EXERCISES 12.7

In Exercises 1–6, find:

(a) the gradient of the given function at the point indicated,

(b) an equation of the plane tangent to the graph of the given function at the point whose x and y coordinates are given, and

(c) an equation of the straight line tangent, at the given point, to the level curve of the given function passing through that point.

1. $f(x, y) = x^2 - y^2$ at $(2, -1)$

2. $f(x, y) = \dfrac{x - y}{x + y}$ at $(1, 1)$

3. $f(x, y) = \dfrac{x}{x^2 + y^2}$ at $(1, 2)$

4. $f(x, y) = e^{xy}$ at $(2, 0)$

5. $f(x, y) = \ln(x^2 + y^2)$ at $(1, -2)$

6. $f(x, y) = \sqrt{1 + xy^2}$ at $(2, -2)$

In Exercises 7–9, find an equation of the tangent plane to the level surface of the given function that passes through the given point.

7. $f(x, y, z) = x^2y + y^2z + z^2x$ at $(1, -1, 1)$

8 $f(x, y, z) = \cos(x + 2y + 3z)$ at $\left(\dfrac{\pi}{2}, \pi, \pi\right)$

9. $f(x, y, z) = y\, e^{-x^2} \sin z$ at $(0, 1, \pi/3)$

In Exercises 10–13, find the rate of change of the given function at the given point in the specified direction.

10. $f(x, y) = 3x - 4y$ at $(0, 2)$ in the direction of the vector $-2\mathbf{i}$

11. $f(x, y) = x^2y$ at $(-1, -1)$ in the direction of the vector $\mathbf{i} + 2\mathbf{j}$

12. $f(x, y) = \dfrac{x}{1 + y}$ at $(0, 0)$ in the direction of the vector $\mathbf{i} - \mathbf{j}$

13. $f(x, y) = x^2 + y^2$ at $(1, -2)$ in the direction making a (positive) angle of $60°$ with the positive x-axis

14. Let $f(x, y) = \ln |\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$. Show that $\nabla f = \dfrac{\mathbf{r}}{|\mathbf{r}|^2}$.

15. Let $f(x, y, z) = |\mathbf{r}|^{-n}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\nabla f = \dfrac{-n\mathbf{r}}{|\mathbf{r}|^{n+2}}$.

16. Show that, in terms of polar coordinates (r, θ) (where $x = r\cos\theta$ and $y = r\sin\theta$), the gradient of a function $f(r, \theta)$ is given by

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\boldsymbol{\theta}},$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, and $\hat{\boldsymbol{\theta}}$ is a unit vector at right angles to $\hat{\mathbf{r}}$ in the direction of increasing θ.

17. In what directions at the point $(2, 0)$ does the function $f(x, y) = xy$ have rate of change -1? Are there directions in which the rate is -3? How about -2?

18. In what directions at the point (a, b, c) does the function $f(x, y, z) = x^2 + y^2 - z^2$ increase at half of its maximal rate at that point?

19. Find $\nabla f(a, b)$ for the differentiable function $f(x, y)$ given the directional derivatives

$$D_{(\mathbf{i}+\mathbf{j})/\sqrt{2}} f(a, b) = 3\sqrt{2} \text{ and } D_{(3\mathbf{i}-4\mathbf{j})/5} f(a, b) = 5.$$

20. If $f(x, y)$ is differentiable at (a, b), what condition should angles ϕ_1 and ϕ_2 satisfy in order that the gradient $\nabla f(a, b)$ can be determined from the values of the directional derivatives $D_{\phi_1} f(a, b)$ and $D_{\phi_2} f(a, b)$?

21. The temperature $T(x, y)$ at points of the xy-plane is given by $T(x, y) = x^2 - 2y^2$.

(a) Draw a contour diagram for T showing some isotherms (curves of constant temperature).

(b) In what direction should an ant at position $(2, -1)$ move if it wishes to cool off as quickly as possible?

(c) If the ant moves in that direction at speed k (units distance per unit time), at what rate does it experience the decrease of temperature?

(d) At what rate would the ant experience the decrease of temperature if it moved from $(2, -1)$ at speed k in the direction of the vector $-\mathbf{i} - 2\mathbf{j}$?

(e) Along what curve through $(2, -1)$ should the ant move in order to continue to experience maximum rate of cooling?

