KOÇ UNIVERSITY

MATH 106 - CALCULUS 1 - FINAL EXAM

Wednesday, December 26, 2018 starting at 08:30

Duration of Exam: 120 minutes

NO QUESTIONS ASKED NO ANSWERS GIVEN

INSTRUCTIONS: No calculators may be used on the test. No books, no notes, and no talking allowed. You must always **explain your answers** and **show your work** to receive **full credit**. Use the back of these pages if necessary. **Print (use CAPITAL LETTERS) and sign the HONOR PLEDGE; indicate your section below.**

Name/Surname/ID: ______________________________

I DID NOT RECEIVE FROM NOR GAVE ASSISTANCE TO ANYONE

Signature: ____________________________________

Section (Check One):

Section 1: Nadim Rustom M-W (16:00 to 17:15) ___
Section 2: Attila Aşkar T-Th (16:00 to 17:15) ___
Section 3: Altan Erdoğan T-Th (11:30 to 12:45) ___
Section 4: Altan Erdoğan T-Th (08:30 to 09:45) ___
Section 5: Nadim Rustom M-W (10:00 to 11:15) ___

PROBLEM	**POINTS**	**SCORE**
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
TOTAL	**120**	

Some information that may be useful:

$$\Big(e^x\Big)' = e^x \qquad \Big(\ln x\Big)' = \frac{1}{x} \qquad \Big(\sin^{-1} x\Big)' = \frac{1}{\sqrt{1-x^2}} \qquad \Big(\tan^{-1} x\Big)' = \frac{1}{1+x^2}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \qquad \sum_{k=0}^{n-1} a^k = \frac{1-a^n}{1-a}$$

1. (20 points) Find the following limits.

(a)

$$\lim_{x\to\infty} \frac{1}{\sqrt{x^2+3x}-\sqrt{x^2-3x}}.$$

Solution.

$$= \lim_{x\to\infty} \frac{\sqrt{x^2+3x}+\sqrt{x^2-3x}}{6x} = \lim_{x\to\infty} \frac{\sqrt{1+3/x}+\sqrt{1-3/x}}{6} = \frac{1}{3}$$

(b)

$$\lim_{x\to 0} \left(1+x^2\right)^{1/x^2}.$$

Solution. Put $y = (1+x^2)^{1/x^2}$. Then

$$\lim_{x\to 0} \ln(y) = \lim_{x\to 0} \frac{\ln(1+x^2)}{x^2} = \lim_{x\to 0} \frac{2x/(1+x^2)}{2x} = 1,$$

$$\lim_{x\to 0} y = e^1 = e.$$

2. (20 points)

(a) Find an equation of the tangent line to the curve

$$y \sin(2x) = x \cos(2y)$$

at the point $(\pi/2, \pi/4)$.

Solution. By implicit differentiation:

$$y' \sin(2x) + 2y \cos(2x) = \cos(2y) - 2xy' \sin(2y)$$

at $(\pi/2, \pi/4)$:

$$0 + 2 \cdot \frac{\pi}{4} \cdot (-1) = 0 - 2 \cdot \frac{\pi}{2} y' \cdot (1) \Rightarrow y' = \frac{1}{2}.$$

Equation of the tangent line is

$$y = \frac{1}{2}(x - \pi/2) + \pi/4 = \frac{x}{2}.$$

(b) Using the definition of the derivative as a limit, find the derivative of the function

$$f(x) = \sqrt{2x+1}.$$

Solution.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} = \lim_{h \to 0} \frac{2h}{h\left(\sqrt{2(x+h)+1} + \sqrt{2x+1}\right)}$$

$$= \frac{1}{\sqrt{2x+1}}.$$

3. (20 points)

(a) Find the area of the region bounded by the graph of $y = \frac{x}{x^4+1}$, $y = 0$, $x = 0$, and $x = 1$.

Solution. When $x \geq 0$, we have $y \geq 0$. So

$$A = \int_0^1 \frac{x}{x^4+1}\, dx.$$

Put $u = x^2$, so

$$du = 2x\ dx,$$

$$x = 0 \Rightarrow u = 0$$

$$x = 1 \Rightarrow u = 1$$

$$A = \frac{1}{2}\int_0^1 \frac{du}{u^2+1} = \left[\frac{1}{2}\arctan(x)\right]_0^1 = \frac{\pi}{8}.$$

(b) Consider the function $F(x)$ defined by

$$F(x) = \int_x^{\pi/2} \frac{\sin(t)}{t}\, dt.$$

Show that

$$I = \int_0^{\pi/2} F(x)\ dx = 1.$$

(Hint: Try to calculate I using integration by parts).

