EXERCISES 12.4

In Exercises 1–6, find all the second partial derivatives of the given function.

1. $z = x^2(1 + y^2)$

2. $f(x, y) = x^2 + y^2$

3. $w = x^3y^3z^3$

4. $z = \sqrt{3x^2 + y^2}$

5. $z = x\,e^y - y\,e^x$

6. $f(x, y) = \ln\big(1 + \sin(xy)\big)$

7. How many mixed partial derivatives of order 3 can a function of three variables have? If they are all continuous, how many different values can they have at one point? Find the mixed partials of order 3 for $f(x, y, z) = x\,e^{xy}\cos(xz)$ that involve two differentiations with respect to z and one with respect to x.

Show that the functions in Exercises 8–12 are harmonic in the plane regions indicated.

8. $f(x, y) = A(x^2 - y^2) + Bxy$ in the whole plane (A and B are constants.)

9. $f(x, y) = 3x^2y - y^3$ in the whole plane (Can you think of another polynomial of degree 3 in x and y that is also harmonic?)

10. $f(x, y) = \dfrac{x}{x^2 + y^2}$ everywhere except at the origin

11. $f(x, y) = \ln(x^2 + y^2)$ everywhere except at the origin

12. $\tan^{-1}(y/x)$ except at points on the y-axis

13. Show that $w = e^{3x+4y}\sin(5z)$ is harmonic in all of $\mathbb{R}^3$, that is, it satisfies everywhere the 3-dimensional Laplace equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0.$$

14. Assume that $f(x, y)$ is harmonic in the xy-plane. Show that each of the functions $z\,f(x, y)$, $x\,f(y, z)$, and $y\,f(z, x)$ is harmonic in the whole of $\mathbb{R}^3$. What condition should the constants a, b, and c satisfy to ensure that $f(ax + by, cz)$ is harmonic in $\mathbb{R}^3$?

15. Let the functions $u(x, y)$ and $v(x, y)$ have continuous second partial derivatives and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Show that u and v are both harmonic.

16. Let $F(x, y) = \begin{cases} \dfrac{2xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}$

Calculate $F_1(x, y)$, $F_2(x, y)$, $F_{12}(x, y)$, and $F_{21}(x, y)$ at points $(x, y) \neq (0, 0)$. Also calculate these derivatives at $(0, 0)$. Observe that $F_{21}(0, 0) = 2$ and $F_{12}(0, 0) = -2$. Does this result contradict Theorem 1? Explain why.

The heat (diffusion) equation

17. Show that the function $u(x, t) = t^{-1/2}e^{-x^2/4t}$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

This equation is called the **one-dimensional heat equation** because it models heat diffusion in an insulated rod (with $u(x, t)$ representing the temperature at position x at time t) and other similar phenomena.

18. Show that the function $u(x, y, t) = t^{-1}e^{-(x^2+y^2)/4t}$ satisfies the **two-dimensional heat equation**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

19. By comparing the results of Exercises 17 and 18, guess a solution to the **three-dimensional heat equation**

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}.$$

Verify your guess. (If you're feeling lazy, use Maple.)

Biharmonic functions

A function $u(x, y)$ with continuous partials of fourth order is **biharmonic** if $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ is a harmonic function.

20. Show that $u(x, y)$ is biharmonic if and only if it satisfies the biharmonic equation

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0$$

21. Verify that $u(x, y) = x^4 - 3x^2y^2$ is biharmonic.

22. Show that if $u(x, y)$ is harmonic, then $v(x, y) = xu(x, y)$ and $w(x, y) = yu(x, y)$ are biharmonic.

Use the result of Exercise 22 to show that the functions in Exercises 23–25 are biharmonic.

23. $x\,e^x\sin y$

24. $y\ln(x^2 + y^2)$

25. $\dfrac{xy}{x^2 + y^2}$

26. Propose a definition of a biharmonic function of three variables, and prove results analogous to those of Exercises 20 and 22 for biharmonic functions $u(x, y, z)$.

27. Use Maple to verify directly that the function of Exercise 25 is biharmonic.

EXERCISES 12.5

In Exercises 1–4, write appropriate versions of the Chain Rule for the indicated derivatives.

