LECTURE 6

In unconstrained optimization we seek $x_* \in \mathbb{R}^n$ such that

$$(i)\ \nabla f(x_*) = 0 \qquad (ii)\ \nabla^2 f(x_*) > 0$$

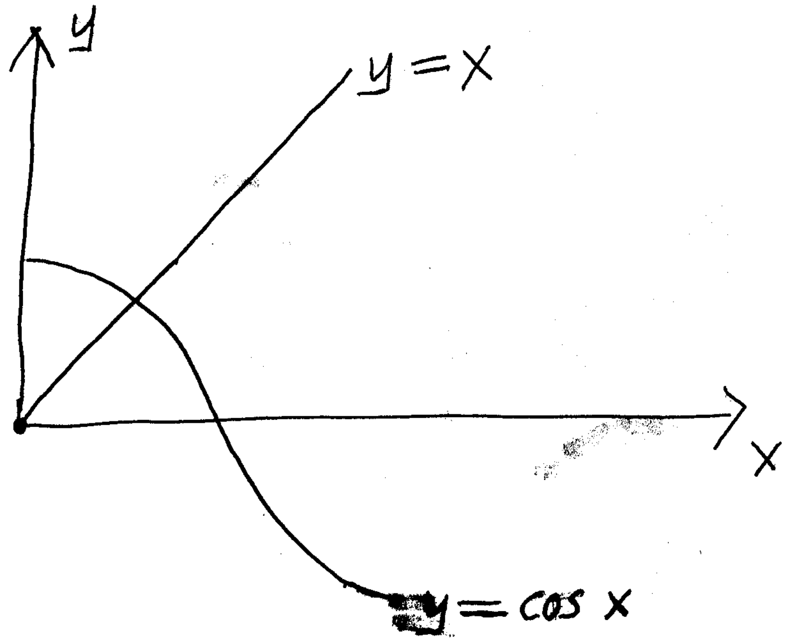
Let $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be continuously differentiable.

x_* such that $F(x_*) = 0$ is called a root of F.

e.g.

$$f(x) = \cos x - x$$

has a root on $(0, \frac{\pi}{2})$



since
$f(0) = 1 > 0$
$f(\pi/2) = -\pi/2 < 0$

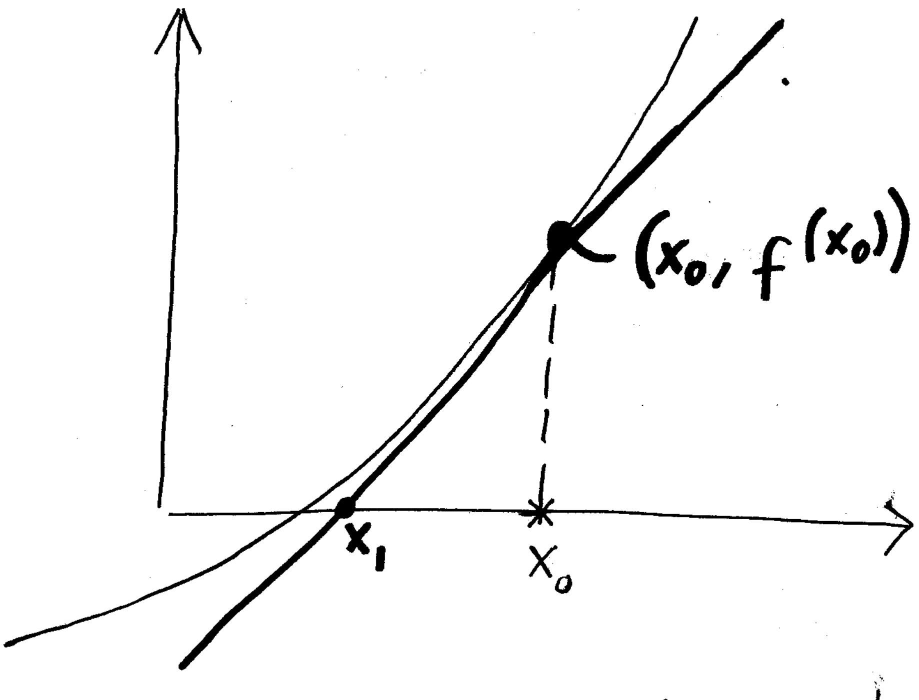
NEWTON'S METHOD FOR UNIVARIATE FUNCTIONS (Gill & Wright 2.2.2)

$f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable

BASIC IDEA

(i) Given an estimate $\tilde{x}$ for a root. Use the linear model that approximat[es] nonlinear f about $\tilde{x}$ well.

