KOÇ UNIVERSITY

MATH 106 - CALCULUS 1

Midterm I March 19, 2015

Duration of Exam: 75 minutes

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print** (use **CAPITAL LETTERS**) **and sign your name, and indicate your section below.**

Name: ______________________

Surname: KEY

Signature: ______________________

Section (Check One):

Section 1: Selda Küçükçifçi M-W (8:30) ___
Section 2: Ayberk Zeytin T-Th(10:00) ___

PROBLEM	POINTS	SCORE
1	22	
2	36	
3	22	
4	13	
5	12	
TOTAL	105	

1. (22 points)

(a) Let $f\colon (0,\infty) \longrightarrow \mathbb{R}$ and $g\colon (0,\infty) \longrightarrow \mathbb{R}$ be two continuous functions. Assume that the equation $f(x) = g(x)$ does not have any solution in $\mathbb{R}$ and that $f(1) > g(1)$. Show that $f(x) > g(x)$ for all $x \in (0,\infty)$.

Let $f:(0,\infty)\to\mathbb{R}$ and $g:(0,\infty)\to\mathbb{R}$ be two continuous function. Suppose $f(x)=g(x)$ does not have any solutions in $\mathbb{R}$ and $f(1)>g(1)$.

Suppose by contradiction $f(a)<g(a)$ for some $a\in(0,\infty)$.

Let $h(x)=f(x)-g(x)$. $h(x)$ is continuous on $(0,\infty)$ too

Since $h(1)>0$ and $h(a)<0$ and $h(a)<0<h(1)$

there is $c\in(1,a)$ (or $(a,1)$ if $a<1$)

such that $h(c)=0$ by Intermediate Value Theorem.

But then $f(c)=g(c)$ for some $c\in(0,\infty)$ gives us a contradiction. Hence $f(x)>g(x)$ for all $x\in(0,\infty)$.

(b) Use the result in (a) to show that $\sqrt{1+x^2} < 1+x$, when $x>0$.

Consider $g(x)=\sqrt{1+x^2}$ and $f(x)=1+x$ in part (a). f & g are continuous on $(0,\infty)$

$f(1)=2>g(1)=\sqrt{2}$ and $f(x)=g(x)$ does not have any solution in $\mathbb{R}$ since otherwise

$$\sqrt{1+x^2}=1+x \Rightarrow 1+x^2=(1+x)^2=1+x^2+2x$$

$\Rightarrow x=0$ but $0\notin(0,\infty)$.

Then by part (a) $f(x)=1+x>g(x)=\sqrt{1+x^2}$ for all $x\in(0,\infty)$.

2. (36 points) Compute the following limits. Do not use l'Hospital's rule.

(a) $\lim\limits_{x\longrightarrow\infty} \dfrac{\cos(x^2)}{x^4+1}$

$$-1 \le \cos(x^2) \le 1$$

$$\frac{-1}{x^4+1} \le \frac{\cos(x^2)}{x^4+1} \le \frac{1}{x^4+1}$$

Since $\lim\limits_{x\to\infty} \dfrac{-1}{x^4+1} = 0$ & $\lim\limits_{x\to\infty} \dfrac{1}{x^4+1} = 0$

$$\lim_{x\to\infty} \frac{\cos(x^2)}{x^4+1} = 0$$

(b) $\lim\limits_{x\longrightarrow 0} \dfrac{x}{|x-1|-|x+1|}$

$$= \lim_{x\to 0} \frac{x}{(-x+1)-(x+1)}$$

$$= \lim_{x\to 0} \frac{x}{-2x} = -\frac{1}{2}$$

(c) $\lim\limits_{x\longrightarrow\infty} (\sqrt{x^2+x+1}-x)$

$$= \lim_{x\to\infty} \frac{(\sqrt{x^2+x+1}-x)(\sqrt{x^2+x+1}+x)}{\sqrt{x^2+x+1}+x}$$

$$= \lim_{x\to\infty} \frac{x^2+x+1-x^2}{\sqrt{x^2+x+1}+x} = \lim_{x\to\infty} \frac{x+1}{\sqrt{x^2+x+1}+x}$$

$$= \lim_{x\to\infty} \frac{1+\overset{\to 0}{\frac{1}{x}}}{\sqrt{1+\underset{0}{\underset{\downarrow}{\frac{1}{x}}}+\underset{0}{\underset{\downarrow}{\frac{1}{x^2}}}}+1} = \frac{1}{2}.$$

(d) $\lim_{x \longrightarrow 0} \frac{(1+x)^{106} - 1}{x}$ $= f'(0)$, where $f(x) = (1+x)^{106}$

$f(0) = 1$.

$f'(x) = 106\,(1+x)^{105}$

So $\lim_{x\to 0} \frac{(1+x)^{106} - 1}{x} = f'(0) = 106$.

3. Compute the following derivatives:

(a) (10 points) $\frac{d}{dx}\left(\ln(\sin^2(x^3)) - \sin^{-1}(\frac{x}{4})\right)$

$= \frac{1}{\sin^2(x^3)} \cdot 2\sin(x^3) \cdot \cos(x^3) \cdot 3x^2 - \frac{\frac{1}{4}}{\sqrt{1-\left(\frac{x}{4}\right)^2}}$

$= \frac{6x^2\cos(x^3)}{\sin(x^3)} - \frac{1}{\sqrt{16-x^2}}$

$= 6x^2\cot(x^3) - \frac{1}{\sqrt{16-x^2}}$.

(b) (12 points) $\frac{d}{dx}((\cos x)^x)$, where $0 < x < \pi/2$.

$y = (\cos x)^x$

$\ln y = x\ln(\cos x)$

$\frac{1}{y} \cdot y' = \ln(\cos x) + x \cdot \frac{1}{\cos x} \cdot (-\sin x)$

$y' = (\cos x)^x\left(\ln(\cos x) - x\tan x\right)$.

4. (13 points) Find an equation of the tangent line to the curve $xe^y + y - 2x = \ln 2$ at the point $(1, \ln 2)$.

$$e^y + xe^y y' + y' - 2 = 0$$

$$y'(xe^y + 1) = 2 - e^y$$

$$y' = \frac{2 - e^y}{xe^y + 1}$$

$$m_T = y'\Big|_{(1, \ln 2)} = \frac{0}{3} = 0$$

equation of the tangent line: $y = \ln 2$

5. (12 points) Let f be a function differentiable in $\mathbb{R}$ and suppose that $3 \leq f'(x) \leq 5$ for any $x \in \mathbb{R}$. Show that $18 \leq f(8) - f(2) \leq 30$.

Let f be a differentiable function in $\mathbb{R}$, so it is continuous everywhere. So f is continuous on $[2, 8]$ and differentiable on $(2, 8)$. Then by Mean Value Theorem there is $c \in (2, 8)$ such that $f'(c) = \frac{f(8) - f(2)}{8 - 2}$

Since $3 \leq f'(x \leq 5$, $3 \leq f'(c) = \frac{f(8) - f(2)}{6} \leq 5$

then $18 \leq f(8) - f(2) \leq 30$.