EXERCISES 13.1

In Exercises 1–17, find and classify the critical points of the given functions.

1. $f(x,y) = x^2 + 2y^2 - 4x + 4y$

2. $f(x,y) = xy - x + y$

3. $f(x,y) = x^3 + y^3 - 3xy$

4. $f(x,y) = x^4 + y^4 - 4xy$

5. $f(x,y) = \dfrac{x}{y} + \dfrac{8}{x} - y$

6. $f(x,y) = \cos(x+y)$

7. $f(x,y) = x \sin y$

8. $f(x,y) = \cos x + \cos y$

9. $f(x,y) = x^2 y\, e^{-(x^2+y^2)}$

10. $f(x,y) = \dfrac{xy}{2 + x^4 + y^4}$

11. $f(x,y) = x\, e^{-x^3+y^3}$

12. $f(x,y) = \dfrac{x^2}{x^2+y^2}$

13. $f(x,y) = \dfrac{xy}{x^2+y^2}$

14. $f(x,y) = \dfrac{1}{1 - x + y + x^2 + y^2}$

15. $f(x,y) = \left(1 + \dfrac{1}{x}\right)\left(1 + \dfrac{1}{y}\right)\left(\dfrac{1}{x} + \dfrac{1}{y}\right)$

! **16.** $f(x,y,z) = xyz - x^2 - y^2 - z^2$

! **17.** $f(x,y,z) = xy + x^2z - x^2 - y - z^2$

! **18.** Show that $f(x,y,z) = 4xyz - x^4 - y^4 - z^4$ has a local maximum value at the point $(1,1,1)$.

19. Find the maximum and minimum values of

$$f(x,y) = xy\, e^{-x^2-y^4}.$$

20. Find the maximum and minimum values of

$$f(x,y) = \frac{x}{(1+x^2+y^2)}.$$

! **21.** Find the maximum and minimum values of $f(x,y,z) = xyz\, e^{-x^2-y^2-z^2}$. How do you know that such extreme values exist?

22. Find the minimum value of $f(x,y) = x + 8y + \dfrac{1}{xy}$ in the first quadrant $x > 0$, $y > 0$. How do you know that a minimum exists?

23. Postal regulations require that the sum of the height and girth (horizontal perimeter) of a package should not exceed L units. Find the largest volume of a rectangular box that can satisfy this requirement.

24. The material used to make the bottom of a rectangular box is twice as expensive per unit area as the material used to make the top or side walls. Find the dimensions of the box of given volume V for which the cost of materials is minimum.

25. Find the volume of the largest rectangular box (with faces parallel to the coordinate planes) that can be inscribed inside the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

26. Find the three positive numbers a, b, and c, whose sum is 30 and for which the expression ab^2c^3 is maximum.

27. Find the critical points of the function $z = g(x, y)$ that satisfies the equation $e^{2zx-x^2} - 3e^{2zy+y^2} = 2$.

28. Classify the critical points of the function g in the previous exercise.

29. Let $f(x, y) = (y - x^2)(y - 3x^2)$. Show that the origin is a critical point of f and that the restriction of f to every straight line through the origin has a local minimum value at the origin. (That is, show that $f(x, kx)$ has a local minimum value at $x = 0$ for every k and that $f(0, y)$ has a local minimum value at $y = 0$.) Does $f(x, y)$ have a local minimum value at the origin? What happens to f on the curve $y = 2x^2$? What does the second derivative test say about this situation?

30. Verify by completing the square (i.e., without appealing to Theorem 8 of Section 10.7) that the quadratic form

$$Q(u, v) = (x, y)\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = Au^2+2Buv+Cv^2$$

is positive definite if $A > 0$ and $\begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0$, negative definite if $A < 0$ and $\begin{vmatrix} A & B \\ B & C \end{vmatrix} > 0$, and indefinite if $\begin{vmatrix} A & B \\ B & C \end{vmatrix} < 0$. This gives independent confirmation of the assertion in the remark preceding Example 7.

31. State and prove (using square completion arguments rather than appealing to Theorem 8 of Section 10.7) a result analogous to that of Exercise 30 for a quadratic form $Q(u, v, w)$ involving three variables. What are the implications of this for a critical point (a, b, c) of a function $f(x, y, z)$ all of whose second partial derivatives are known at (a, b, c)?

the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

26. Find the three positive numbers a, b, and c, whose sum is 30 and for which the expression ab^2c^3 is maximum.

27. Find the critical points of the function $z=g(x,y)$ that satisfies the equation $e^{2zx-x^2}-3e^{2zy+y^2}=2$.

28. Classify the critical points of the function g in the previous exercise.

29. Let $f(x,y)=(y-x^2)(y-3x^2)$. Show that the origin is a critical point of f and that the restriction of f to every straight line through the origin has a local minimum value at the origin. (That is, show that $f(x,kx)$ has a local minimum value at $x=0$ for every k and that $f(0,y)$ has a local minimum value at $y=0$.) Does $f(x,y)$ have a local minimum value at the origin? What happens to f on the curve $y=2x^2$? What does the second derivative test say about this situation?

