Solution Set

MATH 107: Introduction to Linear Algebra

Final - Spring 2017

Duration : 150 minutes

#1	10	
#2	17	
#3	17	
#4	16	
#5	10	
#6	10	
#7	10	
#8	10	
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NAME & LAST NAME ______________

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- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

  SECTION 1 (EMRE MENGI TUTH 11:30-12:45) ______________

  SECTION 2 (EMRE MENGI, TUTH 8:30-9:45) ______________

  SECTION 3 (EMRE MENGI, MW 13:00-14:15) ______________

  SECTION 4 (DOĞAN BILGE, MW 14:30-15:45) ______________

Problem 1. (10 points) Find every polynomial $p \in \mathbb{P}_3$ such that

$$(\times) \quad p(-1) = 4 \quad \text{and} \quad p(2) = 3.$$

Show your work.
(Recall that $\mathbb{P}_n$ denotes the set of polynomials $p : \mathbb{R} \to \mathbb{R}$ of degree at most n.)

p is of the form $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$
and satisfies

$$p(-1) = a_0 - a_1 + a_2 - a_3 = 4$$
$$p(2) = a_0 + 2a_1 + 4a_2 + 8a_3 = 3$$

Form the augmented matrix and row reduce
(with respect to variables a_0, a_1, a_2, a_3)

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 1 & 2 & 4 & 8 & 3 \end{array}\right] \xrightarrow{r_2 := r_2 - r_1} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -1 & 4 \\ 0 & 3 & 3 & 9 & -1 \end{array}\right]$$

$$\xrightarrow[r_1 := r_1 + r_2]{r_2 := r_2/3} \left[\begin{array}{cccc|c} a_0 & a_1 & a_2 & a_3 & \\ 1 & 0 & 2 & 2 & 11/3 \\ 0 & 1 & 1 & 3 & -1/3 \end{array}\right]$$

a_2, a_3 – free
$a_0 = 11/3 - 2a_2 - 2a_3$
$a_1 = -1/3 - a_2 - 3a_3$

The set of all polynomials in $\mathbb{P}_3$ satisfying $(\times)$

$$\left\{ \left(\frac{11}{3} - 2a_2 - 2a_3\right) + \left(\frac{-1}{3} - a_2 - 3a_3\right)t + a_2 t^2 + a_3 t^3 \;\middle|\; a_2, a_3 \in \mathbb{R} \right\}$$

Problem 2. (17 points) The transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(\mathbf{p}) := \frac{d^2\mathbf{p}}{dt^2} + 3\frac{d\mathbf{p}}{dt} + 2\mathbf{p}$$

is linear. Furthermore, let $B = \{1, 1+t, 1+t+t^2\}$ and $C = \{1, t, t^2\}$, both of which are bases for $\mathbb{P}_2$.

b_1 b_2 b_3

Find a matrix M such that

$$[T(\mathbf{p})]_C = M[\mathbf{p}]_B$$

for every $\mathbf{p} \in \mathbb{P}_2$.

M is given by (as discussed in class)

$$M = \begin{bmatrix} [T(b_1)]_C & [T(b_2)]_C & [T(b_3)]_C \end{bmatrix}$$

where

$$T(b_1) = 2$$

$$T(b_2) = 3 + 2(1+t) = 5 + 2t$$

$$T(b_3) = 2 + 3(1+2t) + 2(1+t+t^2)$$

$$= 7 + 8t + 2t^2.$$

Hence

$$M = \begin{bmatrix} [2]_C & [5+2t]_C & [7+8t+2t^2]_C \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 7 \\ 0 & 2 & 8 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 3. Let

$$S = \text{span}\left\{ \underbrace{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix}1\\0\\1\\1\end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}}_{v_3} \right\}.$$

(a) (12 points) Find an *orthonormal basis* for S.

