Question 1. (20 points) Consider the system of three linear equations

$$\begin{aligned} x_1 + ax_2 + 3x_3 &= 0 \\ 3x_1 + (2+4a)x_2 + (b+9)x_3 &= 0 \\ 2x_1 + (b+2a+1)x_2 + 6x_3 &= 0 \end{aligned} \tag{1}$$

in the unknowns x_1, x_2, x_3 depending on two parameters $a, b \in \mathbb{R}$.

Determine the values of $a, b \in \mathbb{R}$ so that the system in (1)

(i) has infinitely many solutions,

(ii) has finitely many solutions, and

(iii) has no solution.

Form the augmented matrix, and perform row-reduction.

$$\begin{bmatrix} 1 & a & 3 & 0 \\ 3 & 2+4a & b+9 & 0 \\ 2 & b+2a+1 & 6 & 0 \end{bmatrix} \xrightarrow[r_3 := r_3 - 2r_1]{r_2 := r_2 - 3r_1} \begin{bmatrix} 1 & a & 3 & 0 \\ 0 & 2+a & b & 0 \\ 0 & b+1 & 0 & 0 \end{bmatrix}$$

Two cases, $a = -2$ (Case 1)

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 0 & b & 0 \\ 0 & b+1 & 0 & 0 \end{bmatrix} \xrightarrow[r_2 \leftrightarrow r_3]{} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & b+1 & 0 & 0 \\ 0 & 0 & b & 0 \end{bmatrix}$$

$a \neq -2$ (Case 2)

$$\begin{bmatrix} 1 & a & 3 & 0 \\ 0 & 2+a & b & 0 \\ 0 & b+1 & 0 & 0 \end{bmatrix} \xrightarrow[r_3 := r_3 - \frac{(b+1)}{(2+a)} r_2]{} \begin{bmatrix} 1 & a & 3 & 0 \\ 0 & 2+a & b & 0 \\ 0 & 0 & \frac{-b(b+1)}{2+a} & 0 \end{bmatrix}$$

(iii) System always has the solution 0

(i) has infinitely many solutions, iff there is a free variable.

In both Case 1 and Case 2, there is a free variable $\iff$ $\cancel{b \neq 0}$ $b = 0$ OR $b = -1$ (OR BOTH)

(ii) has finitely many solutions (namely the trivial solution 0) $\iff$ there is no free variable $\iff$ $b \neq 0$ AND $b \neq -1$

Question 2.
This question has two parts that are independent of each other.

(a) (10 points) Find bases for the column and row spaces of the following matrix.

$$M = \begin{bmatrix} 1 & 2 & -1 & 3 & -2 \\ -1 & 4 & -11 & 6 & 4 \\ 2 & 1 & 4 & 1 & -5 \end{bmatrix}$$

Show the details of your work.

(b) (10 points) Let A be an $m \times n$ matrix such that the linear system

$$A\mathbf{x} = \mathbf{b}$$

has a solution for every $\mathbf{b} \in \mathbb{R}^m$.

What is the dimension of the null space of A? You must support your answer.

(a) Let us perform row reduction of M into echelon form.

$$\begin{array}{ccccc} & & & \text{PIVOT} & \text{COLUMNS} \\ M \xrightarrow[r_3 := r_3 - 2r_1]{r_2 := r_2 + r_1} \begin{bmatrix} 1 & 2 & -1 & 3 & -2 \\ 0 & 6 & -12 & 9 & 2 \\ 0 & -3 & 6 & -5 & -1 \end{bmatrix} & \xrightarrow{r_3 := r_3 + \frac{1}{2} r_2} & \begin{bmatrix} 1 & 2 & -1 & 3 & -2 \\ 0 & 6 & -12 & 9 & 2 \\ 0 & 0 & 0 & -1/2 & 0 \end{bmatrix} \end{array}$$

Echelon form

Basis for column space (Pivot columns of M given by)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}$$

Basis for row space (given by the nonzero rows of the echelon form)

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 6 \\ -12 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1/2 \\ 0 \end{bmatrix} \right\}$$

(b) Since $\underset{m \times n}{A}x = b$ has a solution $\forall b \in \mathbb{R}^m$,

$$\operatorname{Col} A = \mathbb{R}^m \implies \dim \operatorname{Col} A = m$$

Now by using the rank-nullity thm,

$$\underbrace{\dim \operatorname{Col} A}_{= \operatorname{Rank}(A)} + \dim \operatorname{Nul} A = n \implies \boxed{\dim \operatorname{Nul} A = n - m}$$

Question 3.

(a) **(10 points)** Letting

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

find all $\mathbf{x}_* \in \mathbb{R}^3$ such that

$$(\times) \qquad \|\mathbf{b} - A\mathbf{x}_*\| \leq \|\mathbf{b} - A\mathbf{x}\| \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

(b) **(10 points)** Consider the vector space $M_{2\times 2}$ consisting of all 2×2 matrices with real entries (over the set of real numbers) with the inner product and the norm

$$\left\langle \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \right\rangle := f_{11}g_{11} + f_{12}g_{12} + f_{21}g_{21} + f_{22}g_{22}, \quad \text{and}$$

$$\left\| \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \right\| := \sqrt{\left\langle \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \right\rangle} = \sqrt{f_{11}^2 + f_{12}^2 + f_{21}^2 + f_{22}^2},$$

respectively. Moreover, let $\widetilde{M}_{2\times 2}$ denote the subspace of $M_{2\times 2}$ consisting of all 2×2 real symmetric matrices.