(ii) Find the root of the linear model.



linear approximation about x_0

$$l_0(x) = f(x_0) + f'(x_0)(x - x_0)$$

Root of l_0, say x_1, is typically a better estimate than x_0

$$\ell_0(x_1) = 0 = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$\Longrightarrow$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\boxed{\text{NEWTON UPDATE RULE} \quad x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}}$$

The sequence $\{x_k\}$ converges to a root of f under reasonable assumptions.

<u>ALGORITHM</u> (Newton's Method)

Given $x_0 \in \mathbb{R}$, $k = 0$

While $|f(x_k)| > \epsilon$

$\quad x_{k+1} = x_k - f(x_k)/f'(x_k)$

$\quad k = k+1$

end

return x_k

ϵ is a tolerance to decide whether $f(x_k)$ is sufficiently close to 0.

EXAMPLE

$$f(x) = \cos x - x$$

$$f'(x) = -\sin x - 1$$

Newton update rule for f

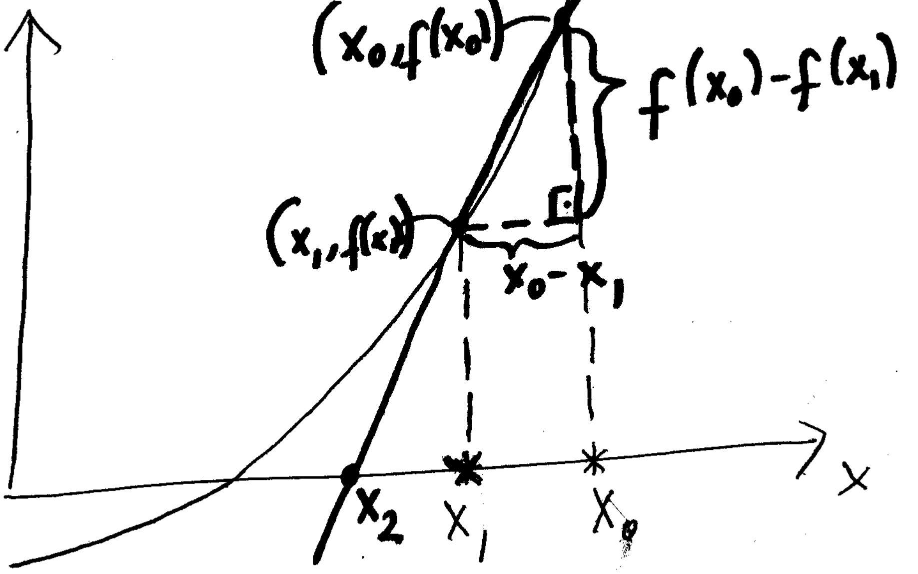
$$x_{k+1} = x_k - \frac{\cos x_k - x_k}{(-\sin x_k - 1)}$$

First four iterations with initial guess $x_0 = 2$

$x_0 = 2$	$\lvert f_0 \rvert = 2.4161468$
$x_1 = 0.7345361$	$\lvert f_1 \rvert = 0.0076055$
$x_2 = 0.7390897$	$\lvert f_2 \rvert = 0.0000077$
$x_3 = 0.7390851$	$\lvert f_3 \rvert \doteq 0.0000000$

SECANT METHOD (Gill & Wright 2.2.3)