$\{u_1, u_2, u_3\}$ as below form an orthonormal basis for S

$$\left\{ \underbrace{\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}}_{u_1 = v_1/\|v_1\|} \right\}$$ — an orthonormal basis for span$\{v_1\}$

Finding an orthonormal basis for span$\{v_1, v_2\}$

$\{u_1, u_2\}$ – u_1 as above

$u_2 = q_2/\|q_2\|$ where

$$q_2 = v_2 - \langle v_2, u_1 \rangle u_1$$
$$= \begin{bmatrix}1\\0\\1\\1\end{bmatrix} - (3/2)\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}1/4\\-3/4\\1/4\\1/4\end{bmatrix}$$

hence $$u_2 = \begin{bmatrix}1/\sqrt{12}\\-3/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\end{bmatrix}$$

Finding an orthonormal basis for $S = \text{span}\{v_1, v_2, v_3\}$

$\{u_1, u_2, u_3\}$ – u_1, u_2 as above

$u_3 = q_3/\|q_3\|$ where

$$q_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$$
$$= \begin{bmatrix}1\\1\\0\\1\end{bmatrix} - (3/2)\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} - \left(\frac{-1}{\sqrt{12}}\right)\begin{bmatrix}1/\sqrt{12}\\-3/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\end{bmatrix} = \begin{bmatrix}1/3\\0\\-2/3\\1/3\end{bmatrix}$$

hence $$u_3 = \begin{bmatrix}1/\sqrt{6}\\0\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$$

(b) (5 points) Find the <u>*orthogonal projection*</u> of the vector $\mathbf{v} = (1, 0, 1, 0)$ onto S.

$$\text{proj}_S v = \langle v, u_1 \rangle u_1 + \langle v, u_2 \rangle u_2 + \langle v, u_3 \rangle u_3$$

$$= (1)\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \left(\frac{2}{\sqrt{12}}\right)\begin{bmatrix} 1/\sqrt{12} \\ -3/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \end{bmatrix} + \left(\frac{-1}{\sqrt{6}}\right)\begin{bmatrix} 1/\sqrt{6} \\ 0 \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 \\ 0 \\ 1 \\ 1/2 \end{bmatrix}$$

(Observe $v - \text{proj}_S v = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ -1/2 \end{bmatrix}$ that is $\text{proj}_S v \perp v - \text{proj}_S v$ as it should be)

Problem 4. Indicate in each part whether the statement is <u>*true or false*</u> for every pair of matrices A and B that are similar. Give a brief explanation for your answer.

(a) (4 points) The determinants of A and B are equal. TRUE

$B = S^{-1}AS$ for some invertible S. Hence,

$$\det(B) = \det(S^{-1})\det(A)\det(S)$$
$$= \det(S^{-1}S)\det(A) = \det(A)$$

(MP) multiplicative property of det

(b) (4 points) A and B have the same column space. FALSE

Because left-multiplication of A with S^{-1} is equivalent to applying row operations to A, and this changes the column space of A.

Ex: $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}^{-1} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, but $\text{Col}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \neq \text{Col}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

(c) (4 points) The set of eigenvalues of A and B are the same. Furthermore, the algebraic multiplicities of λ as an eigenvalue of A and as an eigenvalue of B are equal. TRUE

$$p_B(\lambda) = \det(B - \lambda I) \overset{MP}{=} \det S^{-1} \det(A - \lambda I) \det S$$
$$\overset{MP}{=} \det(S^{-1}S)\det(A - \lambda I) = p_A(\lambda)$$

Eigenvalues of B = Roots of $p_B(\lambda)$ = Roots of $p_A(\lambda)$ = Eigenvalues of A

(d) (4 points) The eigenspaces of A and B are the same. FALSE

$$Bv = \lambda v,\ v \neq 0 \iff A(Sv) = \lambda(Sv),\ Sv \neq 0$$

That is

v is an eigenvector of $B \iff Sv$ is an eigenvector of A

$$E_{\lambda,A} = \text{Nul}(A - \lambda I)$$
$$= \{Sv \mid v \in E_{\lambda,B}\} \neq E_{\lambda,B}$$

where $E_{\lambda,B} = \text{Nul}(B - \lambda I)$.

Problem 5. (10 points) Let us consider the vector space $\mathbb{P}_2$ with the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle := \int_{-1}^{1} \mathbf{p}(t)\mathbf{q}(t)\, dt$$

and the norm

$$\|\mathbf{p}\| := \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} = \sqrt{\int_{-1}^{1} [\mathbf{p}(t)]^2\, dt}.$$

Find $\mathbf{p}_* \in \mathbb{P}_1$ such that $\|t^2 - \mathbf{p}_*\| \le \|t^2 - \mathbf{p}\|$ for all $\mathbf{p} \in \mathbb{P}_1$.

