MATH 107: Introduction to Linear Algebra

Final - Spring 2017

Duration : 150 minutes

#1	10	
#2	17	
#3	17	
#4	16	
#5	10	
#6	10	
#7	10	
#8	10	
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NAME & LAST NAME ____________________

STUDENT ID ____________________

SIGNATURE ____________________

- Put your name, student ID and signature in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book and closed-notes exam.
- Show all of your work; full credit will not be given for unsupported answers.
- Write your solutions clearly; no credit will be given for unreadable solutions.
- Mark your section below.

  SECTION 1 (EMRE MENGI TUTH 11:30-12:45) ____________________

  SECTION 2 (EMRE MENGI, TUTH 8:30-9:45) ____________________

  SECTION 3 (EMRE MENGI, MW 13:00-14:15) ____________________

  SECTION 4 (DOĞAN BILGE, MW 14:30-15:45) ____________________

Problem 1. (10 points) Find every polynomial $\mathbf{p} \in \mathbb{P}_3$ such that

$$\mathbf{p}(-1) = 4 \quad \text{and} \quad \mathbf{p}(2) = 3.$$

Show your work.
(Recall that $\mathbb{P}_n$ denotes the set of polynomials $\mathbf{p} : \mathbb{R} \to \mathbb{R}$ of degree at most n.)

Problem 2. (17 points) The transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined by

$$T(\mathbf{p}) := \frac{d^2\mathbf{p}}{dt^2} + 3\frac{d\mathbf{p}}{dt} + 2\mathbf{p}$$

is linear. Furthermore, let $B = \{1, 1+t, 1+t+t^2\}$ and $C = \{1, t, t^2\}$, both of which are bases for $\mathbb{P}_2$.

Find a matrix M such that

$$[T(\mathbf{p})]_C \ = \ M[\mathbf{p}]_B$$

for every $\mathbf{p} \in \mathbb{P}_2$.

Problem 3. Let

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(a) (12 points) Find an *orthonormal basis* for S.

(b) (5 points) Find the *orthogonal projection* of the vector $\mathbf{v} = (1, 0, 1, 0)$ onto S.

Problem 4. Indicate in each part whether the statement is *true or false* for every pair of matrices A and B that are similar. Give a brief explanation for your answer.

(a) (4 points) The determinants of A and B are equal.

(b) (4 points) A and B have the same column space.

(c) (4 points) The set of eigenvalues of A and B are the same. Furthermore, the algebraic multiplicities of λ as an eigenvalue of A and as an eigenvalue of B are equal.

(d) (4 points) The eigenspaces of A and B are the same.

Problem 5. (10 points) Let us consider the vector space $\mathbb{P}_2$ with the inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle \ := \ \int_{-1}^{1} \mathbf{p}(t)\mathbf{q}(t)\, dt$$

and the norm

$$\|\mathbf{p}\| \ := \ \sqrt{\langle \mathbf{p}, \mathbf{p} \rangle} \ = \ \sqrt{\int_{-1}^{1} [\mathbf{p}(t)]^2 \ dt}.$$

Find $\mathbf{p}_* \in \mathbb{P}_1$ such that $\|t^2 - \mathbf{p}_*\| \ \leq \ \|t^2 - \mathbf{p}\|$ for all $\mathbf{p} \in \mathbb{P}_1$.

Problem 6. Indicate in each part whether the matrix A is *invertible or not*. Explaining your reasoning. Also in each part indicate *the dimension of the null space* of A.

(a) (6 points) $A = \mathbf{u}_1\mathbf{u}_1^T$ where $\mathbf{u}_1 \in \mathbb{R}^n$ is a nonzero vector and $n \geq 2$.

(b) (4 points)

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 & t_1^4 \\ 1 & t_2 & t_2^2 & t_2^3 & t_2^4 \\ 1 & t_3 & t_3^2 & t_3^3 & t_3^4 \\ 1 & t_4 & t_4^2 & t_4^3 & t_4^4 \\ 1 & t_5 & t_5^2 & t_5^3 & t_5^4 \end{bmatrix}$$

where $t_1, t_2, t_3, t_4, t_5 \in \mathbb{R}$ are distinct real numbers (that is $t_i \neq t_j$ for $i, j = 1, \ldots, 5$ such that $i \neq j$).

Problem 7. Every real symmetric $n \times n$ matrix A satisfying $A^T = A$ has n real eigenvalues, say $\lambda_1, \ldots, \lambda_n$. Furthermore, the eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of the symmetric matrix A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ can be chosen as vectors in $\mathbb{R}^n$ such that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is orthonormal (with respect to the standard inner product $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + \cdots + x_n y_n$ on $\mathbb{R}^n$). In parts **(a)** and **(b)** below, you can make use of these facts.

(a) (6 points) Let A be a real symmetric $n \times n$ matrix. Prove that the function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle \;=\; \mathbf{x}^T A \mathbf{y}$$

is an *inner product* on $\mathbb{R}^n$ if and only if all eigenvalues of A are positive.

(b) (4 points) Specifically, is the following function

$$\langle \mathbf{x}, \mathbf{y} \rangle \;=\; 3x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2 \;=\; \mathbf{x}^T \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix} \mathbf{y}$$

an inner product on $\mathbb{R}^2$?

Problem 8. Let S_1, S_2 be two subspaces of a vector space V such that $S_1 + S_2 = V$ and $S_1 \cap S_2 = \{\mathbf{0}\}$, where $S_1 + S_2$ is defined by

$$S_1 + S_2 \;:=\; \{\mathbf{s}_1 + \mathbf{s}_2 \mid \mathbf{s}_1 \in S_1,\ \mathbf{s}_2 \in S_2\}.$$

Every vector $\mathbf{v} \in V$ can be written of the form

$$\mathbf{v} \;=\; \mathbf{v}_{S_1} + \mathbf{v}_{S_2}, \tag{1}$$

for some $\mathbf{v}_{S_1} \in S_1$ and $\mathbf{v}_{S_2} \in S_2$ in a unique way. We refer (1) as the unique decomposition of $\mathbf{v}$.

(a) (5 points) Consider the transformation $P : V \to S_1$ defined by

$$P(\mathbf{v}) \;:=\; \mathbf{v}_{S_1}$$

for every $\mathbf{v} \in V$, where $\mathbf{v}_{S_1}$ is as in (1) in the unique decomposition of $\mathbf{v}$. Prove that P is *linear*.

(b) (5 points) Now suppose an inner product $\langle \cdot, \cdot \rangle$ is defined on V. Recall that a given vector $\mathbf{v} \in V$ can also be decomposed into

$$\mathbf{v} \;=\; \widetilde{\mathbf{v}}_{S_1} + \widetilde{\mathbf{v}}_{S_1^\perp},$$

for some $\widetilde{\mathbf{v}}_{S_1} \in S_1$ and $\widetilde{\mathbf{v}}_{S_1^\perp} \in S_1^\perp$ in a unique way.

Prove

$$\|\widetilde{\mathbf{v}}_{S_1^\perp}\| \;\leq\; \|\mathbf{v}_{S_2}\|$$

where $\mathbf{v}_{S_2}$ is as in (1) (on page 9) in the unique decomposition of $\mathbf{v}$.