MATH 409/509
SPRING 2011

SOLUTIONS TO FINAL

Q1. (a)

Primal LP

$$\begin{aligned} &\underset{x\in\mathbb{R}^4}{\text{minimize}} \quad c^T x \\ &\text{subject to} \\ &Ax = b \\ &x \geq 0 \end{aligned}$$

where

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 1 & -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad c = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

Dual LP

$$\begin{aligned} &\underset{\pi\in\mathbb{R}^2,\, s\in\mathbb{R}^4}{\text{maximize}} \quad b^T \pi \\ &\text{subject to} \\ &A^T\pi + s = c \\ &s \geq 0 \end{aligned}$$

that is

$$\begin{aligned} &\underset{\pi\in\mathbb{R}^2,\, s\in\mathbb{R}^4}{\text{maximize}} \quad \pi_1 + 2\pi_2 \\ &\text{subject to} \end{aligned}$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & -1 \\ \cdots \end{bmatrix}\pi + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \cdots \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ \cdots \end{bmatrix}, \quad \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \cdots \end{bmatrix} \geq 0$$

(b)

$$\underset{\pi^+,\pi^- \in \mathbb{R}^2,\ s \in \mathbb{R}^4}{\text{minimize}} \quad -\pi_1^+ + \pi_1^- - 2\pi_2^+ + 2\pi_2^-$$

subject to

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 2 & 1 & -2 & -1 \\ 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \pi^+ \\ \pi^- \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\pi^+, \pi^-, s \geqslant 0$$

Q2. (a)

$$\underset{x_1, x_2, y_1, y_2 \in \mathbb{R}}{\text{minimize}} \quad 4\,\frac{y_1}{2} + 4\,\frac{y_2+y_1}{2} + \frac{3+y_2}{2}$$

subject to

$$(x_1 - 0)^2 + (y_1 - 0)^2 = 4$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = 4$$

$$(\overset{3}{\cancel{x_3}} - x_2)^2 + (3 - y_2)^2 = 1$$

$$y_2 - \frac{x_2^2}{6} - \frac{x_2}{4} + 1 \geqslant 0$$

$$y_1 - \frac{x_1^2}{6} - \frac{x_1}{4} + 1 \geqslant 0$$

(b) Mixed Logarithmic Barrier-Penalty Function

$$M(x_1, x_2, \cancel{y_1}, y_2; \mu) = 4y_1 + \frac{5y_2}{2} + \frac{3}{2} + \frac{1}{2\mu}\left(x_1^2 + y_1^2 - 4\right)^2$$

