A GLOBAL ATTRACTOR FOR THE SINGULARLY PERTURBED VISCOUS CAHN - HILLIARD EQUATION

VARGA K. KALANTAROV AND SERGEY ZELIK

Abstract. We prove existence of a global attractor of the semigroup, generated by the initial boundary value problem for the 3D singularly perturbed viscous Cahn - Hilliard equation , when the corresponding nonlinear term may grow as a fifth order polynomial.

1. Introduction

In a bounded domain $\Omega \subset \mathbb{R}^3$ with a sufficiently smooth boundary $\partial\Omega$, we consider the following singularly perturbed Cahn - Hilliard equation:

$$\begin{cases} \varepsilon\partial_t^2 u - \gamma\Delta_x\partial_t u + \alpha\partial_t u + \Delta_x(\Delta_x u - f(u) + g) = 0 \\ u\big|_{\partial\Omega} = \Delta_x u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \quad \partial_t u\big|_{t=0} = u_0'. \end{cases} \tag{1.1}$$

Here $u = u(t,x)$ is an unknown function, Δ_x is a Laplacian with respect to the variable x, $\varepsilon > 0$, $\gamma > 0$ and $\alpha \geq 0$ are given parameters where ε is assumed to be small, $g \in L^2(\Omega)$ is a given external force, and $f \in C^1(\mathbb{R}^1, \mathbb{R}^1)$ is a given nonlinearity.

The equation (1.1) models various processes in viscoelasticity, hydrodynamics and phase transitions problems for different type of nonlinearities $f(u)$ (see e.g. [4],[5]).

Existence of a global attractor for the semigroup generated by the 1D problem (1.1) for the nonlinear term $f(u) = u^3 - \lambda u$ in the phase space $H_0^1 \times H_{-1}$ is established in [3],[16]. The existence of an inertial manifold for the 1D problem (1.1) is established in [13], when nonlinear term satisfies the dissipativity conditions (1.3). In [6] it is shown that the 3D problem (1.1) has a global attractor for the nonlinear term $f(u)$ satisfying the growth condition $|f''(u) \leq C(1+|u|)$, $\forall u \in \mathbb{R}$ and the dissipativity condition $\liminf_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1$. The main result obtained in this note can be considered as a development of the result on the attractor obtained in [6].

We assume that f satisfies the following growth

$$|f'(u)| \leq C(1 + |u|^4), \ \forall u \in \mathbb{R}, \tag{1.2}$$

and dissipativity assumptions

$$1) \ f(u) \cdot u \geq -C, \quad 2) \ \ f'(u) \geq -K, \ \forall u \in \mathbb{R}, \tag{1.3}$$

where C and K are some fixed constants.

2000 *Mathematics Subject Classification.* 35B30, 35B35, 35G25.

Let us denote by P the inverse of the operator $-\Delta_x$ under the homogeneous Dirichlet boundary condition.

It is convenient to apply the operator P to both sides of the equation (1.1) and rewrite the problem (1.1) in the following equivalent form:

$$\begin{cases} P\partial_t^2 u + \gamma\partial_t u + \alpha P\partial_t u - \Delta_x u + f(u) = g, \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0, \quad \partial_t u\big|_{t=0} = u_0'. \end{cases} \tag{1.4}$$

As usual, we consider the equation (1.4) in the energy phase space $\mathcal{E}_\varepsilon := H_0^1(\Omega)\times H^{-1}(\Omega)$ endowed by the following norm:

$$\|\xi_u(t)\|^2_{\mathcal{E}_\varepsilon} := \varepsilon\|\partial_t u(t)\|^2_{-1} + \|u(t)\|^2_1, \quad \xi_u(t) := \{u(t), \partial_t u(t)\}. \tag{1.5}$$

Here and in what follows $\|\cdot\|$ and $(\cdot,\cdot)$ will denote the norm and the scalar product in $L^2(\Omega)$, $\|u\|_{-1} := \|P^{\frac{1}{2}}u\|$ and $\|u\|_1 := \|\nabla_x u\|$.
Throughout this paper, the same letter C denotes a generic constant which do not depend on data and parameters ϵ, α.

