KOÇ UNIVERSITY

MATH 106 - CALCULUS 1 - MIDTERM I

Friday, November 2, 2018 starting at 19:00

Duration of Exam: 90 minutes

NO QUESTIONS ASKED NO ANSWERS GIVEN

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign the HONOR PLEDGE; indicate your section below.**

SOLUTIONS

Section (Check One):

Section 1: Nadim Rustom M-W (16:00 to 17:15) ——
Section 2: Attila Aşkar T-Th (16:00 to 17:15) ——
Section 3: Altan Erdoğan T-Th (11:30 to 12:45) ——
Section 4: Altan Erdoğan T-Th (08:30 to 09:45) ——
Section 5: Nadim Rustom M-W (10:00 to 11:15) ——

PROBLEM	POINTS	SCORE
1	20	
2	20	
3	20	
4	20	
5	20	
TOTAL	**100**	

Some information that may be useful: $\pi = 3.14....$ $e = 2.78...$

$$\Big(e^x\Big)' = e^x \qquad \Big(\log x\Big)' = \frac{1}{x} \qquad \Big(\tan x\Big)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x \qquad \Big(\tan^{-1} x\Big)' = \frac{1}{1+x^2}$$

$$\sin(a-b) = \sin a \cos b - \cos a \sin b \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

1. (20 points)

(a) Find the derivative of $f(x) = \sin x$ using the basic definition of derivative with limits

SOLUTION

$$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \to \quad \frac{df}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$\frac{df}{dx} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin(x)}{h} = \lim_{h\to 0}\frac{\sin x\; 1+\cos x\sin h-\sin(x)}{h} = \lim_{h\to 0}\frac{\cos x\sin h}{h}$$

Using the formula given on the front page with h, $\lim_{h\to 0}\frac{\sin h}{h} = 1$:

$$\frac{df}{dx} = \cos x\lim_{h\to 0}\frac{\sin h}{h} = \cos x$$

(b) Compute the following limit, if it exists:

$$\lim_{x\to\infty}\sqrt{x^2-2x}-\sqrt{x^2+2x}$$

SOLUTION: remove the square root in the numerator:

$$f(x) \equiv \sqrt{x^2-2x}-\sqrt{x^2+2x} = \frac{(x^2-2x)-(x^2+2x)}{\sqrt{x^2-2x}+\sqrt{x^2+2x}} = \frac{-4x}{\sqrt{x^2-2x}+\sqrt{x^2+2x}}$$

Now take the limit:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{-4x}{x\left(\sqrt{1-(2/x)}+\sqrt{1+2/x}\right)} = -\frac{4}{2} = -2$$

2. (20 points)

Calculate y' in the following expressions using the rules for differentiation i.e. derivatives.

SOLUTION. The easiest is by logarithmic differentiation.

(a) $y = \left[\frac{(1+x)(x^2+1)}{(x^3+1)(x^4+1)}\right]^{\frac{1}{3}}$ at $x=1$ $\rightarrow$ $\log y = \frac{1}{3}\Big[\log(x{+}1) + \log(x^2{+}1) - \log(x^3{+}1) - \log(x^4{+}1)\Big]$

Taking the derivatives of both sides with respect to x:

$$\frac{y'}{y}\Big|_{x=1} = \frac{1}{3}\left[\frac{1}{(1+x)} + \frac{2x}{(x^2+1)} - \frac{3x^2}{(x^3+1)} - \frac{4x^3}{(x^4+1)}\right]\Bigg|_{x=1} = \frac{1}{3}\left[\frac{1}{2} + \frac{2}{2} - \frac{3}{2} - \frac{4}{2}\right] \rightarrow$$

$$\frac{y'}{y}\Big|_{x=1} = \frac{-2}{3} \qquad y\Big|_{x=1} = \left[\frac{(1+x)(x^2+1)}{(x^3+1)(x^4+1)}\right]^{\frac{1}{3}}\Bigg|_{x=1} = \left[\frac{(2)(2)}{(2)(2)}\right]^{\frac{1}{3}} = 1$$

$$y'\Big|_{x=1} = \frac{-2}{3}$$

(b) $xy^2 + x^3\sin y + xy\log(y^2+1) + e^{-y}\cos x = 1 + x$ at $x = 0$

(Hint: You must calculate the value of y for $x = 0$ from the equation.)