Newton's method, but derivatives are approximated with the slopes of secant lines



secant line passing through $(x_0, f(x_0))$ and $(x_1, f(x_1))$

$$S_0(x) = f(x_1) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_1)$$

Next estimate is the root of $S_0(x)$

$$S_0(x_2) = 0 = f(x_1) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_1)$$

$$\Longrightarrow$$

$$x_2 = x_1 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_1)$$

SECANT UPDATE RULE

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{f(x_k) - f(x_{k-1})} f(x_k)$$

ALGORITHM (Secant Method)

Given $x_0, x_1 \in \mathbb{R}$, $k = 1$

While $|f(x_k)| > \epsilon$

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})}{(f(x_k) - f(x_{k-1}))} f(x_k)$$

$k = k+1$

end

return x_k

EXAMPLE

$$f(x) = \cos x - x$$

Secant update rule

$$x_{k+1} = x_k - \frac{(x_k - x_{k-1})\quad(\cos x_k - x_k)}{(\cos x_k - x_k) - (\cos x_{k-1} - x_{k-1})}$$

First six iterations with initial guesses $x_0 = 3,\ x_1 = 2$

$x_1 = 2.0000000$	$\lvert f_1 \rvert = +2.4161468$
$x_2 = 0.4648134$	$\lvert f_2 \rvert = 0.4290919$
$x_3 = 0.6963356$	$\lvert f_3 \rvert = 0.0708621$
$x_4 = 0.7421335$	$\lvert f_4 \rvert = 0.0051051$
$x_5 = 0.7390558$	$\lvert f_5 \rvert = 0.0000492$
$x_6 = 0.7390851$	$\lvert f_6 \rvert = 0.0000000$

CONVERGENCE OF SEQUENCES
(Gill & Wright 1.5)

Let $\{x_k\}$ be a sequence in $\mathbb{R}^n$.

DEFN (Convergence)

The sequence $\{x_k\}$ converges to x_* if for all $\epsilon > 0$ there exists an integer k_ϵ such that

$$\|x_k - x_*\| < \epsilon \quad \text{for all } k > k_\epsilon .$$

(NOTATION: $\lim x_k = x_*$ if $\{x_k\}$ converges to x_*)

EXAMPLE

Show that the sequence $\left\{\frac{1}{2^i}\right\}$ converges to 0

$$\left|\frac{1}{2^i} - 0\right| < \epsilon \iff \frac{1}{2^i} < \epsilon$$

$$\iff \log_2 \frac{1}{\epsilon} < i$$

Therefore for all $i > k_\epsilon = \left\lceil \log_2 \frac{1}{\epsilon} \right\rceil$

$$\left|\frac{1}{2^i} - 0\right| < \epsilon$$

DEFN (Subsequence)

A subsequence is an ordered set obtained from a sequence by including some of its members and preserving the order.

e.g. $\left\{\frac{1}{4^i}\right\}$ is a subsequence of $\left\{\frac{1}{2^i}\right\}$

$\left\{\frac{1}{16}, \frac{1}{4}, 1, \frac{1}{32}, \dots\right\}$ is not a subsequence of $\{\frac{1}{2^i}\}$

DEFN (Limit Point)

The point x_* is called a <u>limit point</u> of the sequence $S = \{x_k\}$ if there exist a subsequence $\hat{S} = \{\hat{x}_k\}$ of S such that

$$\lim_{k\to\infty} \hat{x}_k = x_*$$

e.g.

$\{(-1)^i\}$ does not converge

But it has two limit points $l_1 = 1$ and $l_2 = -1$.

i.e.

The subsequence $\{(-1)^{2i}\}$ converges to $l_1 = 1$

The subsequence $\{(-1)^{2i-1}\}$ converges to $l_2 = -$

RATE OF CONVERGENCE

Let $\{x_k\}$ be a sequence such that

$$\lim_{k \to \infty} x_k = x_*$$

Linear Convergence

$\{x_k\}$ converges to x_* q-linearly if
(quotient-linearly)

$$\lim_{k \to \infty} \frac{\| x_{k+1} - x_* \|}{\| x_k - x_* \|} = c$$

for some $c \in [0, 1)$.

