Generalized Eigenvalue Problem for Nonsquare Pencils

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- This requires $\tilde{\lambda}$ to be a common root of $\begin{pmatrix} n \\ m \end{pmatrix}$ polynomials.

- Generically $A - \lambda B$ has no eigenvalues.

Problem

- Distance to the nearest pencil with m eigenvalues (Boutry, Elad, Golub and Milanfar - 2005)

$$\inf \left\{ \left\| \begin{bmatrix} \Delta A & \Delta B \end{bmatrix} \right\|_F : (A + \Delta A) - \lambda (B + \Delta B) \text{ has } n \text{ eigenvalues} \right\}$$

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- We consider the following variant

$$\mu_r(A,B) := \inf\left\{\|\Delta A\|_2 : (A+\Delta A) - \lambda B \text{ has } r \text{ eigenvalues}\right\}$$

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e.g.

$$A - \lambda B = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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$\lambda = 2$ is an eigenvalue of algebraic multiplicity two

Motivation

- Solutions of $Bx'(t) = Ax(t)$

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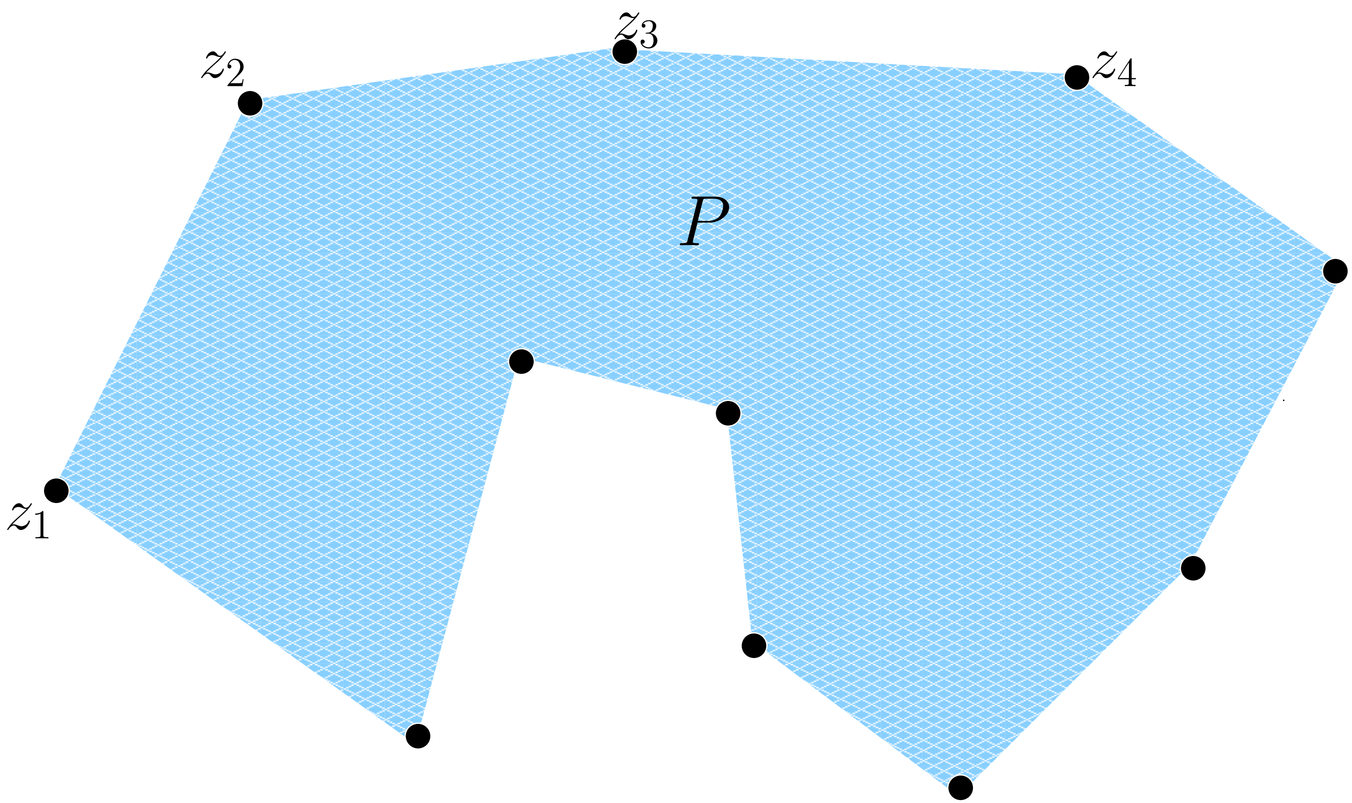
- Estimating a polygon from moments (Elad, Milanfar and Golub - 2004)



Motivation

- Given moments
$$\tau_k = \int\int_P |z|^k \, dx \, dy$$
for $k = 1, \ldots, m$.

