb{C}^{n-r}$.

An application of Theorem 3.1 yields the lower bound

$$\tau_r(\Lambda,A,B) \ge \sigma_{k+1}\left(\mathcal{L}(\Lambda,\Gamma,A,B)\right)$$

for $k := (n-r)(n-1)$ here and hereafter. Only a lower bound is deduced, since the perturbations to the linearized matrix have repeated block diagonal structure. The above inequality holds for all generic $\Gamma \in \mathcal{G}(\Lambda)$, so by continuity we can maximize the right-hand side over all Γ yielding

$$\tau_r(\Lambda,A,B) \ge \kappa_r(\Lambda,A,B) := \sup_{\Gamma}\ \sigma_{k+1}\left(\mathcal{L}(\Lambda,\Gamma,A,B)\right). \tag{3.1}$$

Throughout the rest of the section we establish the other direction of the inequality $\tau_r(\Lambda,A,B) \le \kappa_r(\Lambda,A,B)$ by constructing an optimal perturbation $[\Delta A_*\ \ \Delta B_*]$ of norm $\kappa_r(\Lambda,A,B)$ such that

$$\operatorname{rank}\left(\mathcal{L}(\Lambda,\Gamma_*,A+\Delta A_*,B+\Delta B_*)\right) \le (n-r)(n-1), \tag{3.2}$$

where Γ_* is such that

$$\kappa_r(\Lambda,A,B) = \sigma_{k+1}\left(\mathcal{L}(\Lambda,\Gamma_*,A,B)\right).$$

Here, we use the fact that the supremum in (3.1) must be attained; indeed Γ_* is the point where the supremum is attained. The reasoning is as follows. Due to the arguments in the appendix in [14], all of the r smallest singular values of the square linearized matrix $\mathcal{A}(\Lambda,\Gamma)$ (defined in (2.7)) must decay to zero as $\|\Gamma\| \to \infty$. Since the rank of $\mathcal{L}(\Lambda,\Gamma,A,B)$ is the same as the rank of $\mathcal{A}(\Lambda,\Gamma)$, this implies that the r smallest singular values of $\mathcal{L}(\Lambda,\Gamma,A,B)$ must also approach zero as $\|\Gamma\| \to \infty$. This together with the continuity of the singular value σ_{k+1} with respect to Γ establish that the supremum in (3.1) must be attained.

The derivation here is inspired by the derivations in [18, sect. 2.3–4], [17], and [22]. However, unlike these previous works we extensively utilize the Sylvester equation characterization (Theorem 2.2), and the associated linearized rank characterization with Kronecker structure (Theorem 2.3). Sylvester equations were used in [15] in the derivations for some related distance problems concerning eigenvalues. The upper bound is deduced under mild multiplicity and linear independence assumptions.

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DEFINITION 3.2 (multiplicity qualification). *We say that the multiplicity qualification holds at* (Λ, Γ) *if the multiplicity of* $\sigma_{k+1}(\mathcal{L}(\Lambda, \Gamma, A, B))$ *is one.*

DEFINITION 3.3 (linear independence qualification). *We say that the linear independence qualification holds at* (Λ, Γ) *if there exists a left singular vector* $u = [u_1^T \ u_2^T \ \dots \ u_{n-r}^T]$ *of* $\mathcal{L}(\Lambda, \Gamma, A, B)$ *associated with* $\sigma_{k+1}(\mathcal{L}(\Lambda, \Gamma, A, B))$ *such that the set* $\{u_1, u_2, \dots, u_{n-r}\}$ *is linearly independent where* $u_1, u_2, \dots, u_{n-r} \in \mathbb{C}^n$.

3.1. Derivation of $\tau_r(\Lambda, A, B) = \kappa_r(\Lambda, A, B)$ in the generic case. In this subsection Λ is comprised of distinct elements so that all eigenvalues of $C(\Lambda, \Gamma)$ have algebraic and geometric multiplicities equal to one for all $\Gamma \in \mathbb{C}^{(n-r)(n-r-1)/2}$. This assumption will be removed later at the end of this section.

Suppose $\mathcal{U} \in \mathbb{C}^{n(n-r)}$ and $\mathcal{V} \in \mathbb{C}^{(n+m)(n-r)}$ are the left and right singular vectors associated with $\kappa_r(\Lambda, A, B)$ so that

$$\mathcal{L}(\Lambda, \Gamma_*, A, B)\, \mathcal{V} = \kappa_r(\Lambda, A, B)\, \mathcal{U} \quad \text{and} \quad \mathcal{L}(\Lambda, \Gamma_*, A, B)^*\, \mathcal{U} = \kappa_r(\Lambda, A, B)\, \mathcal{V}, \tag{3.3}$$

where

$$\mathcal{U} = \begin{bmatrix} \mathcal{U}_1^T & \mathcal{U}_2^T & \dots \mathcal{U}_{n-r}^T \end{bmatrix}^T \quad \text{and} \quad \mathcal{V} = \begin{bmatrix} \mathcal{V}_1^T & \mathcal{V}_2^T & \dots \mathcal{V}_{n-r}^T \end{bmatrix}^T$$

with $\mathcal{U}_j \in \mathbb{C}^n$ and $\mathcal{V}_j \in \mathbb{C}^{n+m}$. The optimal perturbation is defined in terms of the block components of $\mathcal{U}$ and $\mathcal{V}$. Let

$$U = \begin{bmatrix} \mathcal{U}_1^* \\ \mathcal{U}_2^* \\ \vdots \\ \mathcal{U}_{n-r}^* \end{bmatrix} \in \mathbb{C}^{(n-r)\times n} \quad \text{and} \quad V = \begin{bmatrix} \mathcal{V}_1^* \\ \mathcal{V}_2^* \\ \vdots \\ \mathcal{V}_{n-r}^* \end{bmatrix} \in \mathbb{C}^{(n-r)\times(n+m)}.$$

