MATH 107: Introduction to Linear Algebra

Spring Semester 2020
Final Exam

#1	20	
#2	20	
#3	20	
#4	20	
#5	20	
$\sum$	100	

Question 1. (20 points) Consider the system of three linear equations

$$\begin{aligned} x_1 + ax_2 + 3x_3 &= 0 \\ 3x_1 + (2+4a)x_2 + (b+9)x_3 &= 0 \\ 2x_1 + (b+2a+1)x_2 + 6x_3 &= 0 \end{aligned} \tag{1}$$

in the unknowns x_1, x_2, x_3 depending on two parameters $a, b \in \mathbb{R}$.

Determine the values of $a, b \in \mathbb{R}$ so that the system in (1)

(i) has infinitely many solutions,

(ii) has finitely many solutions, and

(iii) has no solution.

Question 2.
This question has two parts that are independent of each other.

(a) **(10 points)** Find bases for the column and row spaces of the following matrix.

$$\begin{bmatrix} 1 & 2 & -1 & 3 & -2 \\ -1 & 4 & -11 & 6 & 4 \\ 2 & 1 & 4 & 1 & -5 \end{bmatrix}$$

Show the details of your work.

(b) **(10 points)** Let A be an $m \times n$ matrix such that the linear system

$$A\mathbf{x} = \mathbf{b}$$

has a solution for every $\mathbf{b} \in \mathbb{R}^m$.

What is the dimension of the null space of A? You must support your answer.

Question 3.

(a) **(10 points)** Letting

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

find all $\mathbf{x}_* \in \mathbb{R}^3$ such that

$$\|\mathbf{b} - A\mathbf{x}_*\| \leq \|\mathbf{b} - A\mathbf{x}\| \qquad \text{for all } \mathbf{x} \in \mathbb{R}^3.$$

(b) **(10 points)** Consider the vector space $M_{2\times 2}$ consisting of all 2×2 matrices with real entries (over the set of real numbers) with the inner product and the norm

$$\left\langle \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \right\rangle := f_{11}g_{11} + f_{12}g_{12} + f_{21}g_{21} + f_{22}g_{22}, \text{ and}$$

$$\left\| \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \right\| := \sqrt{\left\langle \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \right\rangle} = \sqrt{f_{11}^2 + f_{12}^2 + f_{21}^2 + f_{22}^2},$$

respectively. Moreover, let $\widetilde{M}_{2\times 2}$ denote the subspace of $M_{2\times 2}$ consisting of all 2×2 real symmetric matrices.

Find $D_* \in \widetilde{M}_{2\times 2}$ such that

$$\left\| \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix} - D_* \right\| \leq \left\| \begin{bmatrix} 3 & -4 \\ -2 & 1 \end{bmatrix} - D \right\| \qquad \text{for all } D \in \widetilde{M}_{2\times 2}.$$

Question 4. The spectral decomposition (or the orthogonal eigenvalue decomposition) of a matrix A whose determinant is zero is given by

$$A = (2) \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} + (-1) \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} + (c) \cdot \mathbf{v}\mathbf{v}^T$$

for some $\mathbf{v} \in \mathbb{R}^3$, and a real number $c \in \mathbb{R}$.

(a) **(5 points)** Find the eigenvalues of A and the value of c. You must justify your answer.

(b) **(5 points)** Find $\mathbf{v}$.

(c) **(5 points)** The matrix A can expressed as $A = UDU^T$ for some 3×3 orthogonal matrix U (that is for some 3×3 matrix U with orthonormal columns) and a 3×3 diagonal matrix D.

If $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then find U.

(d) **(5 points)** Given $a, b \in \mathbb{R}$, find a vector $\mathbf{x} \in \mathbb{R}^3$ in terms of a, b such that

$$\mathbf{x}^T A \mathbf{x} = a^2 - b^2.$$

(Hint: A properly performed change of variable $\mathbf{x} = P\mathbf{y}$ yields $\mathbf{x}^T A\mathbf{x} = c_1 y_1^2 + c_2 y_2^2 + c_3 y_3^2$ for some constants $c_1, c_2, c_3 \in \mathbb{R}$.)

Question 5. Every $n \times n$ real matrix A all of whose eigenvalues are real has a factorization of the form

$$A = QRQ^T \tag{2}$$

for some $n \times n$ orthogonal matrix Q and some $n \times n$ real upper triangular matrix R (that is all entries of R below the diagonal are zero), which is called a Schur factorization of A.

In each part, prove or disprove the statement concerning the factorization of A in (2).

(Note: To disprove the statement, it is sufficient to give a counter example. To prove, you must show that the statement is true for every Schur factorization as in (2).)

(a) **(7 points)** If a scalar $\lambda \in \mathbb{R}$ appears k times along the diagonal of R, then λ is an eigenvalue of A with algebraic multiplicity k.

(b) **(6 points)** Every column of Q is an eigenvector of A.

(c) **(7 points)** If A is symmetric, that is if $A^T = A$, then R is a diagonal matrix.