LECTURE 15

NONLINEAR OPTIMIZATION WITH EQUALITY CONSTRAINTS (PART I)

(NEP) $\displaystyle \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x)$

subject to

$$c_j(x) = 0 \quad j = 1, \ldots, m.$$

$\boxed{\text{OBJECTIVE FUNCTION}}\ f : \mathbb{R}^n \to \mathbb{R}$

$\boxed{\text{CONSTRAINT}}\ c_j : \mathbb{R}^n \to \mathbb{R}$

twice continuously differentiable

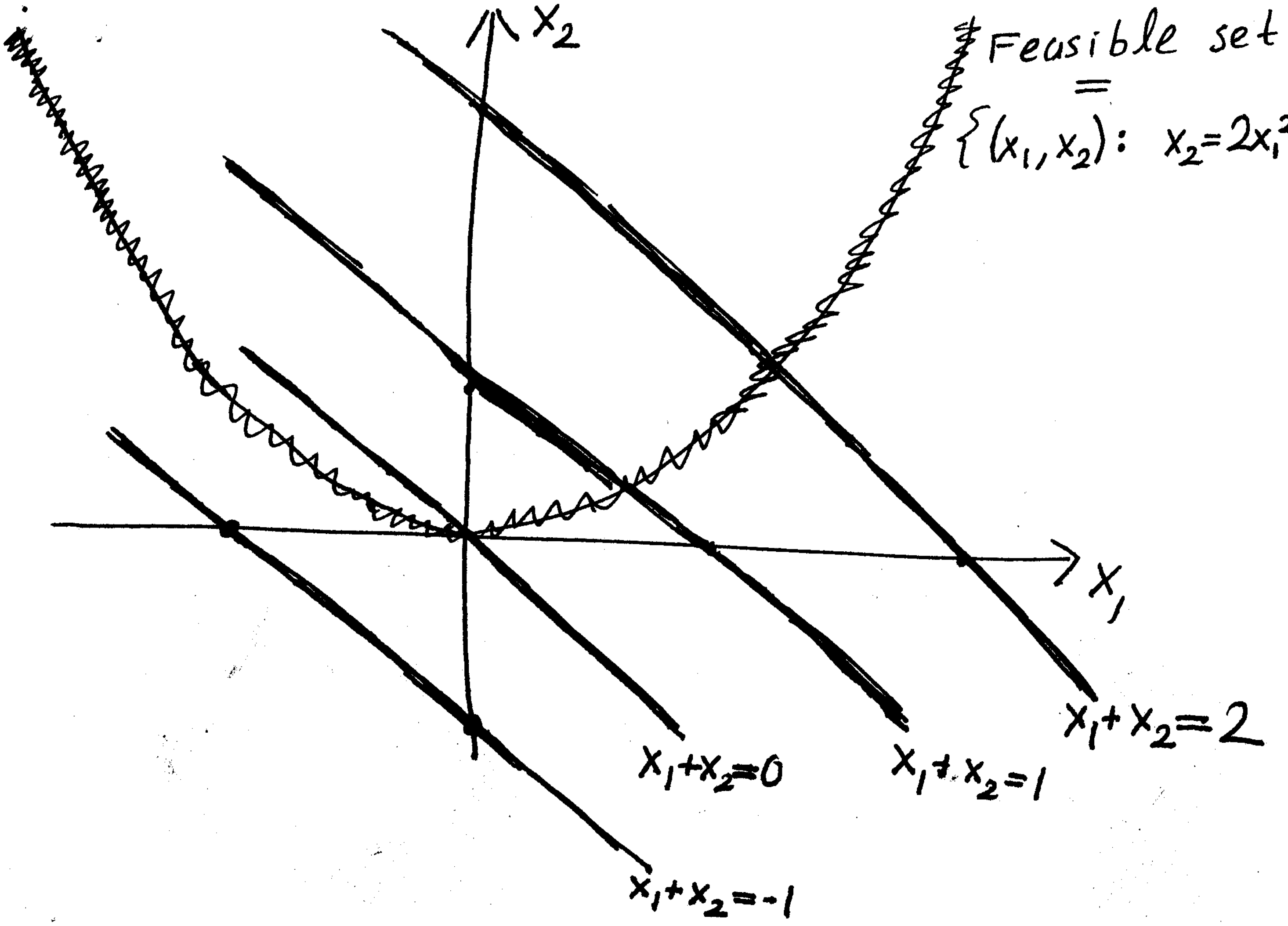
NEP – Nonlinear Equality-Constrained Program