We define

$$\begin{bmatrix} \Delta A_* & \Delta B_* \end{bmatrix} := -\kappa_r(\Lambda, A, B) U^+ V, \tag{3.4}$$

where U^+ denotes the Moore–Penrose pseudoinverse of U. In the next subsection we will show the remarkable property

$$UU^* = VV^* \tag{3.5}$$

provided that the multiplicity qualification holds at (Λ, Γ_*).

To show the optimality of $[\Delta A_* \ \ \Delta B_*]$ as defined in (3.4) we need to deduce that

(i) $\|[\Delta A_* \ \ \Delta B_*]\| = \kappa_r(\Lambda, A, B)$ and

(ii) inequality (3.2) is satisfied.

From [22, Lemma 2] (or a straightforward generalization of [18, Theorem 2.5]) it immediately follows that $\|V^* (U^*)^+)\| = 1$ due to property (3.5).

THEOREM 3.4. *Suppose that property* (3.5) *holds. Then*

$$\left\| \begin{bmatrix} \Delta A_* & \Delta B_* \end{bmatrix} \right\| = \kappa_r(\Lambda, A, B),$$

where $[\Delta A_* \ \ \Delta B_*]$ *is as defined in* (3.4).

Next we prove that inequality (3.2) is satisfied starting from (3.3). Here we benefit from the Sylvester equation point of view on the linearized singular value problems. In particular the right-hand equation in (3.3) can be written as

$$\left(I_{n-r} \otimes \begin{bmatrix} A^* \\ B^* \end{bmatrix} - \overline{C(\Lambda, \Gamma_*)} \otimes \begin{bmatrix} I_n \\ 0 \end{bmatrix} \right) \mathcal{U} = \kappa_r(\Lambda, A, B) \mathcal{V}.$$

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When we express the linear system above as a matrix equation, we obtain

$$\begin{aligned} & \begin{bmatrix} A^* \\ B^* \end{bmatrix} U^* - \begin{bmatrix} I_n \\ 0 \end{bmatrix} U^* C(\Lambda, \Gamma_*)^* = \kappa_r(\Lambda, A, B) V^* \\ \iff \; & U \begin{bmatrix} A & B \end{bmatrix} - C(\Lambda, \Gamma_*) U \begin{bmatrix} I_n & 0 \end{bmatrix} = \kappa_r(\Lambda, A, B) V. \end{aligned}$$

The property $UU^* = VV^*$ implies $V = UU^+V$. (This is evident from the singular value decompositions of U and V.) Consequently,

$$\begin{aligned} & U \begin{bmatrix} A & B \end{bmatrix} - C(\Lambda, \Gamma_*) U \begin{bmatrix} I_n & 0 \end{bmatrix} = \kappa_r(\Lambda, A, B) U U^+ V \\ \implies \quad & U \left(\begin{bmatrix} A & B \end{bmatrix} - \kappa_r(\Lambda, A, B) U^+ V \right) - C(\Lambda, \Gamma_*) U \begin{bmatrix} I_n & 0 \end{bmatrix} = 0 \\ \implies \quad & U \left(\begin{bmatrix} A + \Delta A_* & B + \Delta B_* \end{bmatrix} \right) - C(\Lambda, \Gamma_*) U \begin{bmatrix} I_n & 0 \end{bmatrix} = 0. \end{aligned}$$

Moreover consider the linear subspace $\mathcal{M} := \{X \in \mathbb{C}^{(n-r)\times(n-r)} : C(\Lambda, \Gamma_*)X - XC(\Lambda, \Gamma_*) = 0\}$ consisting of matrices commuting with $C(\Lambda, \Gamma_*)$. The dimension of this subspace is at least $(n - r)$ due to [8, Theorem 1, page 219]. For any matrix $D \in \mathcal{M}$ we have

$$\begin{aligned} & DU \begin{bmatrix} A + \Delta A_* & B + \Delta B_* \end{bmatrix} - DC(\Lambda, \Gamma_*) U \begin{bmatrix} I_n & 0 \end{bmatrix} \\ = \; & (DU) \begin{bmatrix} A + \Delta A_* & B + \Delta B_* \end{bmatrix} - C(\Lambda, \Gamma_*)(DU) \begin{bmatrix} I_n & 0 \end{bmatrix} = 0. \end{aligned}$$