e.g. $\{\frac{1}{2^k}\}$ converges to 0 q-linearly

$$\lim_{k \to \infty} \frac{\left| \frac{1}{2^{k+1}} - 0 \right|}{\left| \frac{1}{2^k} - 0 \right|} = \frac{1}{2}$$

Superlinear Convergence

$\{x_k\}$ converges to x_* q-superlinearly if

$$\lim_{k \to \infty} \frac{\| x_{k+1} - x_* \|}{\| x_k - x_* \|} = 0$$

e.g. $\left\{ \frac{1}{k!} \right\}$ converges to 0 q-superlinearly

$$\lim_{k \to \infty} \frac{| 1/(k+1)! - 0 |}{| 1/k! - 0 |} = \lim_{k \to \infty} \frac{k!}{(k+1)!}$$

$$= \lim_{k \to \infty} \frac{1}{k+1} = 0$$

Quadratic Convergence

$\{x_k\}$ converges to x_* q-quadratically if

$$\lim_{k \to \infty} \frac{\| x_{k+1} - x_* \|}{\| x_k - x_* \|^2} = c$$

for some $c \geq 0$

e.g. $\left\{ \frac{1}{10^{2^i}} \right\}$ converges to 0 q-quadratically

$$\{ \underbrace{0.01}_{10^{-2}}, \underbrace{0.0001}_{10^{-4}}, \underbrace{0.00000001}_{10^{-8}}, \dots \}$$

$$\lim_{k\to\infty} \frac{|1/10^{2^{k+1}} - 0|}{|1/10^{2^k} - 0|^2} = \lim_{k\to\infty} \frac{|1/10^{2^{k+1}}|}{|\underbrace{1/(10^{2^k}\cdot 10^{2^k})}_{1/10^{2\cdot 2^k}}|}$$

$$= 1$$

In general q-order of convergence is $p > 2$ if

$$\lim_{k\to\infty} \frac{\|x_{k+1} - x_*\|}{\|x_k - x_*\|^p} = c$$

for ~~some~~ $c \geq 0$.

RATE OF CONVERGENCE OF NEWTON'S METHOD

Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable.

If the Newton sequence $\{x_k\}$ converges to a root x_* of f, the rate of convergence is q-quadratic, typically.

One exception is when $f'(x_*) = 0$, in which case the rate of convergence is only q-linear.

EXAMPLE (Quadratic Convergence)
It is possible to calculate $1/d$ without performing division.

Apply Newton's method to

$$f(x) = d - \frac{1}{x} \quad \left(f'(x) = \frac{1}{x^2} \right)$$

Newton update rule

$$x_{k+1} = x_k - \frac{(d - 1/x_k)}{(1/x_k^2)} = -dx_k^2 + 2x_k$$

Suppose $\lim\limits_{k\to\infty} x_k = 1/d$. Then

$$\lim_{k\to\infty} \frac{|x_{k+1} - 1/d|}{|x_k - 1/d|^2}$$

$$=$$

$$\lim_{k\to\infty} \frac{|-dx_k^2 + 2x_k - 1/d|}{|x_k - 1/d|^2}$$

$$=$$

$$\lim_{k\to\infty} \frac{|-d\,(x_k^2 - (2/d)\,x_k + 1/d^2|}{|x_k - 1/d|^2}$$

$$=$$

$$\lim_{k\to\infty} \frac{|d|\;|x_k - 1/d|^2}{|x_k - 1/d|^2} = |d|$$

Consequently the q-rate of convergence is quadratic.

<u>EXAMPLE</u>

Suppose Newton's method applied to $g(x) = x^p, \quad p \geq 2 \quad (g'(x) = p x^{p-1}$

converges to the root 0.

Newton update rule

$$x_{k+1} = x_k - \frac{x_k^p}{p x_k^{p-1}} = \left(\frac{p-1}{p}\right) x_k$$

Then

$$\lim_{k\to\infty} \frac{|x_{k+1} - 0|}{|x_k - 0|} = \lim_{k\to\infty} \frac{\left|\left(\frac{p-1}{p}\right) x_k\right|}{|x_k|}$$

$$= (p-1)/p.$$

Consequently the q-rate of convergence is linear.