KOÇ UNIVERSITY

MATH 106 - CALCULUS 1 - MIDTERM II

Friday, December 7, 2018 starting at 19:00

Duration of Exam: 105 minutes

NO QUESTIONS ASKED NO ANSWERS GIVEN

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign the HONOR PLEDGE; indicate your section below.**

Name/Surname/ID: ***KEY***

I DID NOT RECEIVE FROM NOR GAVE ASSISTANCE TO ANYONE

Signature: ____________________

Section (Check One):

Section 1: Nadim Rustom M-W (16:00 to 17:15) ___
Section 2: Attila Aşkar T-Th (16:00 to 17:15) ___
Section 3: Altan Erdoğan T-Th (11:30 to 12:45) ___
Section 4: Altan Erdoğan T-Th (08:30 to 09:45) ___
Section 5: Nadim Rustom M-W (10:00 to 11:15) ___

PROBLEM	POINTS	SCORE
1	20	
2	20	
3	25	
4	20	
5	15	
TOTAL	**100**	

Some information that may be useful:

$$\Big(e^x\Big)' = e^x \qquad \Big(\ln x\Big)' = \frac{1}{x} \qquad \Big(\sin^{-1} x\Big)' = \frac{1}{\sqrt{1-x^2}} \qquad \Big(\tan^{-1} x\Big)' = \frac{1}{1+x^2}$$

$$\sum_{k=1}^{n} k = \frac{1}{2}\,(n+1)n \qquad \sum_{k=1}^{n} k^2 = \frac{1}{6}\,(2n+1)(n+1)n \qquad \sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a}$$

1. (20 points)

(a) **Without using integration**, calculate the following limit:

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\ \Delta x$$

where $f(x) = x^2$, $\Delta x = 2/n$, $x_0 = 0$, and $x_n = 2$. (Hint: some formulas on the front page might be useful).

Solution.

$$x_k = \frac{2k}{n},$$

$$L = \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n} \left(\frac{2k}{n}\right)^2 = \lim_{n\to\infty} \frac{8}{n^3} \sum_{k=1}^{n} k^2 = \lim_{n\to\infty} \frac{8}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{8}{3}.$$

(b) Find the following limit by **evaluating an appropriate integral**.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(i/n)^2}.$$

Solution.

$$\Delta x = \frac{1}{n}, \quad x_i = \frac{i}{n}, \quad f(x) = \frac{1}{1+x^2}, \quad a = x_0 = 0, \quad b = x_n = 1,$$

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(i/n)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_0^1 \frac{1}{1+x^2}\, dx$$

$$= \arctan(1) - \arctan(0) = \frac{\pi}{4}.$$

2. (20 points) Calculate the integrals given below:

(a)

$$I = \int_0^{1/\sqrt{3}} \frac{x\ dx}{\sqrt{1-3x^2}}$$

Solution.

$$u = 1 - 3x^2, \quad du = -6x\ dx \Rightarrow x\ dx = -\frac{1}{6}\ du$$

$$x = 0 \Rightarrow u = 1, \quad x = \frac{1}{\sqrt{3}} \Rightarrow u = 0$$

$$I = -\frac{1}{6}\int_1^0 \frac{du}{\sqrt{u}} = \frac{1}{6}\int_0^1 \frac{du}{\sqrt{u}} = \frac{1}{6}\left[2\sqrt{u}\right]_0^1 = \frac{1}{3}.$$

(b)

$$I = \int_e^\infty \frac{dx}{x\left(\ln x\right)^2}$$

Solution.

$$u = \ln(x) \Rightarrow du = \frac{1}{x}\ dx$$

$$x = e \Rightarrow u = 1, \quad \lim_{x\to\infty} u = \infty.$$

$$I = \int_1^\infty \frac{du}{u^2} = \left[-\frac{1}{u}\right]_1^\infty = \lim_{u\to\infty} -\frac{1}{u} + 1 = 1.$$

(c)

$$I = \int_1^e x \ln(x)\ dx$$

Solution.

$$u = \ln(x) \Rightarrow du = \frac{1}{x}\ dx$$

$$dv = x\ dx \Rightarrow v = \frac{x^2}{2}$$

$$I = \left[\frac{x^2\ln(x)}{2}\right]_1^e - \int_1^e \frac{x}{2}\ dx = \frac{e^2}{2} - \left(\frac{e^2}{4} - \frac{1}{4}\right) = \frac{e^2+1}{4}.$$