Solution. We have $F(\pi/2) = 0$, and by FTC,

$$F'(x) = -\frac{\sin(x)}{x}.$$

Put

$$u = F(x) \Rightarrow du = -\frac{\sin(x)}{x}\ dx$$

$$dv = dx \Rightarrow v = x$$

$$I = \Big[xF(x)\Big]_0^{\pi/2} - \int_0^{\pi/2} -\frac{\sin(x)}{x} \cdot x\ dx = \int_0^{\pi/2} \sin(x)\ dx = \Big[-\cos(x)\Big]_0^{\pi/2} = 1.$$

4. (20 points)

(a) Calculate

$$I = \int_{4/3}^{5/3} (3x-4)^{106}\, dx.$$

Solution. Put $u = 3x - 4$. Then

$$du = 3\ dx$$

$$x = 4/3 \Rightarrow u = 0$$

$$x = 5/3 \Rightarrow u = 1$$

$$I = \int_0^1 \frac{1}{3} u^{106}\, du = \frac{1}{3} \cdot \frac{1}{107} = \frac{1}{321}.$$

(b) Let $a \geq 0$. The arc given by $x^2 + y^2 = a^2$, $x \geq 0$, $-a/2 \leq y \leq a/2$, is rotated about the y-axis. Find the (curved) surface area of the solid it generates. (Note: do not include the top and bottom surfaces.)

Solution.

$$x = f(y) = \sqrt{a^2 - y^2}$$

$$f'(y) = \frac{-y}{\sqrt{a^2 - y^2}}$$

$$A = \int_{y=-a/2}^{y=a/2} 2\pi x\ ds = \int_{-a/2}^{a/2} 2\pi \sqrt{a^2-y^2} \sqrt{1 + \frac{y^2}{a^2 - y^2}} = \int_{-a/2}^{a/2} 2\pi a\ dy$$

$$= \Big[2\pi a y\Big]_{-a/2}^{a/2} = 2\pi a^2.$$

5. (20 points)

(a) Find the power series centred at $c = -1$ representing

$$f(x) = \frac{1}{x^2 + 2x + 2}$$

and determine its radius and interval of convergence. (Hint: express $f(x)$ as a geometric series.)

Solution.

$$f(x) = \frac{1}{x^2 + 2x + 1 + 1} = \frac{1}{(x+1)^2 + 1} = \sum_{n=0}^{\infty} (-1)^n (x+1)^{2n}$$

which holds if and only if

$$|(x+2)^2| < 1 \Leftrightarrow -2 < x < 0.$$

Radius of convergence is $R = 1$, interval of convergence is $I = (-2, 0)$.

(b) Determine the first 3 non-zero terms of the Taylor series of $\sin(x)$ centred at $c = \pi/2$.

Solution 1. Using

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \qquad \forall x,$$

we get

$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(x - \frac{\pi}{2}\right)^{2n}}{(2n)!}$$

$$= 1 - \frac{1}{2}(x - \pi/2)^2 + \frac{1}{24}(x - \pi/2)^4 - \dots$$

Solution 2. $f(x) = \sin(x)$, so

$$f(\pi/2) = 1, f'(\pi/2) = 0, f''(\pi/2) = -1, f'''(\pi/2) = 0, f^{(4)}(\pi/2) = 1,$$

So the Taylor series starts with

$$1 - \frac{1}{2}(x - \pi/2)^2 + \frac{1}{24}(x - \pi/2)^4 - \dots$$

6. (20 points)

(a) Specify whether the series

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(\ln(n))}$$

converges absolutely, converges conditionally, or diverges. Justify your answer by naming the convergence tests used.

Solution. For all $n \geq 1$,

$$0 \leq \ln(\ln(n)) \leq \ln(n) \leq n$$

hence

$$0 \leq \frac{1}{n} \leq \frac{1}{\ln(\ln(n))}.$$

Since $\sum \frac{1}{n}$ diverges (p-series with $p = 1$), by direct comparison the series

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{\ln(\ln(n))} \right| = \sum_{n=3}^{\infty} \frac{1}{\ln(\ln(n))}$$

diverges, hence

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(\ln(n))}$$

does not converge absolutely.
The series

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(\ln(n))}$$

is alternating, $\frac{1}{\ln \ln n}$ is decreasing and goes to 0, so this series converges by the alternating series test. Hence it converges conditionally.

(b) Determine explicitly the sum of the following series and justify your answer:

$$S = \sum_{n=2}^{\infty} \frac{(-1/\sqrt{2})^n}{n!}.$$

Solution. Using

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all x, we have

$$S = e^{-1/\sqrt{2}} - 1 + \frac{1}{\sqrt{2}}.$$