Find $D_* \in \widetilde{M}_{2\times 2}$ such that

$$\left\| \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix} - D_* \right\| \leq \left\| \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix} - D \right\| \qquad \text{for all } D \in \widetilde{M}_{2\times 2}.$$

(a) Let us solve, by solving the normal equations.

(Normal equations)
$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 1 & 1 & -2 \\ -1 & 1 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix}}_{A} x_* = \underbrace{\begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 1 & 1 & -2 \\ -1 & 1 & 1 & 1 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}}_{b}$$

$$(+) \qquad \begin{bmatrix} 4 & 6 & 0 \\ 6 & 10 & -2 \\ 0 & -2 & 4 \end{bmatrix} x_* = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

Set of solutions for $(\times)$ is precisely the set of solutions for $(+)$.

Row-reduction

$$\underbrace{\begin{bmatrix} 4 & 6 & 0 & 3 \\ 6 & 10 & -2 & 4 \\ 0 & -2 & 4 & 1 \end{bmatrix}}_{\text{Augmented matrix}} \xrightarrow[r_2 := r_2 - \frac{3}{2} r_1]{} \begin{bmatrix} 4 & 6 & 0 & 3 \\ 0 & 1 & -2 & -1/2 \\ 0 & -2 & 4 & 1 \end{bmatrix}$$

$$\xrightarrow[r_3 := r_3 + r_2]{} \begin{bmatrix} 4 & 6 & 0 & 3 \\ 0 & 1 & -2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[r_1 := r_1 - 6 r_2]{} \begin{bmatrix} 4 & 0 & 12 & 6 \\ 0 & 1 & -2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[r_1 := \frac{1}{4} r_1]{} \begin{bmatrix} 1 & 0 & 3 & 3/2 \\ 0 & 1 & -2 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

General solution

x_3 is free

$x_1 = 3/2 - 3x_3$

$x_2 = -1/2 + 2x_3$

in vectoral form

$$\left\{ \begin{bmatrix} 3/2 \\ -1/2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

(b) By the best approximation thm, letting $M = \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix}$, we have

$$\| M - \text{proj}_{\tilde{M}_{2\times 2}} M \| \leq \| M - D \| \quad \forall D \in \tilde{M}_{2\times 2}$$

that is $D_* = \text{proj}_{\tilde{M}_{2\times 2}} M$.

$$\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{u_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{u_2}, \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{u_3} \right\}$$ is

an orthogonal basis for $\tilde{M}_{2\times 2}$

(as indeed $\langle u_1, u_2 \rangle = \langle u_1, u_3 \rangle = \langle u_2, u_3 \rangle = 0$
and $\text{span}\{u_1, u_2, u_3\} = \tilde{M}_{2\times 2}$)

Hence, (3c)

$$D_* = \operatorname{proj}_{\tilde{M}_{2\times 2}} M$$

$$= \frac{\langle u_1, M\rangle}{\langle u_1, u_1\rangle} u_1 + \frac{\langle u_2, M\rangle}{\langle u_2, u_2\rangle} u_2 + \frac{\langle u_3, M\rangle}{\langle u_3, u_3\rangle} u_3$$

$$= \frac{3}{1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{(-6)}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{1}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 \\ -3 & 1 \end{bmatrix}$$

Question 4. The spectral decomposition (or the orthogonal eigenvalue decomposition) of a matrix A whose determinant is zero is given by

$$(*)\quad A = \underbrace{(2)}_{\lambda_1}\cdot \underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ -\frac{1}{\sqrt{2}}\end{bmatrix}}_{q_1}\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{bmatrix}}_{q_1^T} + \underbrace{(-1)}_{\lambda_2}\cdot \underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\end{bmatrix}}_{q_2}\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}}_{q_2^T} + \underbrace{(c)}_{\lambda_3}\cdot \underbrace{\mathbf{v}}_{q_3}\underbrace{\mathbf{v}^T}_{q_3^T}$$

for some $\mathbf{v} \in \mathbb{R}^3$, and a real number $c \in \mathbb{R}$.

(a) **(5 points)** Find the eigenvalues of A and the value of c. You must justify your answer.

(b) **(5 points)** Find $\mathbf{v}$.

(c) **(5 points)** The matrix A can expressed as $A = UDU^T$ for some 3×3 orthogonal matrix U (that is for some 3×3 matrix U with orthonormal columns) and a 3×3 diagonal matrix D.

If $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then find U.