Therefore the solution space for the Sylvester equation

$$X \begin{bmatrix} A + \Delta A_* & B + \Delta B_* \end{bmatrix} - C(\Lambda, \Gamma_*)\, X \begin{bmatrix} I_n & 0 \end{bmatrix} = 0$$

includes the subspace $\{DU : D \in \mathcal{M}\}$, which is of dimension at least $(n - r)$ due to the linear independence qualification. Writing the matrix equation

$$\begin{bmatrix} A^* + \Delta A_*^* \\ B^* + \Delta B_*^* \end{bmatrix} X^* - \begin{bmatrix} I_n \\ 0 \end{bmatrix} X^* C(\Lambda, \Gamma_*)^* = 0$$

in vector form yields

$$\begin{aligned} & \left(I_{n-r} \otimes \begin{bmatrix} A^* + \Delta A_*^* \\ B^* + \Delta B_*^* \end{bmatrix} - \overline{C(\Lambda, \Gamma_*)} \otimes \begin{bmatrix} I_n \\ 0 \end{bmatrix} \right) \operatorname{vec}(X^*) = 0 \\ \implies \quad & \operatorname{vec}(X^*)^* \left(I_{n-r} \otimes \begin{bmatrix} A_* + \Delta A_* & B_* + \Delta B_* \end{bmatrix} - C(\Lambda, \Gamma_*)^T \otimes \begin{bmatrix} I_n & 0 \end{bmatrix} \right) = 0 \\ \implies \quad & \operatorname{vec}(X^*)^* \mathcal{L}(\Lambda, \Gamma_*, A + \Delta A_*, B + \Delta B_*) = 0. \end{aligned}$$

This confirms the validity of inequality (3.2) as desired, since we establish that the left null space of $\mathcal{L}(\Lambda, \Gamma_*, A + \Delta A_*, B + \Delta B_*)$ has dimension at least $n - r$. We conclude with $\tau_r(\Lambda, A, B) = \kappa_r(\Lambda, A, B)$.

3.2. Derivation of $UU^* = VV^*$. We derive this property under the assumption that the multiplicity qualification holds at (Λ, Γ_*). Here Γ_* is the point where $\kappa_r(\Lambda, A, B)$ is attained. Without loss of generality we also assume $\kappa_r(\Lambda, A, B) > 0$, since

$$\tau_r(\Lambda, A, B) = 0 \quad \iff \quad \kappa_r(\Lambda, A, B) = 0.$$

The quantity $\kappa_r(\Lambda, A, B)$ is the supremum of the function $f(\Lambda, \Gamma) := \sigma_{k+1}(\mathcal{L}(\Lambda, \Gamma$ $A, B))$ over Γ. Because of the multiplicity qualification and the assumption $f(\Lambda, \Gamma_*) =$

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$\kappa_r(\Lambda, A, B) > 0$, it turns out that f is analytic at (Λ, Γ_*) with respect to Γ (see [18, sect. 2.4] for details). Here we will view f as a mapping from a real domain by replacing each complex $\gamma_{\ell j}$ with its real part $\Re\gamma_{\ell j}$ and imaginary part $\Im\gamma_{\ell j}$. Then by using the analytic formulas for the derivatives [17] of a singular value (when it is differentiable) we have

$$\frac{\partial f}{\partial \Re\gamma_{\ell j}} = \text{Real}\left(\mathcal{U}^* \frac{\partial \mathcal{L}(\Lambda, \Gamma_*, A, B)}{\partial \Re\gamma_{\ell j}} \mathcal{V}\right) = \text{Real}\left(\mathcal{U}_\ell^* \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{V}_j\right) = 0, \tag{3.6}$$

$$\frac{\partial f}{\partial \Im\gamma_{\ell j}} = \text{Real}\left(\mathcal{U}^* \frac{\partial \mathcal{L}(\Lambda, \Gamma_*, A, B)}{\partial \Im\gamma_{\ell j}} \mathcal{V}\right) = \text{Real}\left(i\mathcal{U}_\ell^* \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{V}_j\right) = 0 \tag{3.7}$$

for all $\ell > j$ where $[I_n \;\; 0]$ denotes the $n \times n$ identity matrix appended with the $n \times m$ zero matrix.

LEMMA 3.5. *Suppose that the multiplicity qualification holds at* (Λ, Γ_*) *and* $\kappa_r(\Lambda, A, B) > 0$. *The equality*

$$\mathcal{U}_\ell^* \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{V}_j = 0$$

holds for all $j = 1, \ldots, n-r-1$ *and* $\ell = j+1, \ldots, n-r$.

THEOREM 3.6. *The property* $UU^* = VV^*$ *holds under the assumptions of Lemma* 3.5.