Estimate the vertices z_j for $j = 1, \ldots, n$ (complex scalars) of P.

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Estimate the vertices z_j for $j = 1, \dots, n$ (complex scalars) of P.

- The vertices z_j are the eigenvalues of an $m \times n$ pencil $T_0 - \lambda T_1$ where T_0, T_1 are Hankel matrices defined in terms of τ_k.

Derivation of a Singular Value Characterization

We derive a singular value characterization for

$$\mu_r^{\Omega}(A, B) := \inf \left\{ \|\Delta A\|_2 : (A + \Delta A) - \lambda B \ \text{ has } r \text{ eigenvalues lying in } \Omega \right\}$$

where $\Omega \subseteq \mathbb{C}$. (Note: From here on $A, B \in \mathbb{C}^{m \times n}$ with $m \leq n$.)

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$$\mu_r(A,B) = \mu_r^{\Omega}(A,B) \text{ with } \Omega = \mathbb{C}$$

Derivation of a Singular Value Characterization

Special Case ($r = 2, \;\; B = I, \;\; \Omega = \tilde{\lambda}$)

Distance from $A \in \mathbb{C}^{n \times n}$ to the nearest matrix with $\tilde{\lambda} \in \mathbb{C}$ as a multiple eigenvalue (Malyshev - 1999)

$$\mu_2^{\tilde{\lambda}}(A, I) := \;\; \inf\{\|\Delta A\|_2 : \tilde{\lambda} \text{ is an eigenvalue of } (A + \Delta A) - \lambda I \text{ with} \\ \text{algebraic multiplicity } \; \geq 2\}$$

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$$\operatorname{rank}(A - \tilde{\lambda} I)^2 \leq n - 2$$

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$$\Longleftrightarrow$$

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$$\exists \lambda_j \in \Omega, \quad \forall \gamma_{ik} \neq 0$$

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Derivation of a Singular Value Characterization

Some notation

- Linearized pencil

$$\mathcal{P}_r^{\Lambda,\Gamma}(A,B) := \begin{bmatrix} A-\lambda_1 B & 0 & & & 0 \\ \gamma_{21}B & A-\lambda_2 B & & & 0 \\ & & \ddots & & \\ & & & A-\lambda_{r-1}B & 0 \\ \gamma_{r1}B & \gamma_{r2}B & & \gamma_{r(r-1)}B & A-\lambda_r B \end{bmatrix}$$

with

$$\Lambda = (\lambda_1, \ldots, \lambda_r)\,, \quad \Gamma = \left(\gamma_{21}, \gamma_{31}, \gamma_{32}, \ldots, \gamma_{r(r-1)}\right)$$

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- r-tuples of Ω

$$\Omega^r = \{(\omega_1, \ldots, \omega_r) : \omega_j \in \Omega \;\; j = 1, \ldots, r\}$$

Derivation of a Singular Value Characterization

- Rank definition of $\mu_r^{\Omega}(A,B)$ for all Γ with nonzero components

$$\mu_r^{\Omega}(A,B) = \quad \inf_{\Lambda \in \Omega^r} \quad \inf\{\|\Delta A\|_2 : \operatorname{rank}\left(\mathcal{P}_r^{\Lambda,\Gamma}(A+\Delta A, B)\right) \leq nr - r\}$$

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- By continuity of $\sigma_{nr-r+1}\left(\mathcal{P}_r^{\Lambda,\Gamma}(A,B)\right)$ w.r.t. Γ

$$\mu_r^\Omega(A,B) \ge \inf_{\Lambda\in\Omega^r} \ \sup_{\Gamma\in\mathbb{C}^{r(r-1)/2}} \ \sigma_{nr-r+1}\left(\mathcal{P}_r^{\Lambda,\Gamma}(A,B)\right)$$

