Math 409/509: Optimization

Final - Spring 2011

Duration : 180 minutes

#1	20	
#2	20	
#3	20	
#4	20	
#5	20	
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Name ____________________

Student ID ____________________

Signature ____________________

- Put your name and student ID in the space provided above.
- No calculators or any other electronic devices are allowed.
- This is a closed-book exam. You can use up to 3 pages of double-sided notes.
- Show all of your work; full credit will not be given for unsupported answers.

Question 1. Consider the standard linear program (LP)

$$\begin{array}{lrl} \text{minimize} & -x_1 + 3x_2 + x_3 + x_4 & \\ \text{subject to} & x_1 + 2x_2 + x_3 & = 1 \\ & 3x_1 + x_2 - x_3 + x_4 & = 2 \\ & x_1 & \geq 0 \\ & x_2 & \geq 0 \\ & x_3 & \geq 0 \\ & x_4 & \geq 0. \end{array}$$

(a) (10 points) Write down the dual problem for the LP above.

(b) (10 points) Express the dual problem in part **(a)** as a standard LP.

Question 2. Consider the following variant of the hanging-chain problem. Three bars are connected by two joints in 2-dimensional space. Denote the coordinates of the joint connecting the first bar to the second by (x_1, y_1) and second bar to the third bar by (x_2, y_2). One end of the first bar is fixed at $(x_0, y_0) = (0, 0)$, similarly one end of the third bar is fixed at $(x_3, y_3) = (3, 3)$. Also you are given the lengths of the bars as $L_1 = L_2 = 4$ and $L_3 = 1$ where L_j denotes the length of the jth bar. The end points of the bars are required to be above the floor. The (x, y) coordinates of any point on the floor is related by $y = \frac{x^2}{6} + \frac{x}{4} - 1$. The potential energy of the jth bar is given by

$$L_j \frac{y_j + y_{j-1}}{2}.$$

(a) (10 points) Pose the problem of minimizing the total potential energy of the bars subject to the constraints that

- the distance between the end points of each bar must match the actual length of the bar, and
- the end points of bars must lie above the floor

as a nonlinear program.

(b) (10 points) Write down the mixed penalty, logarithmic-barrier function associated with the nonlinear program in part **(a)**. Assume that μ is the mixed penalty, logarithmic-barrier parameter and is fixed.

Question 3. Consider the nonlinear inequality constrained program

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^2} & x_1 - x_2 \\ \text{subject} & -2x_1^3 - x_2 \geq 0 \\ & -(x_1 - 1)^2 - x_2 \geq 0 \end{array} . \tag{1}$$

(a) (5 points) Find the tangent cone at $\hat{x} = (1/2, -1/4)$ algebraically for (1).

(b) (5 points) Is the constraint qualification satisfied at $\hat{x}$? Show your work.

(c) (5 points) Plot and express the active normal cone at $\hat{x}$ algebraically.

(d) (5 points) Does the first order necessary conditions hold at $\hat{x}$ for (1)? If the first order necessary conditions do not hold at $\hat{x}$ for (1), write down also a feasible descent direction. Show your work.

Question 4. This question concerns a quadratic program of the form

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^n} & \frac{1}{2} x^T H x \\ \text{subject} & Ax = b \\ & x \geq 0 \end{array} \tag{2}$$

where $H \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$.

(a) (10 points) Write down the Karush-Kuhn-Tucker conditions for the quadratic program above.

(b) (10 points) Prove that any point $x \in \mathbb{R}^n$ satisfying the Karush-Kuhn-Tucker conditions is a global minimizer of (2).

Question 5. (20 points) This question concerns the sensitivity of the optimal object value of a linear program with respect to perturbations in the constraints. Consider the standard linear program (LP)

$$\begin{array}{ll} \text{minimize}_{x \in \mathbb{R}^n} & c^T x \\ \text{subject} & Ax = b + \epsilon e_m \\ & x \geq 2\epsilon e_n \end{array} \tag{3}$$

where $A \in \mathbb{R}^{m \times n}$, $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $\epsilon \in \mathbb{R}$ is a perturbation variable. Above also $e_n \in \mathbb{R}^n$ and $e_m \in \mathbb{R}^m$ denote vectors consisting of ones of size n and m, respectively.

View the optimal objective value $f_*(\epsilon)$ for the (LP) above as a function of ϵ. Assume that the active (inequality) constraints remain the same for all ϵ close to zero. Find $f_*'(0)$.