30. Verify by completing the square (i.e., without appealing to Theorem 8 of Section 10.7) that the quadratic form

$$Q(u,v)=(x,y)\begin{pmatrix}A & B\\ B & C\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=Au^2+2Buv+Cv^2$$

is positive definite if $A>0$ and $\begin{vmatrix}A & B\\ B & C\end{vmatrix}>0$, negative definite if $A<0$ and $\begin{vmatrix}A & B\\ B & C\end{vmatrix}>0$, and indefinite if $\begin{vmatrix}A & B\\ B & C\end{vmatrix}<0$. This gives independent confirmation of the assertion in the remark preceding Example 7.

31. State and prove (using square completion arguments rather than appealing to Theorem 8 of Section 10.7) a result analogous to that of Exercise 30 for a quadratic form $Q(u,v,w)$ involving three variables. What are the implications of this for a critical point (a,b,c) of a function $f(x,y,z)$ all of whose second partial derivatives are known at (a,b,c)?

EXERCISES 13.2

1. Find the maximum and minimum values of $f(x, y) = x - x^2 + y^2$ on the rectangle $0 \le x \le 2$, $0 \le y \le 1$.

2. Find the maximum and minimum values of $f(x, y) = xy - 2x$ on the rectangle $-1 \le x \le 1, 0 \le y \le 1$.

3. Find the maximum and minimum values of $f(x, y) = xy - y^2$ on the disk $x^2 + y^2 \le 1$.

4. Find the maximum and minimum values of $f(x, y) = x + 2y$ on the disk $x^2 + y^2 \le 1$.

5. Find the maximum and minimum values of $f(x, y) = xy - x^3y^2$ over the square $0 \le x \le 1, 0 \le y \le 1$.

6. Find the maximum and minimum values of $f(x, y) = xy(1 - x - y)$ over the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

7. Find the maximum and minimum values of $f(x, y) = \sin x \cos y$ on the closed triangular region bounded by the coordinate axes and the line $x + y = 2\pi$.

8. Find the maximum value of $f(x, y) = \sin x \sin y \sin(x + y)$ over the triangle bounded by the coordinate axes and the line $x + y = \pi$.

9. The temperature at all points in the disk $x^2 + y^2 \le 1$ is given by $T = (x + y)\, e^{-x^2-y^2}$. Find the maximum and minimum temperatures at points of the disk.

10. Find the maximum and minimum values of

$$f(x, y) = \frac{x - y}{1 + x^2 + y^2}$$

on the upper half-plane $y \ge 0$.

11. Find the maximum and minimum values of $xy^2 + yz^2$ over the ball $x^2 + y^2 + z^2 \le 1$.

12. Find the maximum and minimum values of $xz + yz$ over the ball $x^2 + y^2 + z^2 \le 1$.

13. Consider the function $f(x, y) = xy\, e^{-xy}$ with domain the first quadrant: $x \ge 0,\ y \ge 0$. Show that $\lim_{x\to\infty} f(x, kx) = 0$. Does f have a limit as (x, y) recedes arbitrarily far from the origin in the first quadrant? Does f have a maximum value in the first quadrant?

14. Repeat Exercise 13 for the function $f(x, y) = xy^2\, e^{-xy}$.

15. In a certain community there are two breweries in competition, so that sales of each negatively affect the profits of the other. If brewery A produces x litres of beer per month and brewery B produces y litres per month, then brewery A's monthly profit $\$P$ and brewery B's monthly profit $\$Q$ are assumed to be

$$P = 2x - \frac{2x^2 + y^2}{10^6},$$
$$Q = 2y - \frac{4y^2 + x^2}{2 \times 10^6}.$$

Find the sum of the profits of the two breweries if each brewery independently sets its own production level to maximize its own profit and assumes its competitor does likewise. Find the sum of the profits if the two breweries cooperate to determine their respective productions to maximize that sum.

16. Equal angle bends are made at equal distances from the two ends of a 100 m long straight length of fence so the resulting three-segment fence can be placed along an existing wall to make an enclosure of trapezoidal shape. What is the largest possible area for such an enclosure?

17. Maximize $Q(x, y) = 2x + 3y$ subject to the constraints $x \geq 0$, $y \geq 0$, $y \leq 5$, $x + 2y \leq 12$, and $4x + y \leq 12$.

18. Minimize $F(x, y, z) = 2x + 3y + 4z$ subject to the constraints $x \geq 0$, $y \geq 0$, $z \geq 0$, $x + y \geq 2$, $y + z \geq 2$, and $x + z \geq 2$.

19. A textile manufacturer produces two grades of fabric containing wool, cotton, and polyester. The deluxe grade has composition (by weight) 20% wool, 50% cotton, and 30% polyester, and it sells for \$3 per kilogram. The standard grade has composition 10% wool, 40% cotton, and 50% polyester, and it sells for \$2 per kilogram. If he has in stock 2,000 kg of wool and 6,000 kg each of cotton and polyester, how many kilograms of fabric of each grade should he manufacture to maximize his revenue?