Derivation of a Singular Value Characterization

Main result

$$\mu_r^{\Omega}(A,B) = \inf \left\{ \|\Delta A\|_2 : (A+\Delta A) - \lambda B \ \text{ has } r \text{ eigenvalues lying in } \Omega \right\}$$

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$$\kappa_r^{\Omega}(A,B) := \inf_{\Lambda \in \Omega^r} \quad \sup_{\Gamma \in \mathbb{C}^{r(r-1)/2}} \ \sigma_{nr-r+1}\left(\mathcal{P}_r^{\Lambda,\Gamma}(A,B)\right)$$

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The direction $\mu_r^{\Omega}(A,B) \leq \kappa_r^{\Omega}(A,B)$ is established by constructing a perturbation ΔA_* satisfying

(i) $\|\Delta A_*\|_2 = \kappa_r^{\Omega}(A,B)$

(ii) $\text{rank}\left(\mathcal{P}_r^{\Lambda,\Gamma}(A+\Delta A_*,B)\right) \leq nr - r \qquad \exists\Gamma \in \mathbb{C}^{r(r-1)/2}, \ \ \exists\Lambda \in \Omega^r$

Corollaries

(i) Distance from $A-\lambda B$ to the nearest pencil with a multiple eigenvalue

$$\inf_{\lambda\in\mathbb{C}} \sup_{\gamma\in\mathbb{C}} \sigma_{2n-1}\left(\begin{bmatrix} A-\lambda B & 0 \\ \gamma B & A-\lambda B \end{bmatrix}\right)$$

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e.g.

$$A-\lambda B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 2 \\ 4 & 2 & -1 \end{bmatrix}$$

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The nearest pencil

$$\begin{bmatrix} 1.91465 & -0.57896 & -1.21173 \\ -1.32160 & 1.93256 & -0.57897 \\ -0.72082 & -1.32160 & 1.91466 \end{bmatrix} - \lambda \begin{bmatrix} -1 & 2 & 3 \\ 2 & -1 & 2 \\ 4 & 2 & -1 \end{bmatrix}$$

at a distance of 0.59299 with the multiple eigenvalue $\lambda_* = -0.85488$

Corollaries

The ϵ pseudospectrum of $A - \lambda B$

$$\Lambda_\epsilon(A, B) = \{\lambda \in \mathbb{C} : \exists \Delta A \ \text{ s.t. } \ \det(A + \Delta A - \lambda B) = 0 \ \text{ and } \ \|\Delta A\|_2 \le \epsilon\}$$