Definition 1.1. A function $\xi_u(t) := \{u(t), \partial_t u(t)\}$ is called a solution of the problem (1.1) if

$$1.\ \xi_u \in C([0,T], \mathcal{E}_\varepsilon), \quad 2.\ \partial_t u \in L^2([0,T], L^2(\Omega)), \quad 3.\ u \in L^2([0,T], H^2(\Omega)) \tag{1.6}$$

for every $T > 0$ and the equation (1.1) is satisfied in the sense of distributions.

Recall now that, due to the standard embedding theorem for the anisotropic Sobolev spaces, see e.g. [11], the assumption (1.6) implies that

$$u \in L^{10}(G_T), \quad \text{where} \quad G_T := (0,T)\times\Omega,$$

and, consequently, due to the growth restriction (1.2),

$$f(u) \in L^2(G_T). \tag{1.7}$$

(exactly this embedding defines the critical exponent 4 in the growth restriction (1.2)). Furthermore, from equation (1.4) we conclude that $P\partial_t^2 u \in L^2(G_T)$ and equation (1.4) can be considered as equality in $L^2(G_T)$.

Notice that the required assumptions 1) and 2) of (1.6) in our definition of an energy solution are standard for the theory of semi-linear hyperbolic equation and are motivated by the usual energy estimate for solutions of (1.1) (which can be obtained by multiplication of (1.4) by $\partial_t u$ and integrating over $(0,T)\times\Omega$, see [1, 10, 15]). However, the third assumption is typical for the theory of parabolic equations and reflects the fact that the problem (1.1) is *parabolic* for $\gamma > 0$ (and can be rewritten as a semi-linear system of parabolic equations, see Remark 2.5 below). Moreover, exactly that parabolic nature of the problem (somehow overpassed in [6]) allows to shift the limit growth exponent from $\frac{n}{n-2} = 2$ (standard hyperbolic critical exponent) till $\frac{n+2}{n-2} = 4$ (usual parabolic one) in the growth assumption (1.2).

The main result of the paper is the following theorem about the existence of a *global attractor*, that is of a compact set which is invariant and attracts uniformly each bounded set of the phase space.

Theorem 1.2. *Let the above assumptions hold. Then, for every $\varepsilon > 0$, problem* (1.1) *is globally well-posed in the phase space $\mathcal{E}_\varepsilon$ and generates a dissipative semigroup $S_\varepsilon(t)$ in that space which possesses a global attractor $\mathcal{A}_\varepsilon$ there.*

Moreover, the above global attractor is uniformly bounded in the space $H^2(\Omega) \times L^2(\Omega)$ as $\varepsilon \to 0$.

2. Proof of the main result

We start from verifying the basic dissipative estimate for solutions of the problem (1.1).

2.1. **Hyperbolic dissipative estimate.** Let us multiply the equation (1.4) by $\partial_t u(t)$ and integrate over Ω. Then, after the integration by parts and standard transformations, we obtain the so-called energy equality

$$1/2\frac{d}{dt}\left(\|\xi_u(t)\|^2_{\mathcal{E}_\varepsilon} + (F(u(t)),1)_{L^2} - 2(g,u)\right) + \gamma\|\partial_t u(t)\|^2 + \alpha\|\partial_t u(t)\|^2_{-1} = 0, \tag{2.1}$$

where $F(u) := \int_0^u f(v)\,dv$ (in contrast to the situation with damped wave equations, all of the terms in the equation (1.4) belong to $L^2(G_T)$, see (1.7), and $\partial_t u$ also belongs to $L^2(G_T)$. Thus, the scalar product of (1.4) is well-defined and not any additional justification is necessary).

