MATH 171B: Numerical Optimization

Instructor: Emre Mengi

Spring Quarter 2009
Final Exam
Wednesday, June 10
Duration : 3 hours

NAME ______________________

#1	15	
#2	15	
#3	15	
#4	17	
#5	18	
#6	20	
#7	*(Bonus) 10*	
Total	100	

- No calculators
- Show your work.
- Put your name in the box above.

Question 1. This question concerns the function

$$f(x) = x^2 - 5.$$

(a) (8 points) Given $x_0 \in \mathbf{R}$ and let $\{x_k\}$ be the sequence generated by Newton's method (for root finding) applied to $f(x)$. Write down the Newton update rule relating the iterates x_k and x_{k+1}.

(b) (7 points) Suppose that a sequence of Newton iterates $\{x_k\}$ converges to the root $x_* = \sqrt{5}$. What is the order of convergence for the sequence $\{x_k\}$? (Note : You can refer to a result discussed in class.)

Question 2. Suppose that a local minimizer of the function

$$f(x_1, x_2) = (x_1^2 + x_2^2)e^{x_1 - x_2}$$

over $\mathbf{R}^2$ is sought starting with the initial guess $x^{(0)} = (x_1^{(0)}, x_2^{(0)}) = (1, 1)$.

(a) (5 points) Apply one iteration of the method of steepest descent with the step-length $\alpha_0 = 1$.

(b) (10 points) Apply one iteration of the BFGS method with the inverse Hessian update rule

$$H_{k+1} = H_k + \left(1 + \frac{y_k^T H_k y_k}{s_k^T y_k}\right) \frac{s_k s_k^T}{s_k^T y_k} - \frac{s_k y_k^T H_k + H_k y_k s_k^T}{s_k^T y_k}$$

where $s_k = x^{(k+1)} - x^{(k)}$ and $y_k = \nabla f(x^{(k+1)}) - \nabla f(x^{(k)})$. Use

$$H_0 = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.5 \end{bmatrix}$$

as the initial inverse Hessian approximation and the step-length $\alpha_0 = 1$. Provide $x^{(1)}$, the new estimate for the local minimizer, and H_1, the updated inverse Hessian approximation.

Question 3. Consider the equality constrained problem

$$\begin{array}{ll} \text{minimize}_{x \in \mathbf{R}^n} & \frac{1}{2} x^T G x + d^T x \\ \text{subject} & Ax = b. \end{array} \tag{1}$$

where $G \in \mathbf{R}^{n \times n}$ is symmetric, $A \in \mathbf{R}^{m \times n}$, $d \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$.

(a) (9 points) Write down the first order necessary conditions that a minimizer x_* of (1) must satisfy.

(b) (6 points) Express the tangent cone $T^0(\bar{x})$ at any given point $\bar{x} \in \mathbf{R}^n$ for the equality constrained problem (1) in terms of the matrix A. (Hint : The constraint qualification holds at $\bar{x}$. Why?)

Question 4. Consider the constrained optimization problem

$$\begin{array}{ll} \text{minimize}_{x\in\mathbf{R}^n} & x_1x_2 \\ \text{subject} & 4x_1^2 + x_2^2 = 4. \end{array} \tag{2}$$

(a) (5 points) Write down the Lagrangian function for (2) and calculate its gradient.

(b) (5 points) Find all stationary points of the Lagrangian function.

(c) (7 points) Apply one iteration of the method of multipliers to problem (2) starting with the initial guesses $x_0 = (0, -1)$ for the local minimizer and $\lambda_0 = 1$ for the optimal Lagrange multiplier. Choose the step-length $\alpha_0 = 1$.

Question 5. Consider the inequality constrained optimization problem

$$\begin{array}{ll} \text{minimize}_{x\in\mathbf{R}^2} & x_1^3 - x_2 \\ \text{subject} & e^{x_1} - x_2 \geq 0 \\ & x_2 - e^{-x_1} \geq 0 \\ & x_1 \geq 0 \\ & -x_1 + 2 \geq 0 \end{array} \tag{3}$$

(a) (3 points) Show that $x(\alpha) = (\alpha, 1)$ is a feasible path at $\bar{x} = (0, 1)$.

(b) (9 points) Draw the feasible region and the tangent cone at $\bar{x} = (0, 1)$. Does the constraint qualification hold at $\bar{x}$?

(c) (6 points) Plot the active normal cone at $\bar{x} = (0, 1)$. Does $\bar{x}$ satisfy the first-order necessary conditions? Is $\bar{x}$ a local minimizer of (3)?

Question 6. Consider a quadratic function of the form

$$q(x) = \frac{1}{2}x^T Ax + b^T x + c$$

where $A \in \mathbf{R}^{n\times n}$ is symmetric, $b \in \mathbf{R}^n$ and c is a scalar.

(a) (10 points) Suppose that A is not positive semidefinite, v is an eigenvector associated with a negative eigenvalue of A and $x_* \in \mathbf{R}^n$ is the stationary point of $q(x)$. Show that v is a direction of decrease for $q(x)$ at x_*, *i.e.* $q(x_* + \alpha v) - q(x_*) < 0$ for all $\alpha > 0$.

(b) (10 points) In particular for the quadratic polynomial

$$\bar{q}(x) = \frac{1}{2}x_1^2 + 5x_1x_2 + \frac{1}{2}x_2^2 - 3x_1 + 9x_2 + 5$$

the stationary point is $\bar{x} = (-2, 1)$. Find a direction of decrease for $\bar{q}(x)$ at $\bar{x}$. (Note : An eigenvector v of A assoc. with eigenvalue λ satisfies $(A - \lambda I)v = 0$.)

Question 7. *(10 points - extra credit)* Consider a twice continuously differentiable function $f : \mathbf{R}^n \to \mathbf{R}$. Suppose that there exists a direction $v \in \mathbf{R}^n$ such that $v^T \nabla^2 f(x) v < 0$ at all x. Show that no stationary point of $f(x)$ is a local minimizer.