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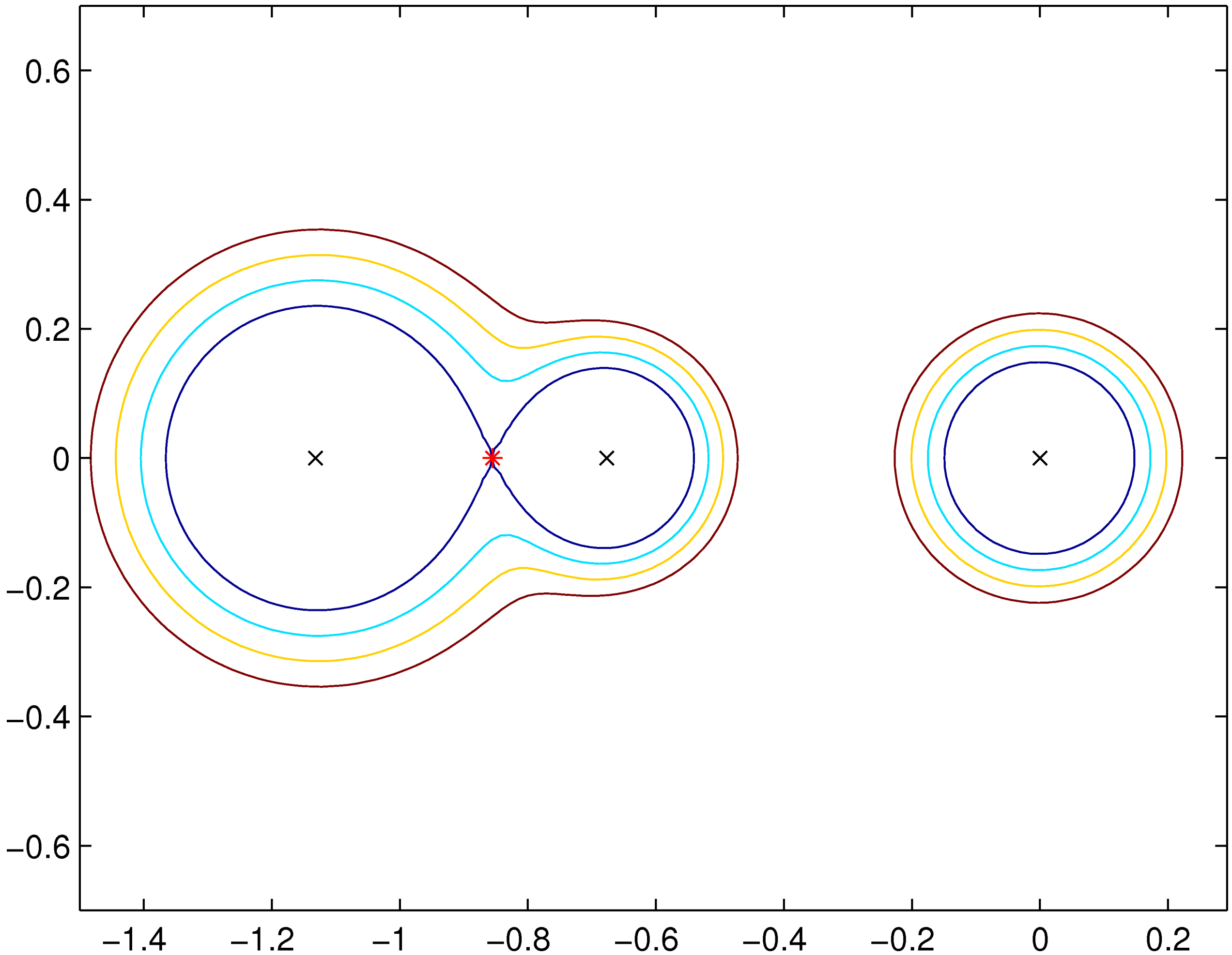

$\Lambda_\epsilon(A, B)$ for $\epsilon = 0.59299$; Red asterisks marks $\lambda_* = -0.85488$.

Corollaries

(ii) Distance from nonsquare $A - \lambda B$ to the nearest pencil with r eigenvalues

$$\inf_{\lambda_j \in \mathbb{C}} \ \sup_{\gamma_{ik} \in \mathbb{C}} \ \sigma_{nr-r+1} \left(\begin{bmatrix} A - \lambda_1 B & 0 & & 0 \\ \gamma_{21} B & A - \lambda_2 B & & 0 \\ & & \ddots & \\ \gamma_{r1} B & \gamma_{r2} B & & A - \lambda_r B \end{bmatrix} \right)$$

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e.g.

$$A-\lambda B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.1 & 2 & 1 \\ 0 & 0 & 0.3 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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The nearest pencil

$$\begin{bmatrix} 0.99847 & 0 & 0 & 0.00007 \\ -0.03697 & 0.08698 & 2.00172 & 1.00095 \\ -0.01283 & 0.03689 & 0.30078 & 2.00376 \end{bmatrix} - \lambda \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

at a distance of 0.03927 with two eigenvalues $\lambda_1^* = 2.55144$ and $\lambda_2^* = 1.45405$.