At the next step, we multiply equation (1.4) by u and integrate over Ω and obtain

$$1/2\frac{d}{dt}\left(2\varepsilon(P\partial_t u, u) + \gamma\|u\|_2 + \alpha\|u\|^2_{-1}\right) + \|u\|_1^2 + (f(u),u) = (g,u). \tag{2.2}$$

Multiplying the equation (2.2) by some sufficiently small number δ which will be specified below, taking a sum with equation (2.1) and using the first dissipativity assumption of (1.3), we arrive at

$$\frac{d}{dt}E_\varepsilon(\xi_u(t)) + \delta_1(\|\partial_t u(t)\|^2 + \|u(t)\|_1^2) \le C_\delta(1 + \|g\|^2) \tag{2.3}$$

with

$$E_\varepsilon(\xi_u) := \|\xi_u\|^2_{\mathcal{E}_\varepsilon} + (F(u),1) - 2(g,u) + 2\delta\varepsilon(P\partial_t u, u) + \gamma\delta\|u\|^2 + \alpha\delta\|u\|^2_{-1} \tag{2.4}$$

and $\delta_1 := \min\{\gamma, \delta\}$. Using again the first dissipativity assumption of (1.3), we infer that

$$F(u) \ge -C_1(|u| + 1)$$

for some constant C_1 independent of u. Applying that estimate to (2.4), we see that, for sufficiently small δ (uniformly with respect to $\varepsilon \to 0$), the following coercivity estimate holds:

$$E_\varepsilon(\xi_u) \ge 1/2\|\xi_u\|^2_{\mathcal{E}_\varepsilon} - C(\|g\|^2 + 1) \tag{2.5}$$

where C is also uniform with respect to $\varepsilon \to 0$. On the other hand, using the growth restriction (1.2) together with the embedding $H^1 \subset L^6$, we see that

$$E_\varepsilon(\xi_u) \le C(1 + \|\xi_u\|^2_{\mathcal{E}_\varepsilon})^3 \tag{2.6}$$

where the constant C is uniform with respect to $\varepsilon \to 0$. Thus, we can rewrite the inequality (2.3) in the form

$$\frac{d}{dt}E_\varepsilon(\xi_u(t)) + \delta_2\|\xi_u(t)\|^2_{\mathcal{E}_\varepsilon} \le C(1 + \|g\|^2). \tag{2.7}$$

Then, the Gronwal-like estimate (see e.g. [2]) applied to the inequality (2.7) gives the required dissipative estimate

$$\|\xi_u(t)\|^2_{\mathcal{E}_\varepsilon} \le Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon})e^{-\alpha t} + Q(\|g\|) \tag{2.8}$$

for some positive constant α and monotone function Q which are uniform with respect to $\varepsilon \to 0$. Finally, integrating the inequality (2.3) over $[T, T+1]$ and using the proven estimate (2.8), we deduce that

$$\int_T^{T+1} \|\partial_t u(t)\|^2\, dt \le Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon})e^{-\alpha T} + Q(\|g\|), \tag{2.9}$$

where Q is also uniform with respect to $\varepsilon \to 0$. Thus, the "hyperbolic" dissipative estimates are obtained.

2.2. **Parabolic dissipative estimate.** Exploiting now the parabolic nature of equation (1.4), we multiply it by $-\Delta_x u$ and integrate over Ω. Then, after integration by parts, we have

$$1/2\frac{d}{dt}\left(2\varepsilon(\partial_t u, u) + \gamma\|u\|_1^2 + \alpha\|u\|^2\right) + \|\Delta_x u\|^2 + (f'(u)\nabla_x u, \nabla_x u) = \|\partial_t u\|^2 - (g, \Delta_x u). \tag{2.10}$$

Integrating this equality with respect to $t \in [T, T+1]$ and using the quasi-monotonicity assumption 2) of (1.3) together with the proven estimates (2.8) and (2.9), one can easily deduce that

$$\int_T^{T+1} \|\Delta_x u(t)\|^2\, dt \le Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon})e^{-\alpha T} + Q(\|g\|), \tag{2.11}$$

where Q and α are also uniform with respect to $\varepsilon \to 0$. Thus, we have proven the following result.