1. $\partial w/\partial t$ if $w = f(x, y, z)$, where $x = g(s, t)$, $y = h(s, t)$, and $z = k(s, t)$

2. $\partial w/\partial t$ if $w = f(x, y, z)$, where $x = g(s)$, $y = h(s, t)$, and $z = k(t)$

3. $\partial z/\partial u$ if $z = g(x, y)$, where $y = f(x)$ and $x = h(u, v)$

4. dw/dt if $w = f(x, y)$, $x = g(r, s)$, $y = h(r, t)$, $r = k(s, t)$, and $s = m(t)$

5. If $w = f(x, y, z)$, where $x = g(y, z)$ and $y = h(z)$, state appropriate versions of the Chain Rule for $\dfrac{dw}{dz}$, $\left(\dfrac{\partial w}{\partial z}\right)_x$, and $\left(\dfrac{\partial w}{\partial z}\right)_{x,y}$.

6. Use two different methods to calculate $\partial u/\partial t$ if $u = \sqrt{x^2 + y^2}$, $x = e^{st}$, and $y = 1 + s^2 \cos t$.

7. Use two different methods to calculate $\partial z/\partial x$ if $z = \tan^{-1}(u/v)$, $u = 2x + y$, and $v = 3x - y$.

8. Use two methods to calculate dz/dt given that $z = txy^2$, $x = t + \ln(y + t^2)$, and $y = e^t$.

In Exercises 9–12, find the indicated derivatives, assuming that the function $f(x, y)$ has continuous first partial derivatives.

9. $\dfrac{\partial}{\partial x} f(2x, 3y)$

10. $\dfrac{\partial}{\partial x} f(2y, 3x)$

11. $\dfrac{\partial}{\partial x} f(y^2, x^2)$

12. $\dfrac{\partial}{\partial y} f\Big(yf(x, t), f(y, t)\Big)$

13. Suppose that the temperature T in a certain liquid varies with depth z and time t according to the formula $T = e^{-t}z$. Find the rate of change of temperature with respect to time at a point that is moving through the liquid so that at time t its depth is $f(t)$. What is this rate if $f(t) = e^t$? What is happening in this case?

14. Suppose the strength E of an electric field in space varies with position (x, y, z) and time t according to the formula $E = f(x, y, z, t)$. Find the rate of change with respect to time of the electric field strength measured by an instrument moving along the helix $x = \sin t$, $y = \cos t$, $z = t$.

In Exercises 15–20, assume that f has continuous partial derivatives of all orders.

15. If $z = f(x, y)$, where $x = 2s + 3t$ and $y = 3s - 2t$, find

(a) $\dfrac{\partial^2 z}{\partial s^2}$, (b) $\dfrac{\partial^2 z}{\partial s \partial t}$, and (c) $\dfrac{\partial^2 z}{\partial t^2}$.

16. If $f(x, y)$ is harmonic, show that $f\left(\dfrac{x}{x^2 + y^2}, \dfrac{-y}{x^2 + y^2}\right)$ is also harmonic.

17. If $x = t \sin s$ and $y = t \cos s$, find $\dfrac{\partial^2}{\partial s \partial t} f(x, y)$.

18. Find $\dfrac{\partial^3}{\partial x \partial y^2} f(2x + 3y, xy)$ in terms of partial derivatives of the function f.

19. Find $\dfrac{\partial^2}{\partial y \partial x} f(y^2, xy, -x^2)$ in terms of partial derivatives of the function f.

20. Find $\dfrac{\partial^3}{\partial t^2 \partial s} f(s^2 - t, s + t^2)$ in terms of partial derivatives of the function f.

21. Suppose that $u(x, y)$ and $v(x, y)$ have continuous second partial derivatives and satisfy the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Suppose also that $f(u, v)$ is a harmonic function of u and v. Show that $f\Big(u(x, y), v(x, y)\Big)$ is a harmonic function of x and y. *Hint:* u and v are harmonic functions by Exercise 15 in Section 12.4.

22. If $r^2 = x^2 + y^2 + z^2$, verify that $u(x, y, z) = 1/r$ is harmonic throughout $\mathbb{R}^3$ except at the origin.

