Challenging Problems 1.

1. Let $f(t,y)$ be a continuous function on $[0,\infty)\times\mathbb{R}$ and satisfies the Lipchitz conditions with respect to the second variable, i.e. there exists $L>0$ such that

$$|f(t,u)-f(t,v)|\le L|u-v|,\quad \forall t\in[0,T],\quad u,v\in\mathbb{R}.$$

Suppose that $y_1(t)$ is a solution of the equation

$$y'(t)=f(t,y(t)),\quad t\in[0,T] \tag{1}$$

that satisfies the initial condition

$$y_1(0)=A_1 \tag{2}$$

and $y_2(t)$ is a solution of the equation (1) that satisfies the initial condition

$$y_2(0)=A_2 \qquad (3).$$

Show that if $|A_1-A_2|$ is small enough thenthe difference of solutions of the problems (1), (2) and (1) (3) is small enough on the interval $[0,T]$.

2. Consider the Cauchy problem

$$\begin{cases} y''(t)+by'(t)+cy(t)=h(t),\quad t>0,\\ y(0)=y_0,\quad y'(0)=y_1, \end{cases} \qquad (B)$$

where $b>0, c>0$ are given positive numbers and $h(t)$ is a given function that is continuous and bounded on $[0,\infty)$.

(i) Show that there exists $M_0>0$ such that for each initial data y_0,y_1 the solution of the problem (B) satisifies

$$[y'(t)]^2+[y(t)]^2\le M_0,\quad \forall t>T_0,$$

where $T_0>0$ depeds on y_0,y_1.

(ii) Show that if $h(t)\to 0$, as $t\to\infty$, then the solution of the problem (B) tends to zero as $t\to\infty$.