20. A 10-hectare parcel of land is zoned for building densities of 6 detached houses per hectare, 8 duplex units per hectare, or 12 apartments per hectare. The developer who owns the land can make a profit of \$40,000 per house, \$20,000 per duplex unit, and \$16,000 per apartment that he builds. Municipal bylaws require him to build at least as many apartments as the total of houses and duplex units. How many of each type of dwelling should he build to maximize his profit?

EXERCISES 13.3

1. Use the method of Lagrange multipliers to maximize x^3y^5 subject to the constraint $x + y = 8$.

2. Find the shortest distance from the point $(3, 0)$ to the parabola $y = x^2$

(a) by reducing to an unconstrained problem in one variable, and

(b) by using the method of Lagrange multipliers.

3. Find the distance from the origin to the plane $x + 2y + 2z = 3$

(a) using a geometric argument (no calculus),

(b) by reducing the problem to an unconstrained problem in two variables, and

(c) using the method of Lagrange multipliers.

4. Find the maximum and minimum values of the function $f(x, y, z) = x + y - z$ over the sphere $x^2 + y^2 + z^2 = 1$.

5. Use the Lagrange multiplier method to find the greatest and least distances from the point $(2, 1, -2)$ to the sphere with equation $x^2 + y^2 + z^2 = 1$. (Of course, the answer could be obtained more easily using a simple geometric argument.)

6. Find the shortest distance from the origin to the surface $xyz^2 = 2$.

7. Find a, b, and c so that the volume $V = 4\pi abc/3$ of an ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ passing through the point $(1, 2, 1)$ is as small as possible.

8. Find the ends of the major and minor axes of the ellipse $3x^2 + 2xy + 3y^2 = 16$.

9. Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 12$.

10. Find the maximum and minimum values of $x + 2y - 3z$ over the ellipsoid $x^2 + 4y^2 + 9z^2 \le 108$.

11. Find the distance from the origin to the surface $xy^2z^4 = 32$.

12. Find the maximum value of $\sum_{i=1}^{n} x_i$ on the n-sphere $\sum_{i=1}^{n} x_i^2 = 1$ in $\mathbb{R}^n$.

13. Find the maximum and minimum values of the function $f(x, y, z) = x$ over the curve of intersection of the plane $z = x + y$ and the ellipsoid $x^2 + 2y^2 + 2z^2 = 8$.

14. Find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ on the ellipse formed by the intersection of the cone $z^2 = x^2 + y^2$ and the plane $x - 2z = 3$.

15. Find the maximum and minimum values of $f(x, y, z) = 4 - z$ on the ellipse formed by the intersection of the cylinder $x^2 + y^2 = 8$ and the plane $x + y + z = 1$.

16. Find the maximum and minimum values of $f(x, y, z) = x + y^2z$ subject to the constraints $y^2 + z^2 = 2$ and $z = x$.

! **17.** Use the method of Lagrange multipliers to find the shortest distance between the straight lines $x = y = z$ and $x = -y$, $z = 2$. (There are, of course, much easier ways to get the answer. This is an object lesson in the folly of shooting sparrows with cannons.)

18. Find the most economical shape of a rectangular box with no top.

19. Find the maximum volume of a rectangular box with faces parallel to the coordinate planes if one corner is at the origin and the diagonally opposite corner lies on the plane $4x + 2y + z = 2$.

20. Find the maximum volume of a rectangular box with faces parallel to the coordinate planes if one corner is at the origin and the diagonally opposite corner is on the first octant part of the surface $xy + 2yz + 3xz = 18$.

21. A rectangular box having no top and having a prescribed volume V m^3 is to be constructed using two different materials. The material used for the bottom and front of the box is five times as costly (per square metre) as the material used for the back and the other two sides. What should be the dimensions of the box to minimize the cost of materials?

! **22.** Find the maximum and minimum values of $xy + z^2$ on the ball $x^2 + y^2 + z^2 \le 1$. Use Lagrange multipliers to treat the boundary case.

! **23.** Repeat Exercise 22 but handle the boundary case by parametrizing the sphere $x^2 + y^2 + z^2 = 1$ using

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi,$$

where $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

? **24.** If α, β, and γ are the angles of a triangle, show that

$$\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \le \frac{1}{8}.$$

For what triangles does equality occur?

! **25.** Suppose that f and g have continuous first partial derivatives throughout the xy-plane, and suppose that $g_2(a, b) \ne 0$. This implies that the equation $g(x, y) = g(a, b)$ defines y implicitly as a function of x near the point (a, b). Use the Chain Rule to show that if $f(x, y)$ has a local extreme value at (a, b) subject to the constraint $g(x, y) = g(a, b)$, then for some number λ the point (a, b, λ) is a critical point of the function

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y).$$

This constitutes a more formal justification of the method of Lagrange multipliers in this case.

? **26.** What is the shortest distance from the point $(0, -1)$ to the curve $y = \sqrt{1 - x^2}$? Can this problem be solved by the Lagrange multiplier method? Why?

? **27.** Example 3 showed that the method of Lagrange multipliers might fail to find a point that extremizes $f(x, y)$ subject to the constraint $g(x, y) = 0$ if $\nabla g = \mathbf{0}$ at the extremizing point. Can the method also fail if $\nabla f = \mathbf{0}$ at the extremizing point? Why?