23. If $x = e^s \cos t$, $y = e^s \sin t$, and $z = u(x, y) = v(s, t)$, show that

$$\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = (x^2 + y^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right).$$

24. **(Converting Laplace's equation to polar coordinates)** The transformation to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, implies that $r^2 = x^2 + y^2$ and $\tan\theta = y/x$. Use these equations to show that

$$\frac{\partial r}{\partial x} = \cos\theta \qquad \frac{\partial r}{\partial y} = \sin\theta$$
$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

Use these formulas to help you express $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ in terms of partials of u with respect to r and θ, and hence re-prove the formula for the Laplace differential equation in polar coordinates given in Example 10.

25. If $u(x, y) = r^2 \ln r$, where $r^2 = x^2 + y^2$, verify that u is a biharmonic function by showing that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

26. If $f(x, y)$ is positively homogeneous of degree k and has continuous partial derivatives of second order, show that

$$x^2 f_{11}(x, y) + 2xyf_{12}(x, y) + y^2 f_{22}(x, y) = k(k-1)f(x, y).$$

27. Generalize the result of Exercise 26 to functions of n variables.

28. Generalize the results of Exercises 26 and 27 to expressions involving mth-order partial derivatives of the function f.

Exercises 29–30 revisit Exercise 16 of Section 12.4. Let

$$F(x, y) = \begin{cases} \dfrac{2xy(x^2 - y^2)}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$$

29. (a) Show that $F(x, y) = -F(y, x)$ for all (x, y).

(b) Show that $F_1(x, y) = -F_2(y, x)$ and $F_{12}(x, y) = -F_{21}(y, x)$ for $(x, y) \neq (0, 0)$.

(c) Show that $F_1(0, y) = -2y$ for all y and, hence, that $F_{12}(0, 0) = -2$.

(d) Deduce that $F_2(x, 0) = 2x$ and $F_{21}(0, 0) = 2$.

30. (a) Use Exercise 29(b) to find $F_{12}(x, x)$ for $x \neq 0$.

(b) Is $F_{12}(x, y)$ continuous at $(0, 0)$? Why?

31. Use the change of variables $\xi = x + ct$, $\eta = x$ to transform the partial differential equation

$$\frac{\partial u}{\partial t} = c\frac{\partial u}{\partial x}, \qquad (c = \text{constant}),$$

into the simpler equation $\partial v/\partial \eta = 0$, where $v(\xi, \eta) = v(x + ct, x) = u(x, t)$. This equation says that $v(\xi, \eta)$ does not depend on η, so $v = f(\xi)$ for some arbitrary differentiable function f. What is the corresponding "general solution" $u(x, t)$ of the original partial differential equation?

32. Having considered Exercise 31, guess a "general solution" $w(r, s)$ of the second-order partial differential equation

$$\frac{\partial^2}{\partial r\,\partial s} w(r, s) = 0.$$

Your answer should involve two arbitrary functions.

33. Use the change of variables $r = x + ct$, $s = x - ct$, $w(r, s) = u(x, t)$ to transform the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

to a simpler form. Now use the result of Exercise 32 to find the *general solution* of this wave equation in the form given in Example 4 in Section 12.4.

34. Show that the initial-value problem for the one-dimensional wave equation

$$\begin{cases} u_{tt}(x, t) = c^2 u_{xx}(x, t) \\ u(x, 0) = p(x) \\ u_t(x, 0) = q(x) \end{cases}$$

has the solution

$$u(x, t) = \frac{1}{2}\Big[p(x - ct) + p(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} q(s)\,ds.$$

(Note that we have used subscripts x and t instead of 1 and 2 to denote the partial derivatives here. This is common usage in dealing with partial differential equations.)

Remark The initial-value problem in Exercise 34 gives the small lateral displacement $u(x, t)$ at position x at time t of a vibrating string held under tension along the x-axis. The function $p(x)$ gives the *initial* displacement at position x, that is, the displacement at time $t = 0$. Similarly, $q(x)$ gives the initial velocity at position x. Observe that the position at time t depends only on values of these initial data at points no further than ct units away. This is consistent with the previous observation that the solutions of the wave equation represent waves travelling with speed c.