EXAMPLE

$\displaystyle \underset{x \in \mathbb{R}^2}{\text{minimize}} \quad x_1 + x_2$

subject to

$$2x_1^2 - x_2 = 0$$



DEFN (Feasible Set)

The set
$$\mathbb{F} = \{x \in \mathbb{R}^n : c_j(x) = 0,\ j = 1, \dots, m\}$$
is called the feasible set.

DEFN (Local Minimizer)

A point $x_* \in \mathbb{F}$ is called a local minimizer of (NEP) if there exists a $\delta > 0$ s.t.
$$f(x_*) \leq f(x) \quad \text{for all } x \in B(x_*, \delta) \cap \mathbb{F}$$

DEFN (Global Minimizer)
A point $x_* \in \mathbb{R}^n$ is called a <u>global minimizer</u> of (NEP) if

$$f(x_*) \leqslant f(x) \quad \text{for all } x \in \mathbb{F}$$

OPTIMALITY CONDITIONS FOR NEP

We use Taylor's thm to distinguish a local minimizer x_* of (NEP).

TAYLOR'S THM
$$f(x_*+p) = f(x_*) + \nabla f(x_*)^T p + \frac{1}{2} p^T \nabla^2 f(x_*+tp)$$
for some $t \in (0,1)$

UNCONSTRAINED OPTIMIZATION

x_* is a local minimizer

$\Longrightarrow$

$f(x_*+p) \geqslant f(x_*)$ for all small $p \in \mathbb{R}$

NEP

x_* is a local minimizer

$\Longrightarrow$

$f(x_*+p) \geqslant f(x_*)$ for all small p [redacted]
such that $x_*+p \in \mathbb{F}$

We need to define small p such that $x_* + p \in \mathbb{F}$ rigorously.

<u>DEFN</u> (Feasible Path)

A feasible path $x(\alpha): \mathbb{R} \to \mathbb{R}^n$ at $x_* \in \mathbb{F}$ is a parametric twice-differentiable directed curve such that

(i) $x(0) = x_*$,

(ii) $x'(0) \neq 0$, and

(iii) $c_j(x(\alpha)) = 0 \quad j = 1, \dots, m$

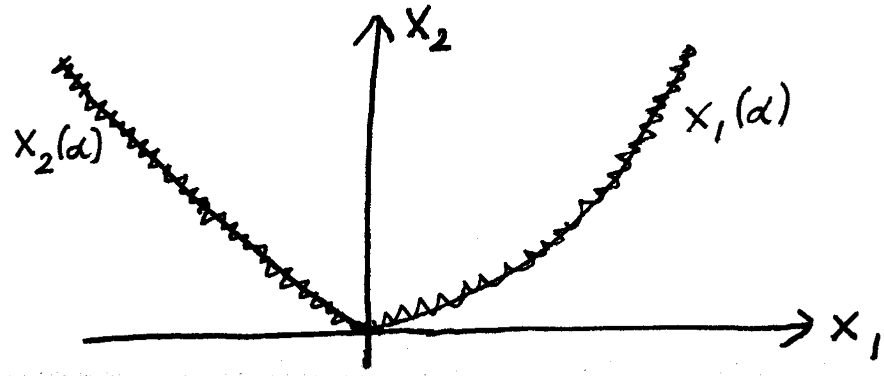
for all $\alpha \in [0, \sigma)$ where $\sigma > 0$.