Proof. We need only deduce $\mathcal{U}_\ell^* \mathcal{U}_j = \mathcal{V}_\ell^* \mathcal{V}_j$ for $\ell \geq j$. For such a pair j, ℓ, from (3.3), we have

$$\gamma_{j1} \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{V}_1 + \cdots + \gamma_{j(j-1)} \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{V}_{j-1} + \begin{bmatrix} A - \lambda_j I & B \end{bmatrix} \mathcal{V}_j = \kappa_r(\Lambda, A, B) \mathcal{U}_j \tag{3.8}$$

and

$$\mathcal{U}_\ell^* \begin{bmatrix} A - \lambda_\ell I & B \end{bmatrix} + \gamma_{(\ell+1)\ell} \mathcal{U}_{\ell+1}^* \begin{bmatrix} I_n & 0 \end{bmatrix} + \cdots + \gamma_{(n-r)\ell} \mathcal{U}_{n-r}^* \begin{bmatrix} I_n & 0 \end{bmatrix} = \kappa_r(\Lambda, A, B) \mathcal{V}_\ell^*. \tag{3.9}$$

Now multiply both sides of (3.8) by $\mathcal{U}_\ell^*$ from left to obtain

$$\begin{aligned} &\gamma_{j1} \mathcal{U}_\ell^* \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{V}_1 + \gamma_{j2} \mathcal{U}_\ell^* \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{V}_2 + \cdots + \gamma_{j(j-1)} \mathcal{U}_\ell^* \begin{bmatrix} I_n & 0 \end{bmatrix} \mathcal{V}_{j-1} \\ &\quad + \mathcal{U}_\ell^* \begin{bmatrix} A - \lambda_\ell I & B \end{bmatrix} \mathcal{V}_j + \mathcal{U}_\ell^* \begin{bmatrix} (\lambda_\ell - \lambda_j) I_n & 0 \end{bmatrix} \mathcal{V}_j = \kappa_r(\Lambda, A, B) \mathcal{U}_\ell^* \mathcal{U}_j. \end{aligned}$$

Note that $\mathcal{U}_\ell^* [(\lambda_\ell - \lambda_j) I_n \;\; 0] \mathcal{V}_j = 0$ (by Lemma 3.5 if $\ell > j$, trivially otherwise). The terms with coefficients γ_{jp}, $p = 1, \ldots, j-1$ on the left are also all zero by Lemma 3.5 yielding

$$\mathcal{U}_\ell^* \begin{bmatrix} A - \lambda_\ell I & B \end{bmatrix} \mathcal{V}_j = \kappa_r(\Lambda, A, B) \mathcal{U}_\ell^* \mathcal{U}_j.$$

Finally, in the last equation we substitute for $\mathcal{U}_\ell^* [A - \lambda_\ell I \;\; B]$ using the expression in (3.9), which gives

$$\begin{aligned} \left(\kappa_r(\Lambda, A, B) \mathcal{V}_\ell^* - \gamma_{(\ell+1)\ell} \mathcal{U}_{\ell+1}^* \begin{bmatrix} I_n & 0 \end{bmatrix} - \cdots - \gamma_{(n-r)\ell} \mathcal{U}_{n-r}^* \begin{bmatrix} I_n & 0 \end{bmatrix}\right) \mathcal{V}_j \\ = \kappa_r(\Lambda, A, B) \mathcal{U}_\ell^* \mathcal{U}_j. \end{aligned}$$

Again the terms with coefficients $\gamma_{p\ell}$, $p = \ell+1, \ldots, n-r$, are zero by Lemma 3.5. Since $\kappa_r(\Lambda, A, B) > 0$ by assumption, we infer $\mathcal{V}_\ell^* \mathcal{V}_j = \mathcal{U}_\ell^* \mathcal{U}_j$ for $\ell \geq j$. ∎

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3.3. Nongeneric case. Now consider Λ with repeating elements, and assume that the multiplicity and linear independence qualifications hold at (Λ, Γ_*) where Γ_* is the point where $\kappa_r(\Lambda, A, B)$ is attained. The geometric multiplicities of all eigenvalues of $C(\Lambda, \Gamma_*)$ do not have to be one, i.e., possibly $\Gamma_* \notin \mathcal{G}(\Lambda)$.

However, there are $\tilde{\Lambda}$ with distinct elements arbitrarily close to Λ. The multiplicity and linear independence qualifications must hold at $(\tilde{\Lambda}, \tilde{\Gamma}_*)$ due to the continuity of singular values and singular vectors. Here $\tilde{\Gamma}_*$ is the point where $\kappa_r(\tilde{\Lambda}, A, B)$ is attained. Consequently, $\tau_r(\tilde{\Lambda}, A, B) = \kappa_r(\tilde{\Lambda}, A, B)$. Finally, the continuity of $\tau_r(\Lambda, A, B)$ and $\kappa_r(\Lambda, A, B)$ with respect to Λ imply $\tau_r(\Lambda, A, B) = \kappa_r(\Lambda, A, B)$ even when Λ is not comprised of distinct elements.

Our ultimate aim is a singular value characterization for $\tau_r(A, B)$. We can infer that $\tau_r(A, B)$ is the infimum of $\kappa_r(\Lambda, A, B)$ over all Λ provided that the multiplicity and linear independence qualifications are satisfied at the optimal (Λ, Γ). In particular this is true even if these qualifications are violated at other Λ. Suppose $\tau_r(\Lambda_*, A, B) = \kappa_r(\Lambda_*, A, B)$, where Λ_* is the Λ value minimizing $\kappa_r(\Lambda, A, B)$. Then from (3.1) for all Λ we have

$$\tau_r(\Lambda, A, B) \geq \kappa_r(\Lambda, A, B) \geq \kappa_r(\Lambda_*, A, B) = \tau_r(\Lambda_*, A, B),$$

that is,

$$\kappa_r(\Lambda_*, A, B) = \tau_r(\Lambda_*, A, B) = \inf_{\Lambda} \tau_r(\Lambda, A, B).$$