22. Find an equation of the curve in the xy-plane that passes through the point $(1, 1)$ and intersects all level curves of the function $f(x, y) = x^4 + y^2$ at right angles.

23. Find an equation of the curve in the xy-plane that passes through the point $(2, -1)$ and that intersects every curve with equation of the form $x^2y^3 = K$ at right angles.

24. Find the second directional derivative of $e^{-x^2-y^2}$ at the point $(a, b) \neq (0, 0)$ in the direction directly away from the origin.

25. Find the second directional derivative of $f(x, y, z) = xyz$ at $(2, 3, 1)$ in the direction of the vector $\mathbf{i} - \mathbf{j} - \mathbf{k}$.

26. Find a vector tangent to the curve of intersection of the two cylinders $x^2 + y^2 = 2$ and $y^2 + z^2 = 2$ at the point $(1, -1, 1)$.

27. Repeat Exercise 26 for the surfaces $x + y + z = 6$ and $x^2 + y^2 + z^2 = 14$ and the point $(1, 2, 3)$.

28. The temperature in 3-space is given by

$$T(x, y, z) = x^2 - y^2 + z^2 + xz^2.$$

At time $t = 0$ a fly passes through the point $(1, 1, 2)$, flying along the curve of intersection of the surfaces $z = 3x^2 - y^2$ and $2x^2 + 2y^2 - z^2 = 0$. If the fly's speed is 7, what rate of temperature change does it experience at $t = 0$?

29. State and prove a version of Theorem 6 for a function of three variables.

30. What is the level surface of $f(x, y, z) = \cos(x + 2y + 3z)$ that passes through (π, π, π)? What is the tangent plane to that level surface at that point? (Compare this exercise with Exercise 8 above.)

31. If $\nabla f(x, y) = 0$ throughout the disk $x^2 + y^2 < r^2$, prove that $f(x, y)$ is constant throughout the disk.

32. Theorem 6 implies that the level curve of $f(x, y)$ passing through (a, b) is smooth (has a tangent line) at (a, b) provided f is differentiable at (a, b) and satisfies $\nabla f(a, b) \neq \mathbf{0}$. Show that the level curve need not be smooth at (a, b) if $\nabla f(a, b) = \mathbf{0}$. (*Hint:* Consider $f(x, y) = y^3 - x^2$ at $(0, 0)$.)

EXERCISES 12.8

In Exercises 1–12, calculate the indicated derivative from the given equation(s). What condition on the variables will guarantee the existence of a solution that has the indicated derivative? Assume that any general functions F, G, and H have continuous first partial derivatives.

1. $\dfrac{dx}{dy}$ if $xy^3 + x^4 y = 2$

2. $\dfrac{\partial x}{\partial y}$ if $xy^3 = y - z$

3. $\dfrac{\partial z}{\partial y}$ if $z^2 + xy^3 = \dfrac{xz}{y}$

4. $\dfrac{\partial y}{\partial z}$ if $e^{yz} - x^2 z \ln y = \pi$

5. $\dfrac{\partial x}{\partial w}$ if $x^2y^2 + y^2z^2 + z^2t^2 + t^2w^2 - xw = 0$

6. $\dfrac{dy}{dx}$ if $F(x, y, x^2 - y^2) = 0$

7. $\dfrac{\partial u}{\partial x}$ if $G(x, y, z, u, v) = 0$

8. $\dfrac{\partial z}{\partial x}$ if $F(x^2 - z^2, y^2 + xz) = 0$

9. $\dfrac{\partial w}{\partial t}$ if $H(u^2 w, v^2 t, wt) = 0$

10. $\left(\dfrac{\partial y}{\partial x}\right)_u$ if $xyuv = 1$ and $x + y + u + v = 0$

11. $\left(\dfrac{\partial x}{\partial y}\right)_z$ if $x^2 + y^2 + z^2 + w^2 = 1$, and $x + 2y + 3z + 4w = 2$

12. $\dfrac{du}{dx}$ if $x^2 y + y^2 u - u^3 = 0$ and $x^2 + yu = 1$

13. If $x = u^3 + v^3$ and $y = uv - v^2$ are solved for u and v in terms of x and y, evaluate

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial y}, \quad \text{and} \quad \frac{\partial(u, v)}{\partial(x, y)}$$

at the point where $u = 1$ and $v = 1$.