SOLUTION. The only way is by implicit function differentiation.

First, set x $= 0$ in order to find the corresponding y:

$$0+0+0+\ e^{-y}\ 1 = 1 \quad\rightarrow\quad e^{-y} = 1 \quad\rightarrow\quad y = 0$$

Now differentiate implicitly w. r. to x:

$$\Big(y^2 + x(2yy')\Big) + \Big(3x^2\sin y + x^3\cos y\ y'\Big) + \Big(y\log(y^2{+}1) + xy'\log(y^2{+}1) + xy\frac{2y}{(y^2+1)}\Big)$$

$$+\Big(-y'\ e^{-y}\cos x\ -\ e^{-y}\sin x\Big)\Big|_{x=0,y=0} = 0+1 \rightarrow$$

$$\Big(0+0\Big) + \Big(0+0\,y'\Big) + \Big(0+0+0\Big) + \Big(-y'\big|_{x=0,y=0} - 0\Big) = 1 \rightarrow -y'\big|_{x=0,y=0} = 1 \rightarrow$$

$$y'\Big|_{x=0,y=0} = -1$$

3. (20 points)

(a) Find an approximate value for $a = (1.728)^{\frac{1}{3}}$ using linearization with the tangent line.

SOLUTION.

Eq. of the tangent line for $y = f(x)$: $y = f(a) + f'(a)(x - a)$. For this example, let

$$f(x) = x^{\frac{1}{3}} \qquad a = 1 \qquad x = 1.728 \quad \rightarrow$$

$$f(a) = f(1) = 1 \qquad f'(a) = \frac{1}{3} \quad \rightarrow \quad \text{Equation of tangent line: } y \;=\; 1 \;+\; \frac{1}{3}\,(x - 1) \quad \rightarrow$$

$$y\Big|_{x = 1.728} \;=\; 1 \;+\; \frac{1}{3}\,(1.728 - 1) \;=\; 1 \;+\; \frac{0.728}{3} \quad \rightarrow$$

$$(1.728)^{\frac{1}{3}} \;\approx\; 1.242$$

or let

$$f(x) = (x + 1)^{\frac{1}{3}} \qquad a = 0 \qquad x = 0.728 \quad \rightarrow$$

$$f(a) = f(1) = 1 \qquad f'(a) = \frac{1}{3} \quad \rightarrow \quad \text{Equation of tangent line: } y \;=\; 1 \;+\; \frac{x}{3} \quad \rightarrow$$

$$y\Big|_{x = 0.728} \;=\; 1 \;+\; \frac{0.728}{3} \quad \rightarrow$$

$$(1.728)^{\frac{1}{3}} \;\approx\; 1.242$$

(b) Find the derivative of $f(x) \;=\; \sin^{-1} x$ using inverse functions method.

SOLUTION.

Take the inverse of both sides and take the derivative with chain rule on the l. h. s.:

$$y \;=\; \sin^{-1} x \quad \rightarrow \quad \sin y = x \quad \rightarrow \quad \cos y\; y' = 1 \quad \rightarrow \quad y' = \frac{1}{\cos y}$$

$$\sin y = x \quad \rightarrow \quad \cos y = \sqrt{1 - \sin^2 y} \;=\; \sqrt{1 - x^2}$$

Substituting $\cos y$ in y':

$$y' = \frac{1}{\sqrt{1 - x^2}}$$

4. (15 points)

A manufacturer wants to design an open box (a box with open top) having a square base with side length x, height h, and total surface area of $108cm^2$. Find the values of x and h that produce a box with maximum volume and the value of the maximum volume.

SOLUTION.