THEOREM 3.7. *Let*

$$\kappa_r(A, B) := \inf_{\Lambda} \kappa_r(\Lambda, A, B) = \inf_{\Lambda} \sup_{\Gamma} \sigma_{k+1} \left(\mathcal{L}(\Lambda, \Gamma, A, B) \right),$$

where $k := (n - r)(n - 1)$. *Suppose that the multiplicity and linear independence qualifications hold at a point* (Λ_*, Γ_*) *where* $\kappa_r(A, B)$ *is attained. Then*

(i) $\tau_r(A, B) = \kappa_r(A, B)$,

(ii) *a nearest system whose reachable subspace has dimension at most* r *is given by* $(A + \Delta A_*, B + \Delta B_*)$ *where* $[\Delta A_* \ \ \Delta B_*]$ *is as defined in* (3.4) *in terms of the singular vectors of* $\mathcal{L}(\Lambda_*, \Gamma_*, A, B)$ *and* $\kappa_r(\Lambda_*, A, B)$.

4. Computational issues. We briefly outline how $\kappa_r(A, B)$ can be computed to a low precision. The technique outlined here is analogous to the technique described in [18, sect. 3] for computing the distance to a nearest matrix with a multiple eigenvalue with specified algebraic multiplicity. The interested reader can see [18] for the details. For the sake of simplicity let us use the notation

$$f(\Lambda, \Gamma) := \sigma_{k+1} \left(\mathcal{L}(\Lambda, \Gamma, A, B) \right) \quad \text{and} \quad g(\Lambda) := \sup_{\Gamma} \ f(\Lambda, \Gamma) \tag{4.1}$$

for $k := (n - r)(n - 1)$ in this section so that

$$\kappa_r(A, B) = \inf_{\Lambda} g(\Lambda).$$

4.1. Evaluation of $f(\Lambda, \Gamma)$. To solve the inner and outer optimization problems it will be necessary to compute $f(\Lambda, \Gamma)$ at various Λ and Γ. This can be achieved at a quadratic cost with respect to n for r close to n in addition to a Schur factorization of A. Let $A^* = QT^*Q^*$ be a Schur factorization so that T is lower triangular.

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We have

$$\mathcal{L}(\Lambda,\Gamma,A,B) = (I_{n-r}\otimes Q)\mathcal{L}(\Lambda,\Gamma,T,Q^*B)\left(I_{n-r}\otimes\begin{bmatrix} Q^* & 0\\ 0 & I_m\end{bmatrix}\right).$$

Therefore the singular values of $\mathcal{L}(\Lambda,\Gamma,A,B)$ and $\mathcal{L}(\Lambda,\Gamma,T,Q^*B)$ are the same. Furthermore, by applying $O(mn(n-r)^2)$ plane rotations at a cost of $O(mn^2(n-r)^3)$ the matrix $\mathcal{L}(\Lambda,\Gamma,T,Q^*B)$ can be converted into

$$\begin{bmatrix} \tilde{T}_1 & 0 & 0 & 0 & 0 & 0 & & 0 & 0\\ A_{2,1} & 0 & \tilde{T}_2 & 0 & 0 & 0 & & 0 & 0\\ A_{3,1} & 0 & A_{3,2} & 0 & \tilde{T}_3 & 0 & & 0 & 0\\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ A_{n-r,1} & 0 & A_{n-r,2} & 0 & & & & \tilde{T}_{n-r} & 0\end{bmatrix}, \tag{4.2}$$

where $\tilde{T}_j,\ j=1,\ldots,n-r$, denote lower triangular matrices. The matrix (4.2) is lower triangular, so its $(k+1)$th largest singular value can typically be retrieved at a cost of $O(n^2(n-r)^2)$ by an iterative singular value solver. Computation of $f(\Lambda,\Gamma)$ at p points would require $O(n^3+pmn^2(n-r)^3)$ floating point operations. The cubic term of n due to the Schur factorization is dominated by the quadratic term of n assuming $p\gg n$.

4.2. Inner maximization. The singular value function $f(\Lambda,\Gamma)$ is generically analytic at a given Λ and Γ. Any smooth numerical optimization algorithm, for instance BFGS, can be used to maximize $f(\Lambda,\Gamma)$ with respect to Γ. Strictly speaking this maximization problem is neither convex nor unimodal. However, it has properties almost as desirable as unimodality. In particular, we can verify whether or not a converged local maximizer is a global maximizer by checking the linear independence and multiplicity qualifications. Any local maximizer where these two qualifications hold is a global maximizer. If one of these qualifications is violated (which we expect to happen very rarely in practice), the smooth optimization algorithm can be restarted with a different initial guess. In our computations we used an implementation of the limited memory BFGS due to Liu and Nocedal [16] for the inner maximization.

BFGS and smooth optimization techniques would require the derivative of $f(\Lambda,\Gamma)$ with respect to Γ in addition to the function value $f(\Lambda,\Gamma)$. The derivative is available by means of the analytic formulas (see (3.6) and (3.7)) in terms of the singular vectors as soon as the singular value $f(\Lambda,\Gamma)$ is evaluated. Thus, the calculation of the derivative does not affect the computational cost in any significant way.