Lemma 2.1. *Let the above assumptions hold and let $\xi_u(t) := (u(t), \partial_t u(t))$ be a solution of the equation* (1.1) *satisfying* (1.6). *Then, the following estimate holds:*

$$\|\xi_u(T)\|^2_{\mathcal{E}_\varepsilon} + \int_T^{T+1} \|\partial_t u(t)\|^2 + \|\Delta_x u(t)\|^2\, dt \le Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon})e^{-\alpha T} + Q(\|g\|), \tag{2.12}$$

where the positive constant α and the monotone function Q are uniform with respect to $\varepsilon \to 0$.

Indeed, the assertion of the lemma is an immediate corollary of estimates (2.8), (2.9) and (2.11).

Remark 2.2. The quasi-monotonicity assumption $f'(u) \ge -K$, $\forall u \in$ is not necessary in the case of *subcritical* growth rate

$$|f'(u)| \le C(1 + |u|^{4-r}), \quad \forall u \in, \quad r > 0.$$

Indeed, using the standard interpolation inequality

$$\|u\|^5_{L^{10}} \le C\|u\|_1^4\|\Delta_x u\| \tag{2.13}$$

together with the subcritical growth restriction we can estimate the L^2-norm of $f(u)$ as follows:

$$\|f(u)\|^2 \le C_\mu + \mu\|u\|^{10}_{L^{10}} \le C_\mu + \mu\|u\|_1^8\|\Delta_x u\|^2,$$

where $\mu > 0$ is arbitrary. In particular, scaling $\nu = \mu/\|u\|_1^8$, we have

$$\|f(u)\|^2 \le Q_\nu(\|u\|_1) + \nu\|\Delta_x u\|^2, \tag{2.14}$$

where $\nu > 0$ is again arbitrarily small.

Since the H^1-norm of u is under the control due to (2.8), this inequality allows indeed to estimate the term $(f(u), \Delta_x u)$ without integration by parts and without employing the quasi-monotonicity assumption.

Unfortunately, in the critical case $r = 0$, we cannot obtain (2.14) with *small* ν and the quasi-monotonicity assumption seems unavoidable. Since, from our point of view, the possibility to treat the critical case is much more important than the relaxation of the quasi-monotonicity assumption, we have formulated our main result under the dissipativity assumptions (1.3).

2.3. **Uniqueness and existence of a semigroup.** Let $u_1(t)$ and $u_2(t)$ be two solutions of the problem (1.1) satisfying (1.6) and let $\bar u(t) := u_1(t) - u_2(t)$. Then, this function solves

$$\varepsilon P\partial_t^2\bar u + \gamma\partial_t\bar u + \alpha\partial_t\bar u - \Delta_x\bar u = -[f(u_2) - f(u_1)]. \tag{2.15}$$

Multiplying this equation by $\partial_t\bar u$ and arguing as in (2.1), we deduce that

$$\frac{d}{dt}\|\xi_{\bar u}(t)\|^2_{\mathcal{E}_\varepsilon} + \gamma\|\partial_t\bar u(t)\|^2 \le C\|f(u_1(t)) - f(u_2(t))\|^2, \tag{2.16}$$

where the constant C depends only on γ. Thus, we only need to estimate the term in the right-hand side. To this end, using Hölder inequality and the embedding $H^1 \subset L^6$ together with the growth restriction (1.2), we infer

$$\|f(u_1) - f(u_2)\|^2 \le C(1 + \|u_1\|^8_{L^{12}} + \|u_2\|^8_{L^{12}})\|\bar u\|^2_{L^6} \le$$
$$\le C_1(1 + \|u_1\|^8_{L^{12}} + \|u_2\|^8_{L^{12}})\|\bar u\|_1^2 := H_{u_1,u_2}(t)\|\bar u\|_1^2 \le H_{u_1,u_2}(t)\|\xi_{\bar u}(t)\|^2_{\mathcal{E}_\varepsilon}. \tag{2.17}$$

