EXAMPLE

$$\underset{x \in \mathbb{R}^3}{\text{minimize}} \quad 4x_1 + x_2 - 3x_3$$

subject to

$$-x_1 - 2x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 2$$
$$x_1, x_2, x_3 \geqslant 0$$

Equivalently

$$\underset{x \in \mathbb{R}^3}{\text{minimize}} \quad c^T x$$
$$Ax = b$$
$$x \geqslant 0$$

where

$$c = \begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 & -2 \\ 1 & 2 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Dual LP

$$\underset{\pi \in \mathbb{R}^m}{\text{maximize}} \quad b^T \pi$$
$$c - A^T \pi \geqslant 0$$

or

$$\underset{\pi \in \mathbb{R}^2}{\text{maximize}} \quad \pi_1 + 2\pi_2$$
$$\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} \geqslant 0$$

Using slack variables

$$\text{minimize} \quad -\Pi_1 - 2\Pi_2$$
$$\Pi \in \mathbb{R}^2, \; s \in \mathbb{R}^3$$

$$\begin{bmatrix} 4 \\ 1 \\ -3 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} + \begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} s_1 \\ s_2 \\ s_3 \end{bmatrix} \geqslant 0$$

- Letting $\Pi_1 = \Pi_1^+ - \Pi_1^-$ and $\Pi_2 = \Pi_2^+ - \Pi_2^-$

Above problem can be posed as

$$\text{minimize} \quad -\Pi_1^+ + \Pi_1^- - 2\Pi_2^+ + 2\Pi_2^-$$
$$\Pi_1^+, \Pi_1^-, \Pi_2^+, \Pi_2^- \in \mathbb{R}$$
$$s_1, s_2, s_3 \in \mathbb{R}$$

DUAL PROBLEM IN STANDARD LP FORM

$$4 + \Pi_1^+ - \Pi_1^- - \Pi_2^+ + \Pi_2^- + s_1 = 0$$
$$1 - 2\Pi_2^+ + 2\Pi_2^- + s_2 = 0$$
$$-3 + 2\Pi_1^+ - 2\Pi_1^- - 3\Pi_2^+ + 3\Pi_2^- + s_3 = 0$$
$$\Pi_1^+, \Pi_1^-, \Pi_2^+, \Pi_2^-, s_1, s_2, s_3 \geqslant 0$$

EXAMPLE

$$\text{(LP)}\quad \underset{x\in\mathbb{R}^2}{\text{minimize}}\ \underbrace{x_1}_{f(x)} \quad \text{subject to} \quad x_1+x_2=1,\quad x_1,x_2\geqslant 0$$

$$c=\begin{bmatrix}1\\0\end{bmatrix}\qquad A=\begin{bmatrix}1 & 1\end{bmatrix}\qquad b=1$$

Because of the constraint $x_1\geqslant 0$, we have $f(x)\geqslant 0$ for all feasible x.

In particular choose $(x_1,x_2)=(0,1)\in\mathbb{F}$. Then $f(x)=0$, which is the minimal objective value.

$$\text{(DLP)}\quad \underset{\pi\in\mathbb{R}}{\text{maximize}}\ \underbrace{\pi}_{b^T\pi}=g(\pi)$$

subject to

$$\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{c}-\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{A^T}\pi\geqslant 0 \quad \left(\text{i.e. } \begin{matrix}1-\pi\geqslant 0\\ -\pi\geqslant 0\end{matrix}\right. \iff \pi\in(-\infty,0]$$

Consequently maximal value of $g(\pi)=0$ over feasible π.

<u>NOTE</u>: $\underset{x\in\mathbb{F}_p}{\text{minimize}}\ f(x)=\underset{\pi\in\mathbb{F}_d}{\text{maximize}}\ g(\pi)$

EXAMPLE (Transportation Problem)

* 2 Warehouses that produce paper
  Detroit, Pittsburgh
* 3 Publishers that use paper
  Boston, New York, Chicago

Production			Usage
Detroit	250 Tons	Boston	50 Tons
Pittsburgh	130 Tons	New York	190 Tons
		Chicago	130 Tons

COST OF TRANPORTING PAPER
(DOLLAR PER TON)

FROM / TO	Boston	New York	Chicago
Detroit	15	20	16
Pittsburgh	25	13	5

Pose the problem of minimizing transportation costs from producing citres to consuming citres as a linear program.

x_{ij} : amount (in tons) that is transported from city i to city j

i (one of the producing cities)

D (Detroit), P (Pittsburgh)

j (one of the consuming cities)

B (Boston), N (New York), C (Chicago)

$$\underset{x \in \mathbb{R}^6}{\text{minimize}} \quad 15x_{DB} + 20x_{DN} + 16x_{DC} + 25x_{PB} + 13x_{PN} + 5x_{PC}$$

subject to

$x_{DB} + x_{DN} + x_{DC} = 250$ (# PRODUCTION IN DETROIT)

$x_{PB} + x_{PN} + x_{PC} = 130$ (# PRODUCTION IN PITTSBURGH)

$x_{DB} + x_{PB} = 60$ (CONSUMPTION IN BOSTON)

$x_{DN} + x_{PN} = 190$ (CONSUMPTION IN NEW YORK)

$x_{DC} + x_{PC} = 130$ (CONSUMPTION IN BOSTON)

$x \geq 0$

where $x = \begin{bmatrix} x_{DB} \\ x_{DN} \\ x_{DC} \\ x_{PB} \\ x_{PN} \\ x_{PC} \end{bmatrix}$