14. Near what points (r, s) can the transformation

$$x = r^2 + 2s, \qquad y = s^2 - 2r$$

be solved for r and s as functions of x and y? Calculate the values of the first partial derivatives of the solution at the origin.

15. Evaluate the Jacobian $\partial(x, y)/\partial(r, \theta)$ for the transformation to polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$. Near what points (r, θ) is the transformation one-to-one and therefore invertible to give r and θ as functions of x and y?

16. Evaluate the Jacobian $\partial(x, y, z)/\partial(R, \phi, \theta)$, where

$$x = R\sin\phi\cos\theta, \quad y = R\sin\phi\sin\theta, \quad \text{and } z = R\cos\phi.$$

This is the transformation from Cartesian to spherical coordinates in 3-space that we discussed in Section 10.6. Near what points is the transformation one-to-one and hence invertible to give R, ϕ, and θ as functions of x, y, and z?

17. Show that the equations

$$\begin{cases} xy^2 + zu + v^2 = 3 \\ x^3 z + 2y - uv = 2 \\ xu + yv - xyz = 1 \end{cases}$$

can be solved for x, y, and z as functions of u and v near the point P_0 where $(x, y, z, u, v) = (1, 1, 1, 1, 1)$, and find $(\partial y/\partial u)_v$ at $(u, v) = (1, 1)$.

18. Show that the equations $\begin{cases} xe^y + uz - \cos v = 2 \\ u\cos y + x^2 v - yz^2 = 1 \end{cases}$ can be solved for u and v as functions of x, y, and z near the point P_0 where $(x, y, z) = (2, 0, 1)$ and $(u, v) = (1, 0)$, and find $(\partial u/\partial z)_{x,y}$ at $(x, y, z) = (2, 0, 1)$.

19. Find dx/dy from the system

$$F(x, y, z, w) = 0, \quad G(x, y, z, w) = 0, \quad H(x, y, z, w) = 0.$$

20. Given the system

$$\begin{aligned} F(x, y, z, u, v) &= 0 \\ G(x, y, z, u, v) &= 0 \\ H(x, y, z, u, v) &= 0, \end{aligned}$$

how many possible interpretations are there for $\partial x/\partial y$? Evaluate them.

21. Given the system

$$\begin{aligned} F(x_1, x_2, \ldots, x_8) &= 0 \\ G(x_1, x_2, \ldots, x_8) &= 0 \\ H(x_1, x_2, \ldots, x_8) &= 0, \end{aligned}$$

how many possible interpretations are there for the partial $\dfrac{\partial x_1}{\partial x_2}$? Evaluate $\left(\dfrac{\partial x_1}{\partial x_2}\right)_{x_4, x_6, x_7, x_8}$.

22. If $F(x, y, z) = 0$ determines z as a function of x and y, calculate $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\partial y$, and $\partial^2 z/\partial y^2$ in terms of the partial derivatives of F.

23. If $x = u + v$, $y = uv$, and $z = u^2 + v^2$ define z as a function of x and y, find $\partial z/\partial x$, $\partial z/\partial y$, and $\partial^2 z/\partial x\partial y$.

24. A certain gas satisfies the law $pV = T - \dfrac{4p}{T^2}$, where p = pressure, V = volume, and T = temperature.

(a) Calculate $\partial T/\partial p$ and $\partial T/\partial V$ at the point where $p = V = 1$ and $T = 2$.

(b) If measurements of p and V yield the values $p = 1 \pm 0.001$ and $V = 1 \pm 0.002$, find the approximate maximum error in the calculated value $T = 2$.

25. If $F(x, y, z) = 0$, show that $\left(\dfrac{\partial x}{\partial y}\right)_z \left(\dfrac{\partial y}{\partial z}\right)_x \left(\dfrac{\partial z}{\partial x}\right)_y = -1.$

Derive analogous results for $F(x, y, z, u) = 0$ and for $F(x, y, z, u, v) = 0$. What is the general case?

26. If the equations $F(x, y, u, v) = 0$ and $G(x, y, u, v) = 0$ are solved for x and y as functions of u and v, show that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(F, G)}{\partial(u, v)} \bigg/ \frac{\partial(F, G)}{\partial(x, y)}.$$