Using now the proper interpolation inequality:

$$\|u\|^8_{L^{12}} \le C\|u\|_1^6\|\Delta_x u\|^2 \tag{2.18}$$

together with the control (2.12) of the $C([0,T], H^1)$ and $L^2([0,T], H^2)$ norms of solutions u_1 and u_2, we deduce that

$$\int_0^T H_{u_1,u_2}(t)\,dt \le Q(\|\xi_{u_1}(0)\|_{\mathcal{E}_\varepsilon} + \|\xi_{u_2}(0)\|_{\mathcal{E}_\varepsilon})(T+1), \tag{2.19}$$

where the function Q is independent of $\varepsilon \to 0$ and $T \ge 0$. Thus, inserting estimate (2.17) into the right-hand side of (2.16), applying the Gronwall's inequality and using the control (2.19), we infer the following Lipschitz continuity:

$$\|\xi_{u_1}(t) - \xi_{u_2}(t)\|_{\mathcal{E}_\varepsilon} \le Ce^{Kt}\|\xi_{u_1}(0) - \xi_{u_2}(0)\|_{\mathcal{E}_\varepsilon}, \tag{2.20}$$

where the constants C and K depend on the norms of $\xi_{u_i}(0)$, but are independent of $\varepsilon \to 0$. Thus, the uniqueness is verified. The existence of a solution can be verified in a standard way, using e.g. the Galerkin approximation method, see e.g. [1].

Therefore, the problem (1.1) is indeed globally well-posed in the phase space $\mathcal{E}_\varepsilon$ and defines a bounded dissipative semigroup $S_\varepsilon(t)$ in it via the standard expression

$$S_\varepsilon(t)\xi_0 := \xi_u(t), \quad \xi_u \text{ solves (1.1) with } \xi_u(0) = \xi_0. \tag{2.21}$$

Thus, for proving the main Theorem 1.2, we only need to verify the existence of a global attractor $\mathcal{A}_\varepsilon$ for that semigroup and obtain its uniform bounds as $\varepsilon \to 0$. As usual, in order to do so, we need to study the smoothing properties of system (1.1).

2.4. **Smoothing property and absorbing sets.** Let us multiply the equation (1.4) by $-t\partial_t\Delta_x u$ and integrate over Ω. Then, after the integration by parts and obvious transformations, we get

$$\frac{d}{dt}\left(t\|\xi_u\|^2_{\mathcal{E}^1_\varepsilon} + t(g,\Delta_x u)\right) + \\ + \gamma t\|\partial_t\nabla_x u\|^2 \le Ct\|f'(u)\nabla_x u\|^2 + \varepsilon\|\partial_t u\|^2 + \|\Delta_x u\|^2 + (g,\Delta_x u), \quad (2.22)$$

where $\|\xi_u\|^2_{\mathcal{E}^1_\varepsilon} := \varepsilon\|\partial_t u\|^2 + \|\Delta_x u\|^2$. Thus, keeping in mind the control (2.12), we see that we only need to estimate the first term in the right-hand side of (2.22). To this end, arguing analogously to (2.17) and using the growth restriction (1.2), we deduce

$$\|f'(u)\nabla_x u\|^2 \le \|f'(u)\|^2_{L^3}\|\nabla_x u\|^2_{L^6} \le \\ \le C(1+\|u\|_{L^{12}})^8\|u\|^2_{H^2} := H_u(t)\|\Delta_x u\|^2 \le H_u(t)\|\xi_u\|^2_{\mathcal{E}^1_\varepsilon}, \quad (2.23)$$

where, due to the interpolation inequality (2.18) and the control (2.12), the function $H_u(t) := C(1+\|u(t)\|^8_{L^{12}})$ satisfies

$$\int_0^1 H_u(t)\,dt \le Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon}) + Q(\|g\|), \quad t\in[0,1] \quad (2.24)$$

Inserting now estimate (2.23) into the right-hand side of (2.22), integrating it by t and using the Gronwall's inequality together with the control (2.24), we get

$$\|\xi_u(t)\|^2_{\mathcal{E}^1_\varepsilon} \le \frac{C}{t}\left(Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon}) + Q(\|g\|)\right), \quad t\in(0,1], \quad (2.25)$$

where the monotone function Q is uniform with respect to $\varepsilon \to 0$. Thus, we have proven the following result.