<u>EXAMPLE</u>

Consider $\quad c(x) = 2x_1^2 - x_2$

Some feasible paths at $(0,0)$

$$x_1(\alpha) = (\alpha, 2\alpha^2)$$
$$x_2(\alpha) = (-\alpha, 2\alpha^2) \qquad \alpha \geqslant 0$$
$$x_3(\alpha) = (3\alpha, 18\alpha^2)$$



OPTIMALITY CHARAC I

x_* is a local minimizer

$$\Longrightarrow$$

$$f(x(\alpha)) \geq f(x_*)$$

for all feasible paths $x(\alpha)$ at x_* and for all small $\alpha > 0$

Define $l(\alpha) := f(x(\alpha))$. By Taylor's thm

$$l(\alpha) = l(0) + l'(0)\alpha + \frac{1}{2} l''(t)\alpha^2$$

for some $t \in (0, \alpha)$

$$\Longleftrightarrow$$

$$f(x(\alpha)) = f(x(0)) + \left(f'(x(\alpha))\,x'(\alpha)\right)\Big|_{\alpha=0} \alpha + c\alpha^2$$

$$= f(x_*) + \nabla f(x_*)^T x'(0)\alpha + c\alpha^2$$

where $c = l''(t)/2$

Now suppose

$\nabla f(x_*)^T x'(0) < 0 \Longrightarrow \alpha \nabla f(x_*)^T x'(0) + c\alpha^2 < 0$ for small $\alpha > 0$

$\Longrightarrow f(x(\alpha)) < f(x_*)$ for small $\alpha > 0$

$\Longrightarrow x_*$ is not a local minimizer

Consequently

OPTIMALITY CHARAC 2

x_* is a local minimizer

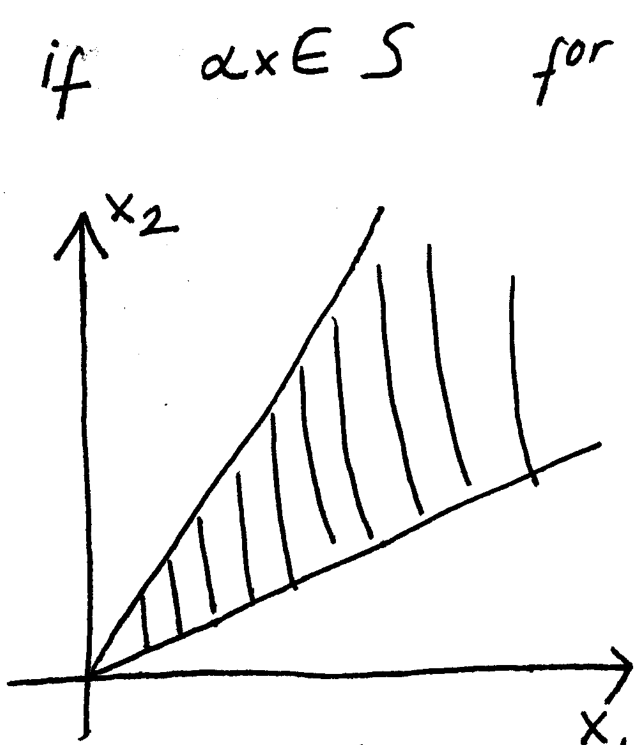
$$\Longrightarrow$$

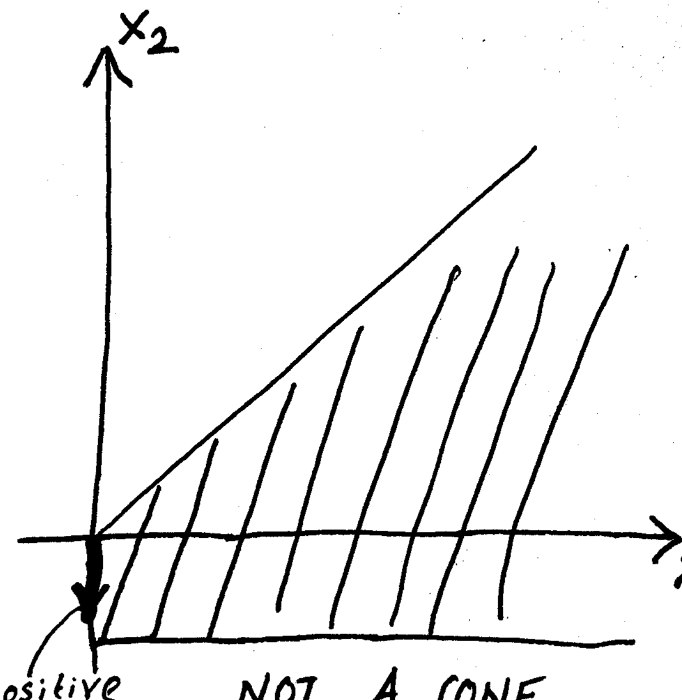
$$\nabla f(x_*)^T x'(0) \geqslant 0$$

for all $x'(0) \in \mathbb{R}^n$ such that $x(\alpha)$ is a feasible path.