Redo the examples and exercises listed in Exercises 35–40 using Maple to do the calculations.

35. Example 10

36. Exercise 16

37. Exercise 19

38. Exercise 20

39. Exercise 23

40. Exercise 34

EXERCISES 12.6

In Exercises 1–6, use suitable linearizations to find approximate values for the given functions at the points indicated.

1. $f(x, y) = x^2y^3$ at $(3.1, 0.9)$
2. $f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$ at $(3.01, 2.99)$
3. $f(x, y) = \sin(\pi xy + \ln y)$ at $(0.01, 1.05)$
4. $f(x, y) = \dfrac{24}{x^2 + xy + y^2}$ at $(2.1, 1.8)$
5. $f(x, y, z) = \sqrt{x + 2y + 3z}$ at $(1.9, 1.8, 1.1)$
6. $f(x, y) = x\,e^{y+x^2}$ at $(2.05, -3.92)$

In Exercises 7–10, write the differential of the given function and use it to estimate the value of the function at the given point by starting with a known value at a nearby point.

7. $z = x^2 e^{3y}$, at $x = 3.05$, $y = -0.02$
8. $g(s, t) = s^2/t$, $g(2.1, 1.9)$
9. $F(x, y, z) = \sqrt{x^2 + y + 2 + z^2}$, $F(0.7, 2.6, 1.7)$
10. $u = x\sin(x + y)$, at $x = \dfrac{\pi}{2} + \dfrac{1}{20}$, $y = \dfrac{\pi}{2} - \dfrac{1}{30}$
11. The edges of a rectangular box are each measured to within an accuracy of 1% of their values. What is the approximate maximum percentage error in
    (a) the calculated volume of the box,
    (b) the calculated area of one of the faces of the box, and
    (c) the calculated length of a diagonal of the box?
12. The radius and height of a right-circular conical tank are measured to be 25 ft and 21 ft, respectively. Each measurement is accurate to within 0.5 in. By about how much can the calculated volume of the tank be in error?
13. By approximately how much can the calculated area of the conical surface of the tank in Exercise 12 be in error?
14. Two sides and the contained angle of a triangular plot of land are measured to be 224 m, 158 m, and 64°, respectively. The length measurements were accurate to within 0.4 m and the angle measurement to within 2°. What is the approximate maximum percentage error if the area of the plot is calculated from these measurements?
15. The angle of elevation of the top of a tower is measured at two points A and B on the ground in the same direction from the base of the tower. The angles are 50° at A and 35° at B, each measured to within 1°. The distance AB is measured to be 100 m with error at most 0.1%. What is the calculated height of the building, and by about how much can it be in error? To which of the three measurements is the calculated height most sensitive?
16. By approximately what percentage will the value of $w = \dfrac{x^2y^3}{z^4}$ increase or decrease if x increases by 1%, y increases by 2%, and z increases by 3%?
17. Find the Jacobian matrix for the transformation $\mathbf{f}(r, \theta) = (x, y)$, where

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

(Although (r, θ) can be regarded as *polar coordinates* in the xy-plane, they are Cartesian coordinates in their own $r\theta$-plane.)

18. Find the Jacobian matrix for the transformation $\mathbf{f}(R, \phi, \theta) = (x, y, z)$, where

$$x = R\sin\phi\cos\theta, \quad y = R\sin\phi\sin\theta, \quad z = R\cos\phi.$$

Here, (R, ϕ, θ) are *spherical coordinates* in xyz-space, as introduced in Section 10.6.

19. Find the Jacobian matrix $D\mathbf{f}(x, y, z)$ for the transformation of $\mathbb{R}^3$ to $\mathbb{R}^2$ given by

$$\mathbf{f}(x, y, z) = (x^2 + yz, y^2 - x\ln z).$$

Use $D\mathbf{f}(2, 2, 1)$ to help you find an approximate value for $\mathbf{f}(1.98, 2.01, 1.03)$.

20. Find the Jacobian matrix $D\mathbf{g}(1, 3, 3)$ for the transformation of $\mathbb{R}^3$ to $\mathbb{R}^3$ given by

$$\mathbf{g}(r, s, t) = (r^2s, r^2t, s^2 - t^2)$$