$$+\frac{1}{2\mu}\left((x_2-x_1)^2+(y_2-y_1)^2-4\right)^2$$

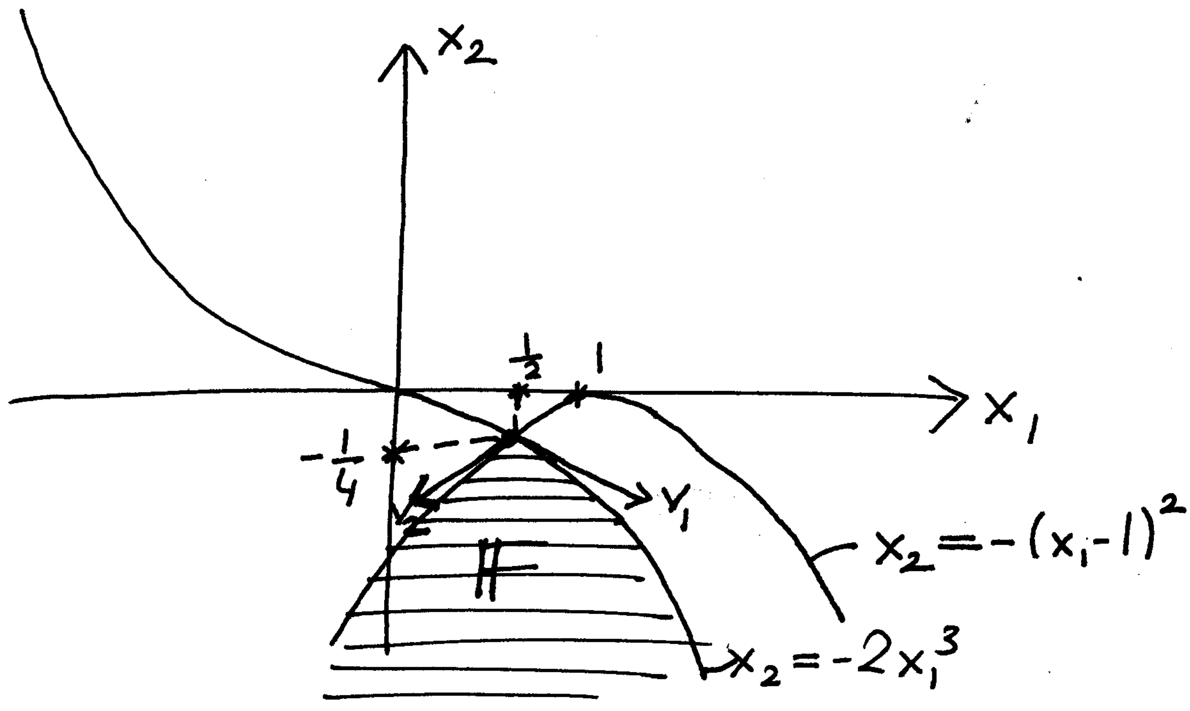
$$+\frac{1}{2\mu}\left((3-x_2)^2+(3-y_2)^2-1\right)^2$$

$$-\mu\ln\left(y_2-\frac{x_2^2}{6}-\frac{x_2}{4}+1\right)$$

$$-\mu\ln\left(y_1-\frac{x_1^2}{6}-\frac{x_1}{4}+1\right)$$

Q3.

(a)



$$-2x_1^3=-(x_1-1)^2 \iff 2x_1^3-x_1^2+2x_1-1=0$$

$$\iff (2x_1-1)(x_1^2+1)=0$$

$$\iff x_1=1/2$$

Line tangent to $x_2=-2x_1^3$ at $(\frac{1}{2},-\frac{1}{4})$ has slope

$$\left.\frac{d(-2x_1^3)}{dx_1}\right|_{x_1=1/2}=-\frac{3}{2}$$

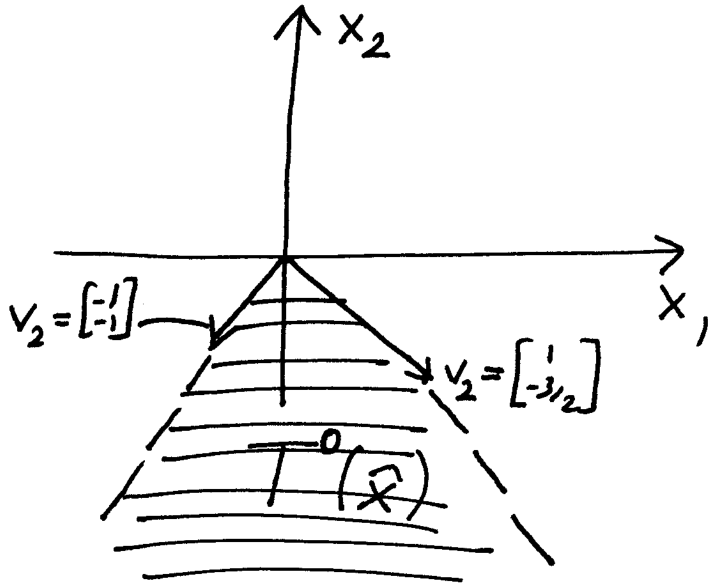
$$V_1 = \left(1, -\frac{3}{2}\right) \quad \text{(OR ANY POSITIVE MULTIPLE)}$$

Line tangent to $x_2 = -(x_1-1)^2$ at $\left(\frac{1}{2}, -\frac{1}{4}\right)$ has slope

$$\left.\frac{d\left(-(x_1-1)^2\right)}{dx_1}\right|_{x_1 = 1/2} = 1$$

$$V_2 = (-1, -1) \quad \text{(OR ANY POSITIVE MULTIPLE)}$$

Tangent cone ~~is the any~~ consists of all nonnegative linear combinations of V_1 and V_2.