Lemma 2.3. *Let the above assumptions hold. Then, the following estimate holds for every solution $u(t)$ of the problem* (1.1)*:*

$$\|\xi_u(T)\|^2_{\mathcal{E}^1_\varepsilon} + \int_T^{T+1}\|\partial_t u(t)\|^2_1\,dt \le \frac{T+1}{T}\left(Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon})e^{-\alpha T} + Q(\|g\|)\right), \quad (2.26)$$

where the positive constant α and monotone function Q are independent of $\varepsilon \to 0$.

Indeed, estimate (2.26) is an immediate corollary of estimates (2.25) and (2.8).
The smoothing estimate (2.26) guarantees the existence of a compact absorbing set for the semigroup $S\varepsilon(t)$ which is sufficient to obtain the existence of its global attractor $\mathcal{A}_\varepsilon$. However, it is not sufficient for obtaining the uniform bounds for $\mathcal{A}_\varepsilon$ as $\varepsilon \to 0$. In order to overcome this difficulty, we need one more smoothing estimate.

Lemma 2.4. *Let the above assumptions hold. Then, the following estimate holds for every solution $u(t)$ of problem* (1.1)*:*

$$\|\partial_t u(T)\|^2 + \varepsilon\int_T^{T+1}\|\partial_t^2 u(t)\|^2_{-1}\,dt \le \frac{T+1}{T}\left(Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon})e^{-\alpha T} + Q(\|g\|)\right), \quad (2.27)$$

where the positive constant α *and monotone function* Q *are independent of* $\varepsilon \to 0$.

Proof. Multiplying equation (1.4) by $t\partial_t^2 u$ and integrating over Ω, we will have

$$\frac{d}{dt}\left(\gamma t\|\partial_t u\|_{L^2}^2 + \alpha t\|\partial_t u\|_{-1}^2 - 2t(\partial_t u, \Delta_x u) + 2t(\partial_t u, f(u)) - 2t(g, \partial_t u)\right) + \\ + 2t\varepsilon\|\partial_t^2 u\|_{-1}^2 = 2t\|\partial_t \nabla_x u\|^2 + 2t(f'(u)\partial_t u, \partial_t u) + \gamma\|\partial_t u\|^2 + \\ + \alpha\|\partial_t u\|_{-1}^2 - 2(\partial_t u, \Delta_x u) + 2t(\partial_t u, f(u)) - 2(g, \partial_t u). \quad (2.28)$$

Analogously to (2.23), we can estimate the term with $f'(u)$ as follows:

$$|(f'(u)\partial_t u, \partial_t u)| \le \|f'(u)\|_{L^3}\|\partial_t u\|\|\partial_t u\|_{L^6} \le \\ \le C(1 + \|u\|_{L^{12}})^4\|\partial_t u\|\|\nabla_x \partial_t u\|_{L^6} \le H_u(t)\|\partial_t u\|^2 + C\|\partial_t \nabla_x u\|^2 \quad (2.29)$$

Moreover, due to the dissipative estimate (2.8),the smoothing property (2.26), the interpolation inequality (2.13) and growth restriction (1.2), for $0 < t \le 1$, we have

$$t\|f(u(t))\|^2 \le Ct(1 + \|u\|_{L^{10}})^{10} \le \\ \le C_1(1 + \|u\|_{H^1})^8[t\|\Delta_x u(t)\|^2] \le Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon}) + Q(\|g\|). \quad (2.30)$$