By the best approximation thm

$$p_* = \text{proj}_{\mathbb{P}_1} t^2.$$

Observe that $\{1, t\}$ is orthogonal, since

$$\langle 1, t \rangle = \int_{-1}^{1} t\, dt = 0$$

Furthermore

$$\|1\|^2 = \int_{-1}^{1} 1\, dt = 2 \implies \|1\| = \sqrt{2}$$

$$\|t\|^2 = \int_{-1}^{1} t^2 dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{2}{3} \implies \|t\| = \sqrt{2/3}$$

Hence $\{1/\sqrt{2}, \sqrt{\tfrac{3}{2}}\, t\}$ is an orthonormal basis for $\mathbb{P}_1$.

$$p_* = \langle t^2, 1/\sqrt{2} \rangle (1/\sqrt{2}) + \langle t^2, \sqrt{\tfrac{3}{2}}\, t \rangle \left(\sqrt{\tfrac{3}{2}}\, t\right)$$

where

$$\langle t^2, 1/\sqrt{2} \rangle = \int_{-1}^{1} \frac{t^2}{\sqrt{2}} dt = \frac{\sqrt{2}}{3}, \quad \langle t^2, \sqrt{\tfrac{3}{2}}\, t \rangle = \int_{-1}^{1} \sqrt{\tfrac{3}{2}}\, t^3 dt = 0$$

t^3 is an odd function

$$\boxed{p_* = \frac{\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = \frac{1}{3}}$$

Problem 6. Indicate in each part whether the matrix A is *invertible or not*. Explaining your reasoning. Also in each part indicate *the dimension of the null space* of A.

(a) (6 points) $A = \mathbf{u}_1\mathbf{u}_1^T$ where $\mathbf{u}_1 \in \mathbb{R}^n$ is a nonzero vector and $n \geq 2$.

$$\text{Col}(A) = \{u_1(\underbrace{u_1^T v}_{\alpha}) \mid v \in \mathbb{R}^n\}$$

$$= \{\alpha u_1 \mid \alpha \in \mathbb{R}\} = \text{span}\{u_1\}$$

Hence,

$$\text{Rank } A = 1 \implies \boxed{\dim \text{Nul } A = n-1} \geq 1$$

$$\implies Ax \neq 0 \text{ for some } x \neq 0$$

$$\implies \boxed{A \text{ is not invertible.}}$$

(b) (4 points)

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 & t_1^4 \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 \\ 1 & t_4 & t_4^2 & t_4^3 & t_4^4 \\ 1 & t_5 & t_5^2 & t_5^3 & t_5^4 \end{bmatrix}$$

where $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}$ are distinct real numbers (that is $t_i \neq t_j$ for $i, j = 1, \ldots, 5$ such that $i \neq j$).

$$Ax = 0 \iff p(t_1) = p(t_2) = p(t_3) = p(t_4) = p(t_5) = 0$$

$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix}$

where

$$(+)\quad p(t) = x_1 + x_2 t + \cdots + x_5 t^4$$

$$\iff p \text{ (as in (+)) has 5 distinct roots}$$

$$\iff p(t) = 0 \text{ for all } t$$

$$\iff x_1 = x_2 = \cdots = x_5 = 0$$

Hence, $Ax = 0$ only for $x = 0$ implying $\boxed{A \text{ is invertible.}}$

Problem 7. Every real symmetric $n \times n$ matrix A satisfying $A^T = A$ has n real eigenvalues, say $\lambda_1, \ldots, \lambda_n$. Furthermore, the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of the symmetric matrix A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ can be chosen as vectors in $\mathbb{R}^n$ such that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is orthonormal (with respect to the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n$ on $\mathbb{R}^n$). In parts (a) and (b) below, you can make use of these facts.

(a) (6 points) Let A be a real symmetric $n \times n$ matrix. Prove that the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$$

is an *inner product* on $\mathbb{R}^n$ if and only if all eigenvalues of A are positive.