4.3. Outer minimization. For the outer minimization problem the following Lipschitzness result whose proof is similar to the proof of Theorem 3.2 in [18] is very helpful.

THEOREM 4.1. *The function $g(\Lambda)$ as defined in* (4.1) *is Lipschitz with Lipschitz constant one, i.e.,*

$$|g(\Lambda+\delta\Lambda)-g(\Lambda)|\le\|\delta\Lambda\|.$$

To optimize Lipschitz functions, derivative free techniques are suggested. Most of these techniques stem from an algorithm due to Piyavskii [21] and Shubert [23]. The Piyavskii–Shubert algorithm approximates a Lipschitz function by a piecewise

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linear function (or a piecewise cone in the multivariate case) lying underneath the function. It can be applied to minimize $g(\Lambda)$ and retrieve $\kappa_r(A, B)$ to a 4–5 decimal digit accuracy. In practice one can use the DIRECT algorithm, which is a sophisticated version of the Piyavskii–Shubert algorithm, that also attempts to estimate the Lipschitz constant locally [7, 12]. We refer to [18, sect. 3.3] for the details.

5. Numerical experiments. All of the numerical experiments are performed in MATLAB running on a MacBook Pro with a 2.8 GHz Intel Duo processor. The numerical algorithm outlined in the previous section is implemented in Fortran. A mex interface file calling the Fortran implementation is available on the author's webpage.[2] Note that the software is not particularly implemented to exploit the multiple arithmetic units in the Intel Duo processor. MATLAB exploits the multiple arithmetic units to a limited degree. For the evaluation of the singular value functions for various Λ and Γ we used the QR algorithm in LAPACK. We only experimented with small systems whose order does not exceed 100. For larger systems one should rather prefer an iterative algorithm such as the Lanczos algorithm.

5.1. Accuracy of the numerical solutions. The reachable subspace of the system with

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1+\epsilon_1 & 1 \\ 1 & 1 \\ 1 & 1-\epsilon_2 \end{bmatrix} \tag{5.1}$$

has dimension one for $\epsilon_1 = \epsilon_2 = 0$, because the left eigenvectors $v_1 = [1 \;\; -1 \;\; 0]^T$ and $v_2 = [0 \;\; 1 \;\; -1]^T$ associated with the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ of A lie in the left null space of B. Therefore $\tau_2(A, B) \leq \min(\epsilon_1, \epsilon_2)$ and $\tau_1(A, B) \leq \max(\epsilon_1, \epsilon_2)$ for $\epsilon_1, \epsilon_2 > 0$.

We computed these distances for $\epsilon_1 = 0.1$ and $\epsilon_2 = 0.4$ using the technique outlined in the previous subsection. The computed values

$$\tau_2(A, B) = 0.0587 \quad \text{and} \quad \tau_1(A, B) = 0.236$$

are roughly half of the upper bounds. The quantity $\tau_2(A, B)$ is the distance to uncontrollability, on which one can find an abundant amount of work in the literature as described in the introduction. What is remarkable here is that $\tau_1(A, B)$ is computed fairly accurately. In particular the computed uncontrollable eigenvalues of the nearest system whose reachable subspace has dimension one are $\lambda_1^* = 0.933$ and $\lambda_2^* = 2.140$. It is easy to see that, in general, when $r = n-2$ (that is, $\mathcal{L}(\Lambda, \Gamma, A, B)$ is a 2×2 block matrix with each of its blocks of size $n \times (n+m)$), we have

$$\begin{bmatrix} I_n & 0 \\ 0 & e^{i\theta} I_n \end{bmatrix} \mathcal{L}(\Lambda, \Gamma, A, B) \begin{bmatrix} I_{n+m} & 0 \\ 0 & e^{-i\theta} I_{n+m} \end{bmatrix} = \mathcal{L}(\Lambda, e^{i\theta}\Gamma, A, B).$$

Therefore the singular values of $\mathcal{L}(\Lambda, \Gamma, A, B)$ and $\mathcal{L}(\Lambda, e^{i\theta}\Gamma, A, B)$ are the same and the inner optimization can be performed over real Γ. The level sets of the function $g(\Lambda)$ over $\Lambda = [\lambda_1 \;\; \lambda_2]^T \in \mathbb{R}^2$ are plotted in Figure 1. Here and hereafter $g(\Lambda)$ is the maximum singular value function defined in (4.1), that corresponds to the distance to the nearest system with Λ as uncontrollable eigenvalues. The asterisk in the figure marks the uncontrollable eigenvalues $(\lambda_1^*, \lambda_2^*)$ of the nearest system.