Integrating now the equation (2.28) over $[0, t]$, $t \le 1$ and using the estimates (2.29) and (2.30) together with the controls (2.12) and (2.26), we infer

$$t\|\partial_t u(t)\|^2 + \int_0^t \varepsilon s\|\partial_t^2 u(s)\|_{-1}^2\,ds \le \\ \le \int_0^t H_u(s)s\|\partial_t u(s)\|^2\,ds + Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon}^2) + Q(\|g\|), \quad (2.31)$$

where the function $H_u(t)$ satisfies the inequality (2.24) and $t \le 1$. Applying finally the Gronwall's inequality to (2.31), we get

$$t\|\partial_t u(t)\|^2 + \varepsilon\int_0^1 s\|\partial_t^2 u(s)\|_{-1}^2\,ds \le Q(\|\xi_u(0)\|_{\mathcal{E}_\varepsilon}) + Q(\|g\|),\ t \in (0, 1] \quad (2.32)$$

which together with the dissipative bounds (2.8) give the required estimate (2.27) and finishes the proof of the lemma. ☐

Remark 2.5. We see that equation (1.1) possesses smoothing estimates (2.26) and (2.27) which are typical for second order parabolic equations. This analogy is not surprising since this equation can be easily transformed to a semi-linear second order parabolic system. Indeed, introducing a new variable $v(t) := -P\partial_t u(t)$, we get

$$\begin{cases} \partial_t u = \Delta_x v, \\ \varepsilon\partial_t v = \gamma\Delta_x v - \alpha v - \Delta_x u + f(u) - g. \end{cases} \quad (2.33)$$

Or rewriting it in a vector form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = A_\varepsilon \Delta_x \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ -\alpha v + f(u) - g \end{pmatrix}, \quad A_\varepsilon := \begin{pmatrix} 0 & ; & 1 \\ -\frac{1}{\varepsilon} & ; & \frac{\gamma}{\varepsilon} \end{pmatrix}. \quad (2.34)$$

Since the eigenvalues $\lambda_{1,2} = \frac{\gamma \pm \sqrt{\gamma^2 - 4\varepsilon}}{2\varepsilon}$ of the matrix A_ε are strictly positive, the obtained system (2.34) is indeed uniformly parabolic. In particular, this means that, in contrast to the case of hyperbolic equations, we have not only the maximal regularity estimates in $L^2(\Omega)$ deduced above, but also the L^p-maximal regularity estimates for every $1 < p < \infty$, see e.g. [11].

2.5. **Attractors and concluding remarks.** Estimates (2.26) and (2.27) show that the R-ball B_R of the space $H^2(\Omega) \times L^2(\Omega)$ will be the absorbing set for the semigroup $S_\varepsilon(t)$ generated by the problem (1.1) if R is large enough. Moreover, this R is uniform with respect to $\varepsilon \to 0$. Due to (2.20), the semigroup $S_\varepsilon(t)$ is Lipschitz continuous in $\mathcal{E}_\varepsilon$ and due to the Lemma 2.3 it is a compact semigroup in $\mathcal{E}_\varepsilon$. Therefore the standard global attractor's existence theorem (see e.g.,[1]) implies existence of a global attractor $\mathcal{A}_\varepsilon$ together with the embedding $\mathcal{A}_\varepsilon \subset B_R$ and finishes the proof of Theorem 1.2.

Remark 2.6. It follows from eqref1.ene that the problem (1.1) possesses a Lyapunov functional

$$L(u) := \varepsilon \|\partial_t u(t)\|_{-1}^2 + \|u(t)\|_1^2 + (F(u(t)), 1) - 2(g, u).$$

Hence the global attractor $\mathcal{A}_\varepsilon$ consists of the finitely may stationary solutions of the problem (1.1) and trajectories joining them (see for instance [10]).