$$T^0(\hat{x}) = \left\{ \alpha_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -3/2 \end{bmatrix} : \alpha_1, \alpha_2 \geq 0 \right\}$$

(b) $$J_a(\hat{x}) = \begin{bmatrix} \nabla c_1(\hat{x})^T \\ \nabla c_2(\hat{x})^T \end{bmatrix} = \begin{bmatrix} -3/2 & -1 \\ 1 & -1 \end{bmatrix}$$

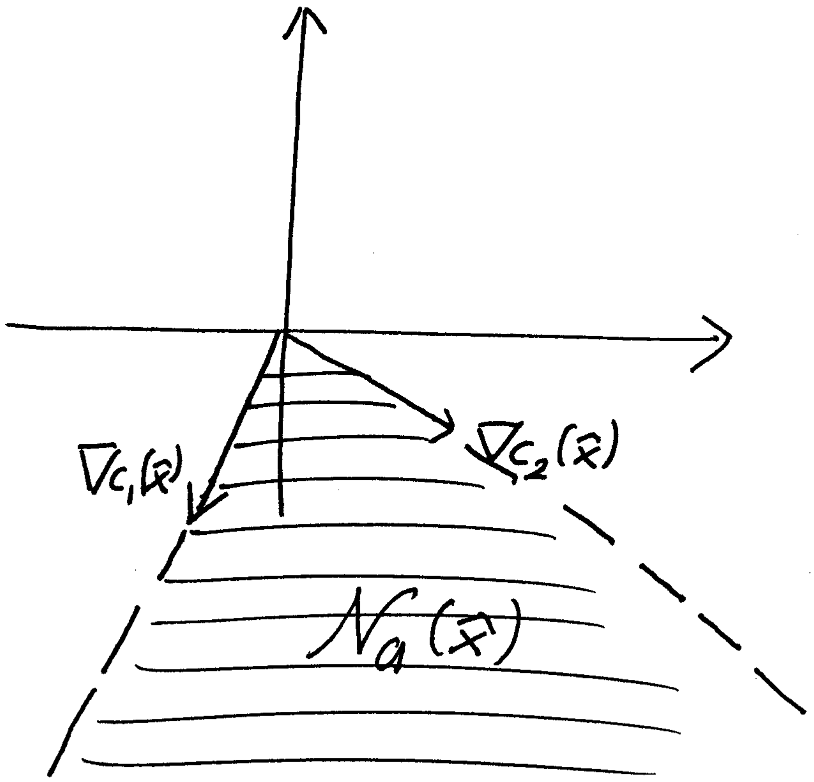
Since $\{\nabla c_1(\hat{x}), \nabla c_2(\hat{x})\}$ is linearly independent, LICQ holds. Consequently constraint qualification holds, that is

$$T^{\circ}(\hat{x}) = \{ p \in \mathbb{R}^2 : J_a(\hat{x})\, p \geq 0 \}$$

(c)

$$\mathcal{N}_a(\hat{x}) = \{ \alpha_1 \nabla c_1(\hat{x}) + \alpha_2 \nabla c_2(\hat{x}) : \alpha_1, \alpha_2 \geq 0 \}$$

$$= \left\{ \alpha_1 \begin{bmatrix} -3/2 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} : \alpha_1, \alpha_2 \geq 0 \right\}$$



(d) $\nabla f(\hat{x}) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

(1) $\nabla f(\hat{x}) = \lambda_1 \nabla c_1(\hat{x}) + \lambda_2 \nabla c_2(\hat{x})$

holds for $\lambda_1 = 0$ and $\lambda_2 = 1 \geq 0$

(that is $\nabla f(\hat{x}) = J_a(\hat{x})^T \lambda$ for some $\lambda \geq 0$)

(2) $c_1(\hat{x}) = c_2(\hat{x})$

Consequently first order necessary conditions holds.

Q4. (a) Since constraints are linear, constraint qualification holds at all x.