Corollaries

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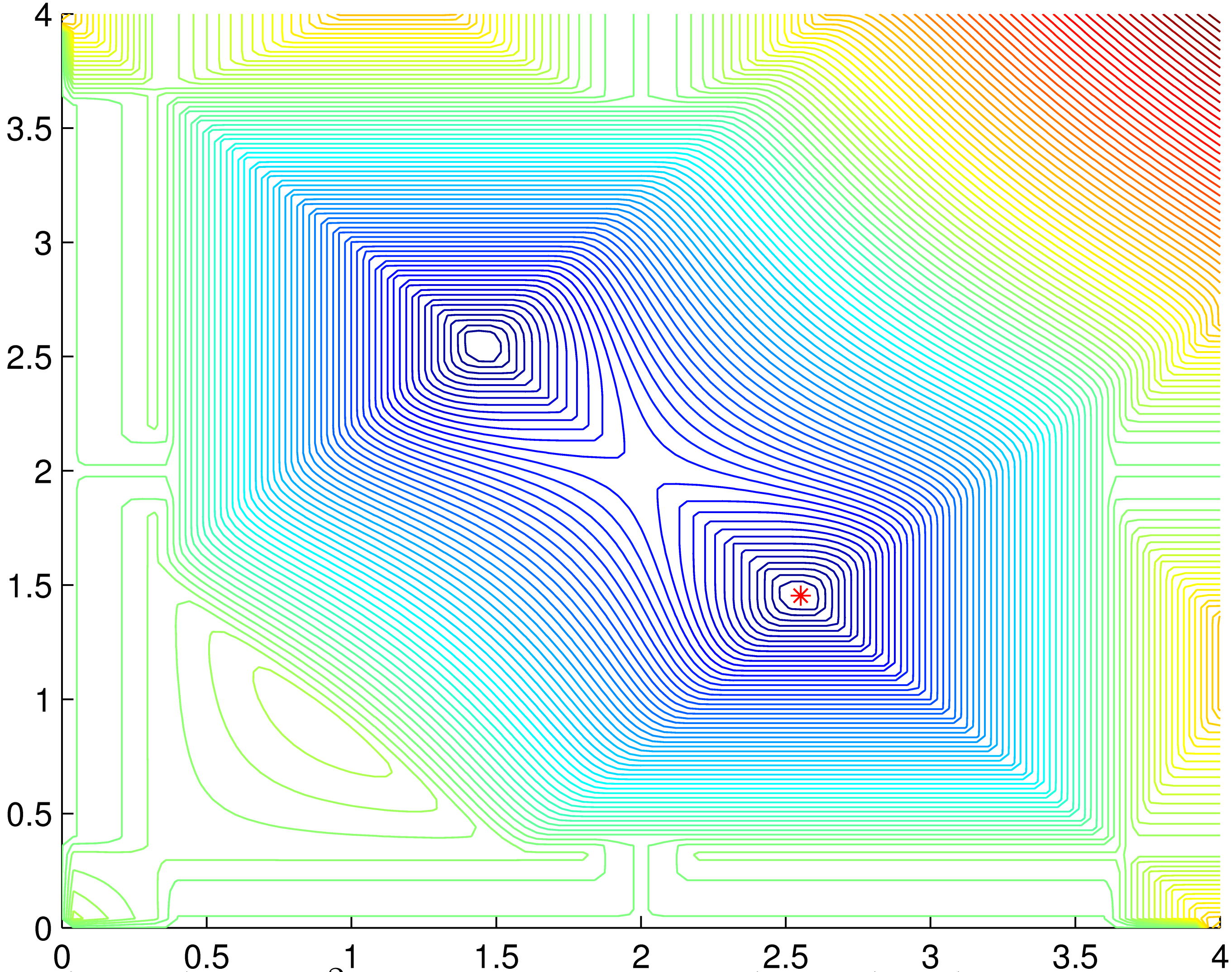

Level sets of $g(\lambda_1, \lambda_2)$ over $\mathbb{R}^2$; Red asterisks marks $(\lambda_1^*, \lambda_2^*) = (2.55144, 1.45405)$.

Corollaries

(iii) Distance from $A - \lambda B$ to the nearest stable pencil

$$\inf_{\lambda_j \in \mathbb{C}^-} \sup_{\gamma_{ik} \in \mathbb{C}} \sigma_{n^2-n+1} \left(\begin{bmatrix} A - \lambda_1 B & 0 & & 0 \\ \gamma_{21} B & A - \lambda_2 B & & 0 \\ & & \ddots & \\ \gamma_{n1} B & \gamma_{n2} B & & A - \lambda_n B \end{bmatrix} \right)$$

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e.g.

$$A = \begin{bmatrix} 0.6 - \frac{i}{3} & -0.2 + \frac{4i}{3} \\ -0.1 + \frac{2i}{3} & 0.5 + \frac{i}{3} \end{bmatrix}$$