DEFN (Cone)

A set $S \subseteq \mathbb{R}^n$ is called a cone if $\alpha x \in S$ for all $x \in S$ and all $\alpha \geqslant 0$.





DEFN (Tangent Cone)

The set

$$T^0 = \{x'(0) \in \mathbb{R}^n : x(\alpha) \text{ is a feasible path at } x_*\} \cup \{0\}$$

is called the <u>tangent cone</u> at x_*.

Let $\alpha = c\bar{\alpha}$ where $c > 0$ is fixed.
Define $y(\bar{\alpha}) := x(c\bar{\alpha}) = x(\alpha)$. ($y$ is the same directed parametric curve as x. Only a different parametrization is used.)

$$x(\alpha) \text{ is a feasible path}$$

$$\Longleftrightarrow$$

$$c_j(x(\alpha)) = 0 \; j = 1, \dots, m \quad \text{for all small } \alpha > 0$$

$$\Longleftrightarrow$$

$$c_j(x(c\bar{\alpha})) = 0 \; j = 1, \dots, m \quad \text{for all small } \bar{\alpha} > 0$$

$$\Longleftrightarrow$$

$$c_j(y(\bar{\alpha})) = 0 \; j = 1, \dots, m \quad \text{for all small } \bar{\alpha} > 0$$

$$\Longleftrightarrow$$

$$y(\bar{\alpha}) \text{ is a feasible path}$$

Now

$$y'(0) = \left(x'(c\bar{\alpha})\, c\right)\Big|_{\bar{\alpha}=0} = x'(0)\, c$$

consequently

$$x'(0) \in T^0 \Longrightarrow c\,x'(0) \in T^0 \text{ for all } c > 0$$

In other word T° is indeed a cone.

OPTIMALITY CHARAC 3

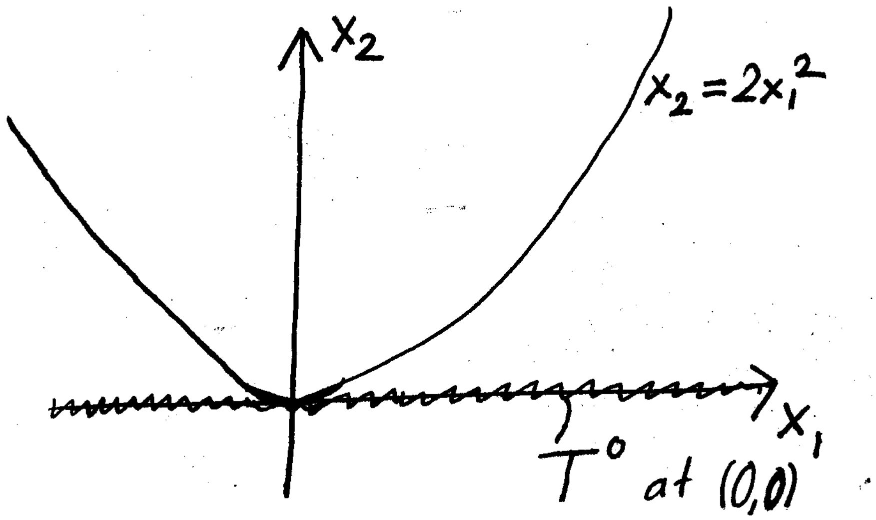
x_* is a local minimizer

$$\Longrightarrow$$

$\nabla f(x_*)^T p \geqslant 0$ for all $p \in T^{\circ}$

EXAMPLE

$$c(x) = 2x_1^2 - x_2$$



Tangent cone at $x=0$

$T^{\circ} = \{\alpha \begin{bmatrix}1\\0\end{bmatrix} : \alpha \in \mathbb{R}$