Surface area: $S = x^2 + 4xh \;=\; 108.$ Volume: $V \;=\; x^2 h.$

Substitue h from the given S. On the righ hand side, with $V \;=\; x\;(xh)$, it is more convenient to replace xh.

$$x^2 + 4xh \;=\; 108 \quad\rightarrow\quad 4xh \;=\; 108 \;-\; x^2 \quad\rightarrow\quad V(x) = \frac{1}{4}\,x\,(108 \;-\; x^2) \quad\rightarrow$$

$$V(x) = \frac{1}{4}\,(108x - x^3) \quad\rightarrow\quad V'(x) = \frac{1}{4}\,(108 - 3x^2) \;=\; 0 \quad\rightarrow\quad x^2 \;=\; \frac{108}{3} \;=\; 36 \quad\rightarrow\quad x = 6$$

With this x:

$$x^2 + 4xh \;=\; 108 \quad\rightarrow\quad 36 + 4*6h \;=\; 108 \quad\rightarrow\quad 24h \;=\; 72 \quad\rightarrow\quad h \;=\; 3$$

$$V \;=\; x^2 h \quad\rightarrow\quad V \;=\; 36*3 = 108$$

Verify: $V = \frac{1}{4}\,(108x \;-\; x^3) \;=\; \frac{1}{4}\,(108*6 \;-\; 36*6) \;=\; \frac{6}{4}\,(108 \;-\; 36) \;=\; \frac{3}{2}\,(72) \;=\; 108$

5. (25 points) Let

$$f(x) = \frac{x^3 - 3x^2 + 1}{x^3} \equiv 1 - \frac{3}{x} + \frac{1}{x^3}$$

(a) Find all vertical, horizontal, and oblique asymptotes of $y = f(x)$ (if any).

SOLUTION.

$$\text{Vertical asymptote:} \quad f(x) \to \infty \quad \to \quad x = 0$$

$$\text{Horizontal asymptote:} \quad \lim_{x\to\infty} f(x) = \frac{x^3 - 3x^2 + 1}{x^3} = 1 \quad \to \quad x = 1$$

Oblique asymptote: does not exist, because of the limit above being constant $x = 1$.

(b) Find all local maxima and minima of $f(x)$ (if any).

SOLUTION. Better to use the second expression for $f(x)$:

$$f(x) = 1 - \frac{3}{x} + \frac{1}{x^3} \quad \to \quad f'(x) = 0 + \frac{3}{x^2} - \frac{3}{x^4} = \frac{3}{x^4}(x^2 - 1) = 0 \quad \to \quad \text{Extrema: } x = -1 \quad x = 1$$

To determine the nature of extremum points:

$$f'(x) = 0 + \frac{3}{x^2} - \frac{3}{x^4} \quad \to \quad f''(x) = -\frac{6}{x^3} + \frac{12}{x^5}$$

$$f''(x)|_{x=-1} = 6 - 12 = -6 < 0 \quad \text{Maximum at } x = -1$$

$$f''(x)|_{x=1} = -6 + 12 = 6 > \quad \text{Minimum at } x = 1$$

(c) Find all inflection points of $y = f(x)$ (if any). Also, Find all intervals on which $y = f(x)$ is concave upwards and all intervals on which $y = f(x)$ is concave downwards (if any).

SOLUTION.

$$\text{Inflection point:} \quad f''(x) = -\frac{6}{x^3} + \frac{12}{x^5} = -\frac{6}{x^5}(x^2 - 2) = 0 \quad \to$$

$$x = -\sqrt{2} \qquad x = \sqrt{2}$$

For:

$$x < -\sqrt{2} \quad \to \quad f''(x) > 0 \quad \to \quad \text{Concave UP}$$

$$-\sqrt{2} < x < 0 \quad \to \quad f''(x) < 0 \quad \to \quad \text{Concave DOWN}$$

$$0 < x < \sqrt{2} \quad \to \quad f''(x) > 0 \quad \to \quad \text{Concave UP}$$

$$x > \sqrt{2} \quad \to \quad f''(x) < 0 \quad \to \quad \text{Concave DOWN}$$