[2] http://home.ku.edu.tr/∼emengi/software.html

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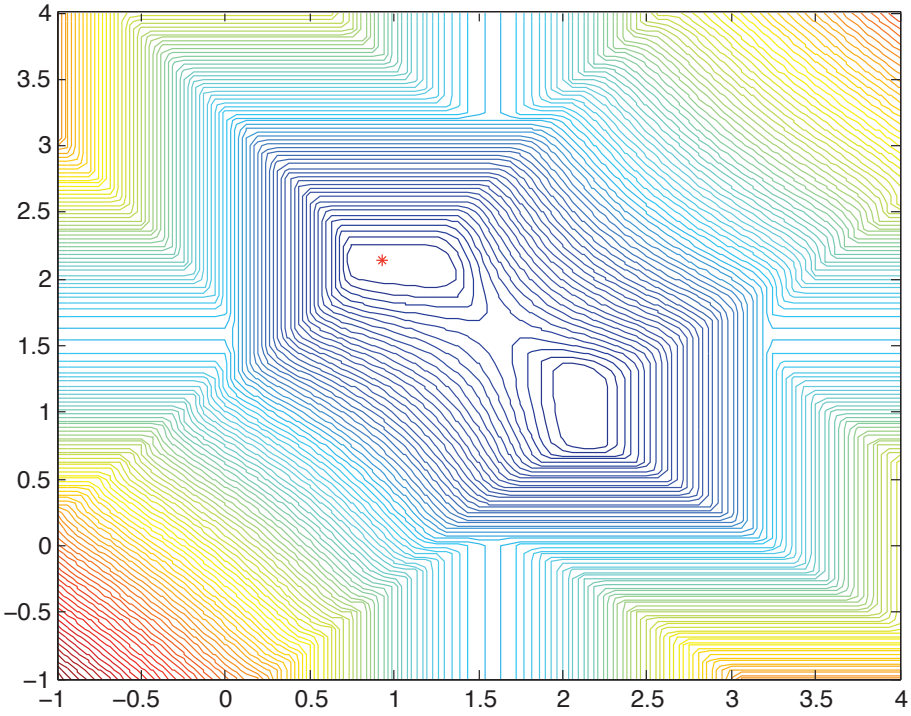

FIG. 1. *The level sets of $g(\Lambda)$ for the matrix pair (A, B) as given by* (5.1) *and $r = 1$ (the darker colors indicate smaller functions values).*

As a second example consider a pair (A, B) where A is 5×5 and originates from a discretization of the convection diffusion operator, and B is 5×2 with entries selected from a normal distribution with zero mean and variance equal to one. In this case $\tau_3(A, B)$ (that is, we seek a nearest system whose reachable subspace has dimension 3 and unreachable subspace has dimension 2) is computed as 1.712. In Figure 2 the level sets of $g(\Lambda)$ are illustrated over $\Lambda \in \mathbb{R}^2$ in which the asterisk is used to mark the computed uncontrollable eigenvalues $(\lambda_1^*, \lambda_2^*) = (-0.576, -0.576)$ of the nearest system.

From the level sets of $g(\Lambda)$ it is evident that g is symmetric with respect to Λ. For instance in the 2-dimensional case $g(\lambda_1, \lambda_2) = g(\lambda_2, \lambda_1)$. This obviously is not a coincidence, since both of the function values $g(\lambda_1, \lambda_2)$ and $g(\lambda_2, \lambda_1)$ correspond to the distance to a nearest system whose uncontrollable eigenvalues are λ_1 and λ_2.

5.2. Efficiency of the numerical algorithm. We test how the running time to compute $\tau_r(A, B)$ varies with respect to r and n on pairs (A, B) where $A \in \mathbb{R}^{n \times n}$ is obtained from a five point finite difference discretization of the Poisson equation and $B \in \mathbb{R}^{n \times m}$ has entries selected from a normal distribution with mean zero and variance equal to one. In particular, when $n = 9$ and $r = 7$ we seek a nearest system with two uncontrollable eigenvalues. In this case the level sets of $g(\Lambda)$ over $\mathbb{R}^2$ are depicted in Figure 3 together with the uncontrollable eigenvalues $(\lambda_1^*, \lambda_2^*) = (5.43, 4.03)$ of the nearest system marked with an asterisk.

The running times of the algorithm presented in the previous section for various n and r values are listed in Table 1. Eventually, as n gets larger, the variation in the

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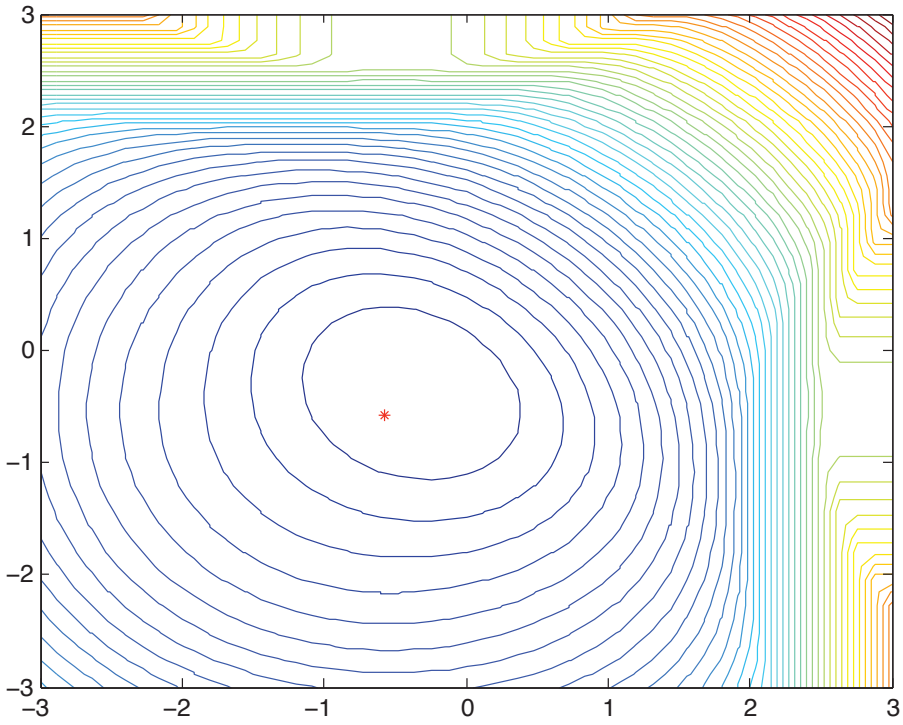