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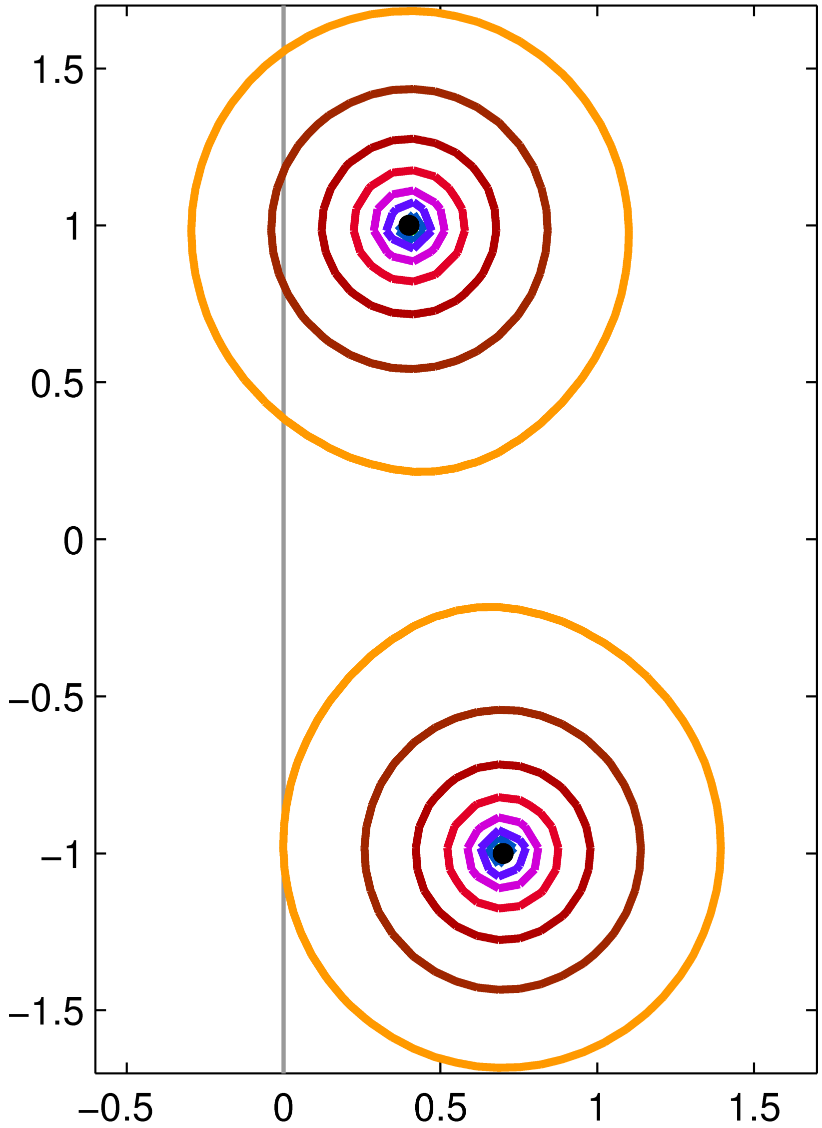
is unstable with eigenvalues $0.7 - i$ and $0.4 + i$ at a distance of 0.6610 to stability. The matrix

$$A + \Delta A_* = \begin{bmatrix} 0.0681 - 0.3064i & -0.4629 + 1.2524i \\ 0.2047 + 0.5858i & -0.1573 + 0.3064i \end{bmatrix}$$

at a distance of 0.6610 has the eigenvalues $-0.9547i$ and $-0.0885 + 0.9547i$.

Corollaries



$\Lambda_\epsilon(A)$ for $\epsilon = 0.6610$

Computation

Key observations for the computation of $\mu_r^{\Omega}(A, B)$

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- Outer minimization:
  The function

$$g(\Lambda) = \sup_{\Gamma \in \mathbb{C}^{r(r-1)/2}} \sigma_{nr-r+1} \left(\mathcal{P}_r^{\Lambda,\Gamma}(A, B) \right)$$

  is Lipschitz continuous (due to Weyl's theorem).

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- Inner maximization:
  Any local maximizer of the inner problem is typically a global maximizer. Algebraic conditions to check whether a local maximizer is a global maximizer exist. Therefore the inner maximization can be performed by means of a Newton-based method, e.g. BFGS.

Computation

Computing the distance from A to a nearest matrix with a multiple eigenvalue for random A

$$\inf_{\lambda \in \mathbb{C}} \ \sup_{\gamma \in \mathbb{C}} \ \sigma_{2n-1}\left(\begin{bmatrix} A - \lambda I & 0 \\ \gamma I & A - \lambda I \end{bmatrix} \right)$$

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size	100	200	300	500	800	1000
time	11	34	124	271	1367	1218
fevals	115	95	127	111	215	147

References

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