3. (25 points) Calculate the integrals given below:

(a)

$$(a)\quad I=\int xe^x\,dx.$$

Solution.

$$u=x\Rightarrow du=dx,\quad dv=e^x\,dx\Rightarrow v=e^x$$

$$I=xe^x-\int e^x\,dx=xe^x-e^x+C.$$

$$(a)\quad I=\int\frac{dx}{x^2+3x+2}$$

Solution.

$$I=\int\frac{dx}{(x+1)(x+2)}$$

$$\frac{1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

$$A=\lim_{x\to-1}(x+1)\cdot\frac{1}{(x+1)(x+2)}=1.$$

$$B=\lim_{x\to-2}(x+2)\cdot\frac{1}{(x+1)(x+2)}=-1.$$

Therefore

$$I=\ln|x+1|-\ln|x+2|+C.$$

$$(b)\quad I=\int_1^2\frac{dx}{x^2-2x+2}$$

Solution.

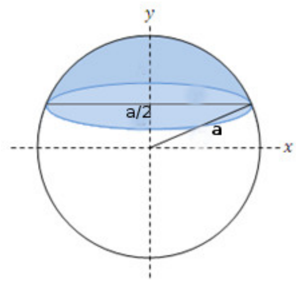
$$x^2-2x+2=(x-1)^2+1$$

$$u=x-1\Rightarrow du=dx$$

$$x=1\Rightarrow u=0,\quad x=2\Rightarrow u=1$$

$$I=\int_0^1\frac{du}{u^2+1}=\arctan(1)-\arctan(0)=\frac{\pi}{4}.$$

4. (20 points) A spherical cap is generated by rotating the arc given by $x^2 + y^2 = a^2$, $\frac{a}{2} \le y \le a$ about the y-axis (see figure).



(a) Find the volume of the spherical cap.

Solution. It is enough to consider the arc $a/2 \le y \le a$, $x \ge 0$, which is the graph of

$$x = \sqrt{a^2 - y^2}$$

rotated about the y-axis.

$$V = \int_{a/2}^{a} \pi x^2 \, dy = \int_{a/2}^{a} \pi(a^2 - y^2) \, dy = \pi \left[a^2 y - \frac{y^3}{3} \right]_{a/2}^{a} = \frac{5\pi a^3}{24}.$$

(b) Find the surface area of the top (curved) surface of the spherical cap.

(Note: do not include the bottom surface)

Solution.

$$\frac{dx}{dy} = \frac{-y}{\sqrt{a^2 - y^2}}$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \sqrt{1 + \frac{y^2}{a^2 - y^2}} \, dy = \frac{a}{\sqrt{a^2 - y^2}} \, dy$$

$$A = \int_{y=a/2}^{y=a} 2\pi x \, ds = \int_{a/2}^{a} 2\pi \sqrt{a^2 - y^2} \frac{a}{\sqrt{a^2 - y^2}} \, dy = \int_{a/2}^{a} 2\pi a \, dy = [2\pi a y]_{a/2}^{a} = \pi a^2.$$

5. (15 points) Consider the sequence defined recursively by $a_1 = 1$,

$$a_{n+1} = \frac{5 + a_n}{3} \text{ for } n \geq 1.$$

(a) By induction, show that $a_n \leq 5/2$ for all $n \geq 1$.

Solution. By induction:

- $n = 1$: $a_1 = 1 \leq 5/2$.
- Assume $n > 1$ and $a_{n-1} \leq 5/2$. Then

$$5 + a_{n-1} \leq 15/2$$

$$a_n = \frac{5 + a_{n-1}}{3} \leq 15/6 = 5/2.$$

(b) Using the result of (a) show that a_n is increasing.

Solution. Compute

$$a_{n+1} - a_n = \frac{5 + a_n}{3} - a_n = \frac{5 - 2a_n}{3} \geq 0$$

since by part (a) we have $a_n \leq 5/2$.

(c) Using (a) and (b), deduce that a_n is convergent and find its limit.

Solution. By part (b), a_n is increasing and by part (a), a_n is bounded above. So a_n converges. Let $L = \lim_{n\to\infty} a_n$. Then

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{5 + a_n}{3} = \frac{5 + L}{3} \Rightarrow L = \frac{5}{2}.$$