EXERCISES 11.1

In Exercises 1–14, find the velocity, speed, and acceleration at time t of the particle whose position is $\mathbf{r}(t)$. Describe the path of the particle.

1. $\mathbf{r} = \mathbf{i} + t\mathbf{j}$
2. $\mathbf{r} = t^2\mathbf{i} + \mathbf{k}$
3. $\mathbf{r} = t^2\mathbf{j} + t\mathbf{k}$
4. $\mathbf{r} = \mathbf{i} + t\mathbf{j} + t\mathbf{k}$
5. $\mathbf{r} = t^2\mathbf{i} - t^2\mathbf{j} + \mathbf{k}$
6. $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}$
7. $\mathbf{r} = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + ct\mathbf{k}$
8. $\mathbf{r} = a\cos\omega t\,\mathbf{i} + b\mathbf{j} + a\sin\omega t\,\mathbf{k}$
9. $\mathbf{r} = 3\cos t\,\mathbf{i} + 4\cos t\,\mathbf{j} + 5\sin t\,\mathbf{k}$
10. $\mathbf{r} = 3\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + t\mathbf{k}$
11. $\mathbf{r} = ae^t\mathbf{i} + be^t\mathbf{j} + ce^t\mathbf{k}$
12. $\mathbf{r} = at\cos\omega t\,\mathbf{i} + at\sin\omega t\,\mathbf{j} + b\ln t\,\mathbf{k}$
13. $\mathbf{r} = e^{-t}\cos(e^t)\mathbf{i} + e^{-t}\sin(e^t)\mathbf{j} - e^t\mathbf{k}$
14. $\mathbf{r} = a\cos t\sin t\,\mathbf{i} + a\sin^2 t\,\mathbf{j} + a\cos t\,\mathbf{k}$
15. A particle moves around the circle $x^2 + y^2 = 25$ at constant speed, making one revolution in 2 s. Find its acceleration when it is at $(3, 4)$.
16. A particle moves to the right along the curve $y = 3/x$. If its speed is 10 when it passes through the point $\left(2, \frac{3}{2}\right)$, what is its velocity at that time?
17. A point P moves along the curve of intersection of the cylinder $z = x^2$ and the plane $x + y = 2$ in the direction of increasing y with constant speed $v = 3$. Find the velocity of P when it is at $(1, 1, 1)$.
18. An object moves along the curve $y = x^2$, $z = x^3$, with constant vertical speed $dz/dt = 3$. Find the velocity and acceleration of the object when it is at the point $(2, 4, 8)$.
19. A particle moves along the curve $\mathbf{r} = 3u\mathbf{i} + 3u^2\mathbf{j} + 2u^3\mathbf{k}$ in the direction corresponding to increasing u and with a constant speed of 6. Find the velocity and acceleration of the particle when it is at the point $(3, 3, 2)$.
20. A particle moves along the curve of intersection of the cylinders $y = -x^2$ and $z = x^2$ in the direction in which x increases. (All distances are in centimetres.) At the instant when the particle is at the point $(1, -1, 1)$, its speed is 9 cm/s, and that speed is increasing at a rate of 3 cm/s^2. Find the velocity and acceleration of the particle at that instant.
21. Show that if the dot product of the velocity and acceleration of a moving particle is positive (or negative), then the speed of the particle is increasing (or decreasing).
22. Verify the formula for the derivative of a dot product given in Theorem 1(c).
23. Verify the formula for the derivative of a 3×3 determinant in the second remark following Theorem 1. Use this formula to verify the formula for the derivative of the cross product in Theorem 1.
24. If the position and velocity vectors of a moving particle are always perpendicular, show that the path of the particle lies on a sphere.
25. Generalize Exercise 24 to the case where the velocity of the particle is always perpendicular to the line joining the particle to a fixed point P_0.
26. What can be said about the motion of a particle at a time when its position and velocity satisfy $\mathbf{r} \bullet \mathbf{v} > 0$? What can be said when $\mathbf{r} \bullet \mathbf{v} < 0$?

In Exercises 27–32, assume that the vector functions encountered have continuous derivatives of all required orders.

27. Show that $\dfrac{d}{dt}\left(\dfrac{d\mathbf{u}}{dt} \times \dfrac{d^2\mathbf{u}}{dt^2}\right) = \dfrac{d\mathbf{u}}{dt} \times \dfrac{d^3\mathbf{u}}{dt^3}$.
28. Write the Product Rule for $\dfrac{d}{dt}\Big(\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w})\Big)$.
29. Write the Product Rule for $\dfrac{d}{dt}\Big(\mathbf{u} \times (\mathbf{v} \times \mathbf{w})\Big)$.

30. Expand and simplify: $\dfrac{d}{dt}\left(\mathbf{u} \times \left(\dfrac{d\mathbf{u}}{dt} \times \dfrac{d^2\mathbf{u}}{dt^2}\right)\right)$.

31. Expand and simplify: $\dfrac{d}{dt}\left((\mathbf{u} + \mathbf{u}'') \bullet (\mathbf{u} \times \mathbf{u}')\right)$.

32. Expand and simplify: $\dfrac{d}{dt}\left((\mathbf{u} \times \mathbf{u}') \bullet (\mathbf{u}' \times \mathbf{u}'')\right)$.

33. If at all times t the position and velocity vectors of a moving particle satisfy $\mathbf{v}(t) = 2\mathbf{r}(t)$, and if $\mathbf{r}(0) = \mathbf{r}_0$, find $\mathbf{r}(t)$ and the acceleration $\mathbf{a}(t)$. What is the path of motion?