(d) **(5 points)** Given $a, b \in \mathbb{R}$, find a vector $\mathbf{x} \in \mathbb{R}^3$ in terms of a, b such that

$$\mathbf{x}^T A\mathbf{x} \;=\; a^2 - b^2.$$

(Hint: A properly performed change of variable $\mathbf{x} = P\mathbf{y}$ yields $\mathbf{x}^T A\mathbf{x} = c_1y_1^2 + c_2y_2^2 + c_3y_3^2$ for some constants $c_1, c_2, c_3 \in \mathbb{R}$.)

(a) The orthogonal eigenvalue decomposition in $(*)$ is of the form

$$A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_3^T + \lambda_3 q_3 q_3^T$$

where

$$\lambda_1 = 2,\quad \lambda_2 = -1,\quad \lambda_3 = c$$

are the eigenvalues of A, and

$$q_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2}\end{bmatrix},\quad q_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2}\end{bmatrix},\quad q_3 = v$$

are the corresponding eigenvectors such that $\{q_1, q_2, v\}$ is ~~orthogonal~~ orthonormal.

Since $\det A = 0$, A is not invertible implying 0 must be an eigenvalue of A. Hence, $\boxed{c=0}$. To summarize,

$$\lambda_1 = 2, \quad \lambda_2 = -1, \quad \lambda_3 = c = 0$$

are the eigenvalues of A.

(b) v is ~~the~~ an unit eigenvector corresponding to $\lambda_3 = c = 0$, that is $v \neq 0$ satisfies

$$Av = 0$$

$$\Longleftrightarrow \left((2)\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} + (-1)\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \right) v = 0$$

$$\Longleftrightarrow \begin{bmatrix} 1/2 & 0 & -3/2 \\ 0 & 0 & 0 \\ -3/2 & 0 & 1/2 \end{bmatrix} v = 0$$

$$\Longleftrightarrow v = c\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad c \neq 0.$$

Since, $\|v\| = 1$, we have

$$v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{OR} \quad v = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

(c) $$A = U D U^T$$

where

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

$$U = [q_2 \;\; v \;\; q_1]$$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

(d) If we let $x = U\underline{y}$, then

$$x^T A x = \underline{y}^T U^T A U \underline{y}$$

$$= \underline{y}^T \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underline{y}$$

$$= -y_1^2 + 2y_3^2$$

Setting $a^2 - b^2 = 2y_3^2 - y_1^2$,

we can for instance choose $y_3 = \frac{a}{\sqrt{2}}$, $y_1 = b$.

Hence, for $x = U \underbrace{\begin{bmatrix} b \\ 0 \\ a/\sqrt{2} \end{bmatrix}}_{y} = b\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} + \frac{a}{\sqrt{2}}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$

$$= \begin{bmatrix} b/\sqrt{2} + a/2 \\ 0 \\ b/\sqrt{2} - a/2 \end{bmatrix}$$

we have $x^T A x = a^2 - b^2$

Question 5. Every $n \times n$ real matrix A all of whose eigenvalues are real has a factorization of the form

$$A = QRQ^T \tag{2}$$

for some $n \times n$ orthogonal matrix Q and some $n \times n$ real upper triangular matrix R (that is all entries of R below the diagonal are zero), which is called a Schur factorization of A.

In each part, prove or disprove the statement concerning the factorization of A in (2).

(Note: To disprove the statement, it is sufficient to give a counter example. To prove, you must show that the statement is true for every Schur factorization as in (2).)

(a) **(7 points)** If a scalar $\lambda \in \mathbb{R}$ appears k times along the diagonal of R, then λ is an eigenvalue of A with algebraic multiplicity k.

(b) **(6 points)** Every column of Q is an eigenvector of A.

(c) **(7 points)** If A is symmetric, that is if $A^T = A$, then R is a diagonal matrix.

THE STATEMENT IS TRUE.

(a) Since, Q is orthogonal, $Q^T = Q^{-1}$ and A, R are similar matrices. Consequently, A and R have the same characteristic polynomial.

Now, suppose λ appears k times along the diagonal of R

$\Longrightarrow$ λ is a root of the characteristic polynomial of R with multiplicity k

$\Longrightarrow$ λ is a root of the characteristic polynomial of A with multiplicity k

$\Longrightarrow$ λ is an eigenvalue of A with algebraic multiplicity k.

(b) THE STATEMENT IS FALSE
(THE FIRST COLUMN OF Q IS AN EIGENVECTOR OF A, BUT THE OTHER COLUMNS OF Q ARE NOT NECESSARILY EIGENVECTORS OF A.)

As a counter example, take

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

with Schur factorization

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{Q^T}.$$

Second column of Q, that is $q_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is not eigenvector of A, as indeed

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{q_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\neq c \begin{bmatrix} 0 \\ 1 \end{bmatrix} \forall c \right)$$

(c) THE STATEMENT IS TRUE

Suppose $A = QRQ^T$ is symmetric. Then

$$QRQ^T = A = A^T = (QRQ^T)^T = QR^TQ^T$$

We have $QRQ^T = QR^TQ^T$. Multiplying this equation with Q^T from left and with Q from right yields $\boxed{R = R^T}$

Hence, R is an upper triangular, symmetric matrix implying R is diagonal.