Suppose A has an eigenvalue $\lambda_j \leqslant 0$ and $v_j \neq 0$ be a corresponding eigenvector.

$$\langle v_j, v_j \rangle = v_j^T A v_j = \lambda_j v_j^T v_j$$
$$= \lambda_j \|v_j\|^2 \leqslant 0$$

Hence, $\langle x, x \rangle > 0$ for all $x \neq 0$ does not hold implying $\langle \cdot, \cdot \rangle$ is not an inner product

Now suppose eigenvalues $\lambda_1, \ldots, \lambda_n$ are all positive with corresponding eigenvectors $v_1, \ldots, v_n$ such that $\{v_1, \ldots, v_n\}$ is orthonormal. (Continues on a separate page)

(b) (4 points) Specifically, is the following function

$$\langle \mathbf{x}, \mathbf{y} \rangle = 3x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2 = \mathbf{x}^T \underbrace{\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}}_{A} \mathbf{y}$$

an inner product on $\mathbb{R}^2$?

Let us compute the eigenvalues of A.

$$p(\lambda) = \det(A - \lambda I_2) = \det \begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix}$$
$$= (3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8$$

$\lambda_1 = 4$ and $\lambda_2 = 2$ are the eigenvalues of A. (roots of $p(\lambda)$)

Since both eigenvalues are positive, from part (a), $\langle \cdot, \cdot \rangle$ is an inner product.

Problem 7(a), continues

Since $\{v_1, \ldots, v_n\}$ is orthonormal in $\mathbb{R}^n$, it is indeed an orthonormal basis for $\mathbb{R}^n$.

Every $x \in \mathbb{R}^n$ can be written of the form

$$x = \alpha_1 v_1 + \ldots + \alpha_n v_n$$

for some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$. But then

$$x^T A x$$
$$= (\alpha_1 v_1 + \ldots + \alpha_n v_n)^T A (\alpha_1 v_1 + \ldots + \alpha_n v_n)$$
$$= (\alpha_1 v_1 + \ldots + \alpha_n v_n)^T (\alpha_1 \lambda_1 v_1 + \ldots + \alpha_n \lambda_n v_n)$$
$$= (\alpha_1^2 \lambda_1 \|v_1\|^2 + \ldots + \alpha_n^2 \lambda_n \|v_n\|^2)$$

due to orthogonality; $\lambda_1 \geq 0$, $\lambda_n \geq 0$

$$\geq 0$$

as desired. (Hence positivity) ☒

Furthermore,

$$\langle x, y \rangle = x^T A y = x^T A^T y$$
$$= (Ax)^T y = y^T A x = \langle y, x \rangle$$

so symmetry holds.

Finally, let us show bilinearity

(x)
$$\begin{aligned}\langle x, y+z\rangle &= x^T A (y+z) \\ &= x^T [A(y+z)] \\ &= x^T [Ay + Az] \\ &= x^T A y + x^T A z \\ &= \langle x, y\rangle + \langle x, z\rangle\end{aligned}$$

(xx)
$$\begin{aligned}\langle x, \alpha y\rangle &= x^T [A(\alpha y)] \\ &= x^T (\alpha [Ay]) \\ &= \alpha\, x^T A y = \alpha \langle x, y\rangle,\end{aligned}$$

additionally

$$\begin{aligned}\langle x+z, y\rangle &\overset{\text{symmetry}}{=} \langle y, x+z\rangle \\ &\overset{\text{from (x)}}{=} \langle y, x\rangle + \langle y, z\rangle \\ &\overset{\text{symmetry}}{=} \langle x, y\rangle + \langle z, y\rangle\end{aligned}$$

$$\begin{aligned}\langle \alpha x, y\rangle &\overset{\text{symmetry}}{=} \langle y, \alpha x\rangle \\ &\overset{\text{from (xx)}}{=} \alpha \langle y, x\rangle \\ &\overset{\text{symmetry}}{=} \alpha \langle x, y\rangle\end{aligned}$$

Problem 8. Let S_1, S_2 be two subspaces of a vector space V such that $S_1 + S_2 = V$ and $S_1 \cap S_2 = \{0\}$, where $S_1 + S_2$ is defined by

$$S_1 + S_2 := \{\mathbf{s}_1 + \mathbf{s}_2 \mid \mathbf{s}_1 \in S_1, \mathbf{s}_2 \in S_2\}.$$

Every vector $\mathbf{v} \in V$ can be written of the form

$$\mathbf{v} = \mathbf{v}_{S_1} + \mathbf{v}_{S_2}, \tag{1}$$

for some $\mathbf{v}_{S_1} \in S_1$ and $\mathbf{v}_{S_2} \in S_2$ in a unique way. We refer (1) as the unique decomposition of $\mathbf{v}$.

