LECTURE 27
LOGARITHMIC BARRIER METHOD

(NIP)
$$\begin{aligned} &\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad f(x) \\ &\text{subject to} \\ &c_i(x) \geq 0 \quad i=1,\dots,m \end{aligned}$$

Logarithmic Barrier Function

$$L(x,\mu) = f(x) \underbrace{- \mu \sum_{i=1}^{m} \ln c_i(x)}_{\text{Logarithmic Barrier Term}}$$

For a fixed $\mu > 0$

* logarithmic barrier term is large if x is close to the boundary of the feasible region i.e. $c_i(x) \approx 0$.

PENALIZE X CLOSE TO ~~THE~~ BOUNDARY

* For feasible x away from the boundary logarithmic barrier term is small.

Algorithm

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \quad L(x,\mu)$$

~~Subject~~

Solve the minimization problem above for various $\mu > 0$

* starting with large μ
* letting $\mu \to 0$

EXAMPLE

$$\underset{x \in \mathbb{R}^2}{\text{minimize}} \quad x_1^2 + x_2^2$$

subject to

$$x_1 + x_2 \geq 4$$

$\boxed{x_* = (2,2) \text{ is the global minimizer}}$

Logarithmic Barrier Function

$$L(x,\mu) = x_1^2 + x_2^2 - \mu \ln(x_1 + x_2 - 4)$$

Finding a local minimizer $x(\mu)$

$$\nabla_x L(x,\mu) = \begin{bmatrix} 2x_1 - \dfrac{\mu}{x_1+x_2-4} \\ 2x_2 - \dfrac{\mu}{x_1+x_2-4} \end{bmatrix}$$

* Stationary points

$$x(\mu) = \left(1 + \frac{\sqrt{4-\mu}}{2},\ 1 + \frac{\sqrt{4-\mu}}{2}\right)$$

$$\left(\text{as well as } \tilde{x}(\mu) = \left(1 - \frac{\sqrt{4-\mu}}{2},\ 1 - \frac{\sqrt{4-\mu}}{2}\right)\right)$$

Notice that

$$\lim_{\mu \to 0} x(\mu) = (2,2)$$

APPROACHES GLOBAL MINIMIZER

ALGORITHM (Logarithmic Barrier Method)

Given $x_0^s \in \mathbb{F}^0$ (Strictly feasible) and $\mu_0 > 0$. Let $k=0$.

While $\mu_k > \varepsilon$

(1) Choose an initial guess x_k^s for Newton's method using x_{k-1}. Use Newton's method to find an approximate local minimizer x_k of $L(x, \mu_k)$ satisfying

$$\| \nabla_x L(x, \mu_k) \| < \tau_k$$

(2) Choose a $\mu_{k+1} \in (0, \mu_k)$ and Let $k=k+1$.

end

Connection with Primal-Dual methods

Suppose $x(\mu) \in \mathbb{F}^{\circ}$ is a local minimizer of

$$L(x, \mu) = f(x) - \mu \sum_{i=1}^{m} \ln c_i(x)$$

Then

$$\nabla L(x(\mu), \mu) = \nabla f(x(\mu)) - \sum_{i=1}^{m} \frac{\mu}{c_i(x(\mu))} \nabla c_i(x(\mu))$$
$$= 0.$$

Letting $\lambda_i = \mu / c_i(x(\mu))$ we obtain the following.

$$\text{(1)} \quad c_i(x(\mu)) \geqslant 0$$

CENTERALITY CONDITIONS

$$\text{(2)} \quad \lambda_i \geqslant 0$$

$$\text{(3)} \quad \nabla f(x(\mu)) = J(x(\mu))^T \lambda$$

$$\text{(4)} \quad c_i(x(\mu)) \lambda_i = \mu$$

where

$$J(x(\mu)) = \begin{bmatrix} \nabla c_1(x(\mu))^T \\ \vdots \\ \nabla c_m(x(\mu))^T \end{bmatrix}$$

REMARK

As indicated by (1)–(4) the logarithmic barrier method is indeed a primal-dual method.

MIXED PENALTY AND LOGARITHMIC BARRIER METHOD

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$$

subject to

$$c_i(x) \geqslant 0 \quad i=1,\dots,m$$

$$\tilde{c}_j(x) = 0 \quad j=1,\dots,l$$

Mixed penalty, logarithmic-barrier function

$$M(x,\mu) = f(x) + \frac{1}{2\mu} \sum_{j=1}^{l} \tilde{c}_j^{\,2}(x)$$

$$- \mu \sum_{j=1}^{m} \ln c_j(x)$$

This needs to be started with an x satisfying $c_j(x) \geqslant 0 \quad j=1,\dots,m$.
To avoid finding such an x introduce slack variables

$$\underset{x \in \mathbb{R}^n,\ s \in \mathbb{R}^m}{\text{minimize}} \quad f(x)$$

subject to

$$c_i(x) - s_i = 0 \qquad i = 1, \dots, m$$

$$\tilde{c}_j(x) = 0 \qquad j = 1, \dots, \ell$$

$$s_i \geqslant 0 \qquad i = 1, \dots, m$$

Mixed penalty, logarithmic-barrier function

$$M(x, \mu) = f(x) + \frac{1}{2\mu} \sum_{j=1}^{m} \left(c_i(x) - s_i \right)^2 + \frac{1}{2\mu} \sum_{j=1}^{\ell} \tilde{c}_j^{\,2}(x) - \mu \sum_{j=1}^{m} \ln s_j$$

Start with an s such that $s_j \geqslant 0$.