Remark 2.7. By using the concavity method (see [12],[9]) one can show that if the nonlinear term f satisfies the conditions

$$kF(u) - f(u)u \le C, \quad \forall u \in \mathbb{R},$$

with some $k > 2$ and $C \ge 0$, then for a wide class of initial data the corresponding solutions of the problem (1.1) blow up in a finite time.

The paper was completed in 2008 and was not published, because we learned that the result on exitence of an exponential attractor for seingularly perturbed viscous cahn-Hilliard equation was had been published in [A] :

[A] A.Bonfoh, Existence and continuity of uniform exponential attractors for a singular perturbation of a generalized Cahn-Hilliard equation, Asymptot. Anal.**43** (2005)233-247.

References

[1] A.V. Babin, M.I. Vishik, *Attractors of evolution equations*, North-HollandAmsterdam1992

[2] V. Belleri, V. Pata, Attractors for semilinear strongly damped wave equation on $\mathbb{R}^3$, Discrete Contin. Dynam. Systems,(7)(2001)719–735.

[3] A. Eden, V.K. Kalantarov, Finite dimensional attractors for a class of semilinear wave equations, Turkish Journal of mathematics, (20) 3(1996)425-450.

[4] A. Eden, A. Milani, B. Nicolaenko, Local exponential attractors for models of phase change for compressible gas dynamics. Nonlinearity (6) 1 (1993) 93–117

[5] P. Galenko, Phase-field models with relaxation of the diffusion flux in nonequilibrium solidification of a binary system, Phys. Lett. A (287) (2001)190-197.

[6] S.Gatti, M. Grasselli, V. Pata , A. Miranville, Hyperbolic relaxation of the viscous Cahn-Hilliard equation in 3-D, Mathematical Models and Methods in Applied Sciences, (15) 2 (2005)165-198

[7] K.-H. Hoffmann, P.Rybka, On convergence of solutions to the equation of viscoelasticity with capillarity. Comm. Partial Differential Equations (25) 9-10 (2000) 1845–1890.
[8] K.-H. Hoffmann, P.Rybka, Analyticity of the nonlinear term forces convergence of solutions to two equations of continuum mechanics. Nonlinear Anal, 50 (2002), no. 3, Ser. A: Theory Methods, 409–424.
[9] V.K. Kalantarov, Blow-up theorems for second order nonlinear evolutionary equations, in O.Boratav, A.Eden and A.Erzan .(Eds.) "Turbulence modeling and vortex dynamics" Lecture Not. in Physics, Springer Verlag , 1997, pp.169-181.
[10] O.A. Ladyzhenskaya , *Attractors for semigroups and evolution equations*, Cambridge University, 1991.
[11] O.A. Ladyzhenskaya, V.A. Solonnikov, N.N. Uraltseva, *Linear and quasilinear equations of parabolic type*, Nauka, 1967.
[12] H.A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations. SIAM J. Math. Anal.(5) (1974)138–146.
[13] B. Nicolaenko, W. Qian, Inertial manifolds for nonlinear viscoelasticity equations, Nonlinearity (11) 4 (1998) 1075–1093.
[14] P.Rybka, K.-H. Hoffmann, Convergence of solutions to the equation of quasi-static approximation of viscoelasticity with capillarity, J. Math. Anal. Appl. (226) 1 (1998) 61–81.
[15] R. Temam, it Infinite-dimensional dynamical systems in mechanics and physics, SpringerNew York1988
[16] S. Zheng, A. Milani, Exponential attractors and inertial manifolds for singular perturbations of the Cahn-Hilliard equations. Nonlinear Anal. (57)5-6(2004)843–877
[17] S. Zheng, A. Milani, Global attractors for singular perturbations of the Cahn-Hilliard equations. J. Differential Equations (209) 1 (2005) 101–139.

(V.K. Kalantarov) Department of Mathematics,
Koç University Istanbul, Turkey

(S. Zelik) Department of mathematics,
University of Surrey, Guildford, GU2 7XH UK