(a) (5 points) Consider the transformation $P : V \to S_1$ defined by

$$P(\mathbf{v}) := \mathbf{v}_{S_1}$$

for every $\mathbf{v} \in V$, where $\mathbf{v}_{S_1}$ is as in (1) in the unique decomposition of $\mathbf{v}$. Prove that P is <u>linear</u>.

For every $v, w \in V$, suppose the decompositions are given by

$$v = v_{s_1} + v_{s_2} \quad \text{and} \quad w = w_{s_1} + w_{s_2}$$

where $v_{s_1}, w_{s_1} \in S_1$ and $v_{s_2}, w_{s_2} \in S_2$.

But then

$$v + w = (v_{s_1} + w_{s_1}) + (v_{s_2} + w_{s_2})$$

$$\alpha v = (\alpha v_{s_1}) + (\alpha v_{s_2})$$

where $v_{s_1} + w_{s_1}, \alpha v_{s_1} \in S_1$ and $v_{s_2} + w_{s_2}, \alpha v_{s_2} \in S_2$.

It follows that

$$P(v+w) = v_{s_1} + w_{s_1} = P(v) + P(w)$$

that is the additivity holds, and

$$P(\alpha v) = \alpha v_{s_1} = \alpha P(v)$$

that is the homogeneity holds. Hence P is linear.

(b) (5 points) Now suppose an inner product $\langle \cdot, \cdot \rangle$ is defined on V. Recall that a given vector $\mathbf{v} \in V$ can also be decomposed into

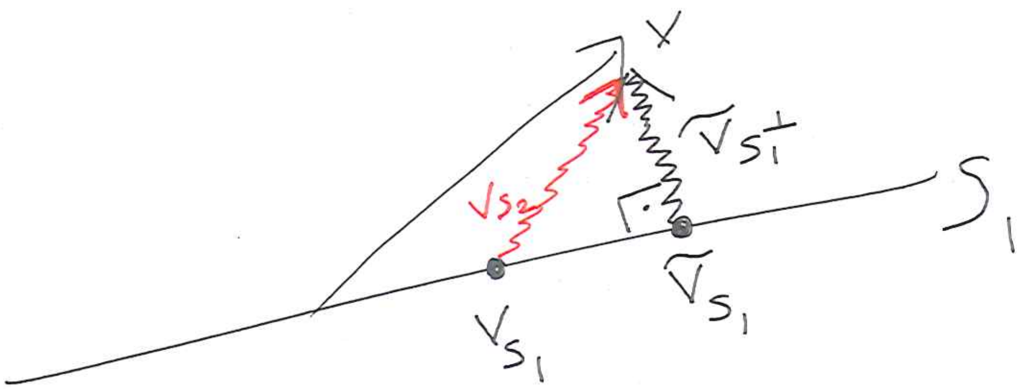
$$\mathbf{v} = \widetilde{\mathbf{v}}_{S_1} + \widetilde{\mathbf{v}}_{S_1^\perp},$$

for some $\widetilde{\mathbf{v}}_{S_1} \in S_1$ and $\widetilde{\mathbf{v}}_{S_1^\perp} \in S_1^\perp$ in a unique way.

Prove

$$\|\widetilde{\mathbf{v}}_{S_1^\perp}\| \leq \|\mathbf{v}_{S_2}\|$$

where $\mathbf{v}_{S_2}$ is as in (1) (on page 9) in the unique decomposition of $\mathbf{v}$.



Notice that

$$v_{S_1}, \widetilde{v}_{S_1} \in S_1 \implies \widetilde{v}_{S_1} - v_{S_1} \in S_1$$

also

$$\widetilde{v}_{S_1^\perp} = v - \widetilde{v}_{S_1} \in S_1^\perp.$$

Now

$$\|v_{S_2}\|^2 = \|v - v_{S_1}\|^2$$

$$= \|\underbrace{(v - \widetilde{v}_{S_1})}_{S_1^\perp} + \underbrace{(\widetilde{v}_{S_1} - v_{S_1})}_{S_1}\|^2$$

$$= \|\underbrace{v - \widetilde{v}_{S_1}}_{\widetilde{v}_{S_1^\perp}}\|^2 + \underbrace{\|\widetilde{v}_{S_1} - v_{S_1}\|^2}_{\geq 0}$$

Hence $\|v_{S_2}\| \geq \|\widetilde{v}_{S_1^\perp}\|$.