FIG. 2. *The level sets of $g(\Lambda)$ for the matrix pair (A, B) where $A \in \mathbb{R}^{5\times 5}$ is obtained from a discretization of the convection diffusion operator, $B \in \mathbb{R}^{5\times 2}$ and $r = 3$.*

running time with respect to n appears to be cubic. This would normally drop to n^2 had we used the iterative eigenvalue solvers rather than the direct solvers. It seems that the running time increases more drastically with respect to r (as r decreases). We did not attempt to compute $\tau_r(A, B)$ for $r = n - 3$ when $n = 64$ and $n = 121$. This is due to the fact that computations take an excessive amount of time.

6. Summary and concluding remarks. The main contribution of this work is the derivation of a singular value characterization for the distance from a system of order n to a nearest system whose reachable subspace is of dimension $r < n$ or smaller. The characterization is deduced under a mild multiplicity and linear independence assumptions. But our numerical and analytical experience indicates that the singular value characterization holds even when these assumptions are violated.

As a by-product of the derivation a generalization of the PBH test is provided to verify whether the dimension of the reachable subspace of a system of order n is r or smaller. A numerical algorithm exploiting the Lipschitzness of singular values is outlined and implemented in Fortran to solve the derived singular value optimization problems. All the discussions in this work extend to observability. In particular a singular value characterization similar to that given by Theorem 3.7 holds for a nearest system whose observable subspace is of dimension r or smaller.

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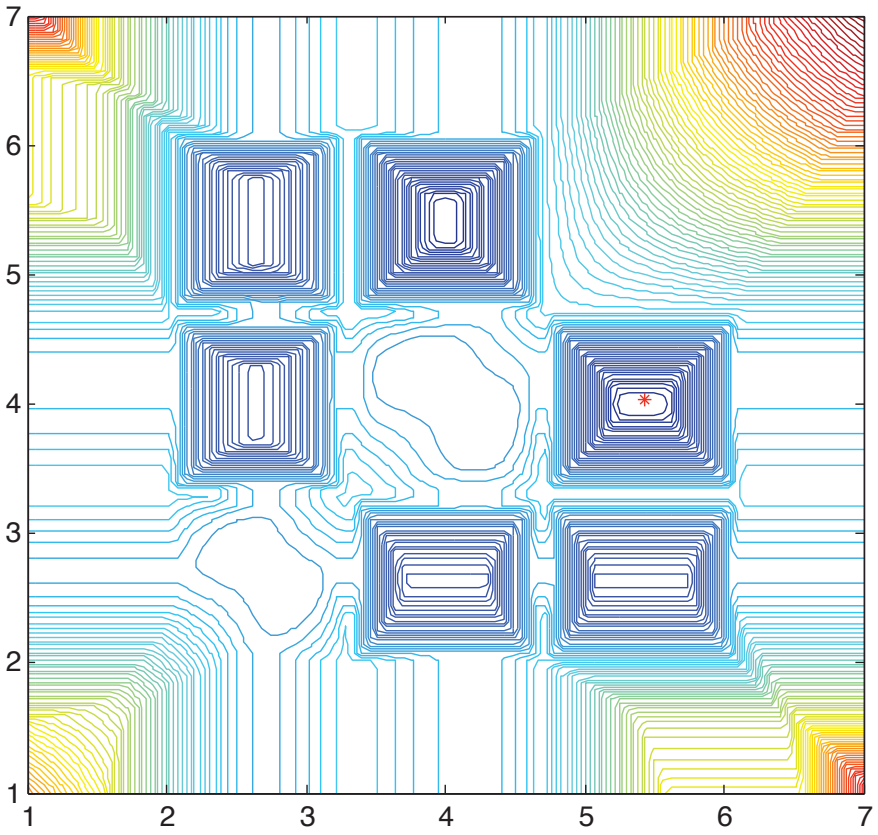

FIG. 3. *The level sets of $g(\Lambda)$ for the matrix pair (A, B) where $A \in \mathbb{R}^{9\times 9}$ is obtained from a finite difference discretization of the Poisson equation by using the five point formula, $B \in \mathbb{R}^{9\times 3}$, and $r = 7$.*

TABLE 1
CPU-times in seconds for the Poisson, random matrix pair.

(n, m) / r	$(n-1)$	$(n-2)$	$(n-3)$
$(9, 3)$	1	16	108
$(25, 8)$	2	63	1400
$(64, 20)$	4	284	
$(121, 40)$	10	1687	

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