If x_* is a local minimizer, then following KKT conditions holds for some $\lambda \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$:

<u>KKT conditions</u>

(1) $Ax_* = b$

(2) $x_* \geqslant 0$

(3) $\underbrace{Hx_*}_{\nabla f(x_*)} = A^T\lambda + s$

(4) $s \geqslant 0$

(5) $s^T x_* = 0$ $\boxed{\text{COMPLEMENTARITY}}$

(b)

Suppose x is any feasible point and x_* is a point satisfying the KKT conditions. Then

$$f(x) = \frac{1}{2} x^T H x$$

$$= \frac{1}{2}\Big(x_*^T H x_* + (x-x_*)^T H (x-x_*) + \underbrace{2(x-x_*)^T H x_*}_{\boxed{= (x-x_*)^T H x_* + x_*^T H (x-x_*) \ \text{(since } H \text{ is symmetric)}}} \Big)$$

$$= \frac{1}{2} x_*^T H x_* + \frac{1}{2} \underbrace{(x-x_*)^T H (x-x_*)}_{\geq 0 \ \boxed{\text{SINCE } H \succeq 0}}$$

$$+ \underbrace{(x-x_*)^T (A^T \lambda + s)}_{\big(\underbrace{x^T A^T}_{b \ \boxed{\text{SINCE } x \text{ IS FEASIBLE i.e. } Ax=b}} - \underbrace{x_*^T A^T}_{b \ \boxed{\text{SINCE } x_* \text{ IS FEASIBLE}}}\big)\lambda + \underbrace{x^T s}_{\geq 0 \ \boxed{\text{SINCE } x \text{ IS FEASIBLE, i.e. } x \geq 0}} - \underbrace{x_*^T s}_{0 \ \boxed{\text{COMPLEMEN}}}}$$

$$\geq f(x_*) = \frac{1}{2} x_*^T H x_* .$$

Consequently x_* is a global minimizer.

Q5.

First consider the unperturbed problem.

$$f_*(0) = c^T x_*(0)$$

where $x_*(0)$ satisfies

(1) $A x_*(0) = b$

(2) $x_*(0) \geqslant 0$

(3) $c = A^T \pi + s$

(4) $s \geqslant 0$

(5) $s^T x_*(0) = 0$

for some $\pi \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$.

From (3)

$$\begin{aligned} f_*(0) &= (A^T\pi + s)^T x_*(0) \\ &= \pi^T A x_*(0) + \underbrace{s^T x_*(0)}_{0\ (4)} \\ &= \pi^T b \end{aligned}$$

Focus on the perturbed problem

$$\begin{aligned} f_*(\epsilon) &= c^T x_*(\epsilon) \\ &= (A^T\pi + s)^T x_*(\epsilon) \\ &= \pi^T A x_*(\epsilon) + s^T x_*(\epsilon) \end{aligned}$$

$$= \pi^T (b + \epsilon e_m) + s^T x_*(\epsilon)$$

Notice that

$$(s^T x_*(\epsilon))_j = \begin{cases} 0 & s_j = 0 \ \left(\begin{array}{c}\text{CONSTRAINT}\\ \text{IS INACTIVE}\end{array}\right) \\ s_j (x_*(\epsilon))_j & s_j \neq 0 \ \left(\begin{array}{c}\text{CONSTRAINT}\\ \text{IS ACTIVE}\end{array}\right) \end{cases}$$

$$= \begin{cases} 0 & s_j = 0 \\ s_j (2\epsilon) & s_j \neq 0 \end{cases}$$

$$\Longrightarrow$$

$$s^T x_*(\epsilon) = (2\epsilon)\, s^T e_n$$

Consequently

$$f_*(\epsilon) = \pi^T (b + \epsilon e_m) + (2\epsilon)\, s^T e_n$$

and the derivative is given by

$$f_*'(0) = \lim_{\epsilon \to 0} \frac{f_*(\epsilon) - f_*(0)}{\epsilon} = \lim_{\epsilon \to 0} \frac{(\epsilon)\pi^T e_m + (2\epsilon) s^T e_n}{\epsilon}$$

$$= \underline{\underline{\pi^T e_m + 2 s^T e_n}}$$