* **34.** Verify that $\mathbf{r} = \mathbf{r}_0 \cos(\omega t) + (\mathbf{v}_0/\omega)\sin(\omega t)$ satisfies the initial-value problem

$$\frac{d^2\mathbf{r}}{dt^2} = -\omega^2\mathbf{r}, \qquad \mathbf{r}'(0) = \mathbf{v}_0, \qquad \mathbf{r}(0) = \mathbf{r}_0.$$

(It is the unique solution.) Describe the path $\mathbf{r}(t)$. What is the path if $\mathbf{r}_0$ is perpendicular to $\mathbf{v}_0$?

* **35. (Free fall with air resistance)** A projectile falling under gravity and slowed by air resistance proportional to its speed has position satisfying

$$\frac{d^2\mathbf{r}}{dt^2} = -g\mathbf{k} - c\frac{d\mathbf{r}}{dt},$$

where c is a positive constant. If $\mathbf{r} = \mathbf{r}_0$ and $d\mathbf{r}/dt = \mathbf{v}_0$ at time $t = 0$, find $\mathbf{r}(t)$. (*Hint:* Let $\mathbf{w} = e^{ct}(d\mathbf{r}/dt)$.) Show that the solution approaches that of the projectile problem given in this section as $c \to 0$.

EXERCISES 11.3

In Exercises 1–4, find the required parametrization of the first quadrant part of the circular arc $x^2 + y^2 = a^2$.

1. In terms of the y-coordinate, oriented counterclockwise
2. In terms of the x-coordinate, oriented clockwise
3. In terms of the angle between the tangent line and the positive x-axis, oriented counterclockwise
4. In terms of arc length measured from $(0, a)$, oriented clockwise
5. The cylinders $z = x^2$ and $z = 4y^2$ intersect in two curves, one of which passes through the point $(2, -1, 4)$. Find a parametrization of that curve using $t = y$ as parameter.
6. The plane $x + y + z = 1$ intersects the cylinder $z = x^2$ in a parabola. Parametrize the parabola using $t = x$ as parameter.

In Exercises 7–10, parametrize the curve of intersection of the given surfaces. *Note*: The answers are not unique.

7. $x^2 + y^2 = 9$ and $z = x + y$
8. $z = \sqrt{1 - x^2 - y^2}$ and $x + y = 1$
9. $z = x^2 + y^2$ and $2x - 4y - z - 1 = 0$
10. $yz + x = 1$ and $xz - x = 1$
11. The plane $z = 1 + x$ intersects the cone $z^2 = x^2 + y^2$ in a parabola. Try to parametrize the parabola using as parameter: (a) $t = x$, (b) $t = y$, and (c) $t = z$. Which of these choices for t leads to a single parametrization that represents the whole parabola? What is that parametrization? What happens with the other two choices?
12. ! The plane $x + y + z = 1$ intersects the sphere $x^2 + y^2 + z^2 = 1$ in a circle $\mathcal{C}$. Find the centre $\mathbf{r}_0$ and radius r of $\mathcal{C}$. Also find two perpendicular unit vectors $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ parallel to the plane of $\mathcal{C}$. (*Hint:* To be specific, show that $\hat{\mathbf{v}}_1 = (\mathbf{i} - \mathbf{j})/\sqrt{2}$ is one such vector; then find a second that is perpendicular to $\hat{\mathbf{v}}_1$.) Use your results to construct a parametrization of $\mathcal{C}$.
13. Find the length of the curve $\mathbf{r} = t^2\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ from $t = 0$ to $t = 1$.
14. For what values of the parameter λ is the length $s(T)$ of the curve $\mathbf{r} = t\mathbf{i} + \lambda t^2\mathbf{j} + t^3\mathbf{k}$, $(0 \le t \le T)$ given by $s(T) = T + T^3$?
15. Express the length of the curve $\mathbf{r} = at^2\,\mathbf{i} + bt\,\mathbf{j} + c\ln t\,\mathbf{k}$, $(1 \le t \le T)$, as a definite integral. Evaluate the integral if $b^2 = 4ac$.
16. Describe the parametric curve $\mathcal{C}$ given by

$$x = a\cos t\sin t, \qquad y = a\sin^2 t, \qquad z = bt.$$

What is the length of $\mathcal{C}$ between $t = 0$ and $t = T > 0$?

EXERCISES 11.4

Find the unit tangent vector $\hat{\mathbf{T}}(t)$ for the curves in Exercises 1–4.

1. $\mathbf{r} = t\mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k}$

2. $\mathbf{r} = a \sin \omega t\ \mathbf{i} + a \cos \omega t\ \mathbf{k}$

3. $\mathbf{r} = \cos t \sin t\ \mathbf{i} + \sin^2 t\ \mathbf{j} + \cos t\ \mathbf{k}$

4. $\mathbf{r} = a \cos t\ \mathbf{i} + b \sin t\ \mathbf{j} + t\mathbf{k}$

5. Show that if $\kappa(s) = 0$ for all s, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a straight line.

6. Show that if $\tau(s) = 0$ for all s, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a plane curve. *Hint:* Show that $\mathbf{r}(s)$ lies in the plane through $\mathbf{r}(0)$ with normal $\hat{\mathbf{B}}(0)$.

7. Show that if $\kappa(s) = C$ is a positive constant and $\tau(s) = 0$ for all s, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a circle. *Hint:* Find a circle having the given constant curvature. Then use Theorem 3.

8. Show that if the curvature $\kappa(s)$ and the torsion $\tau(s)$ are both nonzero constants, then the curve $\mathbf{r} = \mathbf{r}(s)$ is a circular helix. *Hint:* Find a helix having the given curvature and torsion.