Math 204(2011) Midterm 2.
Problems and Solutions.

Problem 1. Verify that $y_1(t) = \frac{1}{t}$ is a solution of the differential equation

$$y'' + \frac{3}{t}y' + \frac{1}{t^2}y = 0, \ t > 0.$$

Then find the general solution of this differential equation.

Solution.

$$y_1'(t) = -\frac{1}{t^2}, \quad y_2'' = \frac{2}{t^3}.$$

Thus

$$\frac{2}{t^3} + \frac{3}{t}(-\frac{1}{t^2}) + \frac{1}{t^2}\frac{1}{t} = 0,$$

i.e. $y_1(t) = \frac{1}{t}$ is a solution of the equation. We look for the second solution in the form

$$y_2(t) = \frac{1}{t}v(t).$$

$$y_2'(t) = -\frac{1}{t^2}v(t) + \frac{1}{t}v'(t),$$

$$y_2''(t) = \frac{2}{t^3}v(t) - 2\frac{1}{t^2}v'(t) + \frac{1}{t}v''(t)).$$

Inserting into equation:

$$\frac{2}{t^3}v(t) - 2\frac{1}{t^2}v'(t) + \frac{1}{t}v''(t) + \frac{3}{t}(-\frac{1}{t^2}v(t) + \frac{1}{t}v'(t)) + \frac{1}{t^3}v(t) = 0.$$

From this equation we get

$$v''(t) + \frac{1}{t}v'(t) = 0.$$

Solving this equation we find

$$v(t) = \ln t.$$

Thus $y_2(t) = \frac{1}{t}\ln t$,
and the general solution has the form

$$y(t) = (C_1 + C_2 \ln t)\frac{1}{t}.$$

Problem 2. Find the general solution of the equation

$$y'' + 4y' + 4y = \frac{e^{-2t}}{t^2}.$$

Solution. First we find the general solution of the homogeneous equation

$$y'' + 4y' + 4y = 0.$$

The corresponding characteristic equation

$$r^2 + 4r + 4 = 0$$

has a double root $r_1 = r_2 = -2$. Thus the general solution of the homogeneous equation is

$$y_h(t) = (C_1 + C_2 t)e^{-2t}.$$

By using the method of variation of parameters we find the particular solution of the nonhomogeneous equation

$$y_p(t) = -y_1(t)\int \frac{y_2(t)\frac{e^{-2t}}{t^2}}{W[y_1, y_2](t)}dt + y_2(t)\int \frac{y_1(t)\frac{e^{-2t}}{t^2}}{W[y_1, y_2](t)}dt.$$

$$W[y_1, y_2](t) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} = e^{-4t},$$

and

$$y_p(t) = -e^{-2t}\int \frac{dt}{t} + te^{-2t}\int \frac{dt}{t^2} = -e^{-2t}\ln t - e^{-2t}.$$

Thus the general solution of the nonhomogeneous equation has the form

$$y(t) = (C_1 + C_2 t)e^{-2t} - e^{-2t}\ln t.$$

Problem 3. Find the power series solution of the differential equation

$$y'' - xy' - y = 0$$

around $x_0 = 0$. Give the first 3 nonzero terms of each solution. Also determine the recursion formula satisfied by coefficients.

Solution.

$$y = \sum_{n=0}^{\infty} a_n x^n,\ y' = \sum_{n=1}^{\infty} n a_n x^{n-1},\ y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n.$$

thus we have

$$\sum_{n=2}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Or

$$2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - n a_n - a_n\right]x^n = 0$$

Thus we have the recursion formula

$$a_{n+2} = \frac{a_n}{n+2},\ n = 0, 1, 2, ...$$

By using this formula we get

$$a_2 = \frac{1}{2}a_0,\ a_4 = \frac{1}{4}a_2 = \frac{1}{8}a_0, ...$$

$$a_3 = \frac{1}{3}a_1,\ a_5 = \frac{1}{5}a_3 = \frac{1}{15}a_1, ...$$

So we have $y(x) = a_0 y_1(x) + a_1 y_2(x)$, where

$$y_1(x) = 1 + \frac{x^2}{2} + \frac{x^4}{8} + ...$$

$$y_2(x) = x + \frac{x^3}{3} + \frac{x^5}{15} + ...$$

Problem 4a. Find the Laplace transform of the function $f : [0, \infty) \to \mathbb{R}$ defined as

$$f(t) = \begin{cases} 0 \text{ if } 0 \leqslant t < 4, \\ t^2 \text{ if } t \geqslant 4. \end{cases}$$

Solution.

$$f(t) = u_4(t)g(t-4),$$

where $g(t) = (t+4)^2$. Thus

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u_4(t)g(t-4)\} = e^{-4s}\mathcal{L}\{u_4(t)g(t)\} =$$

$$e^{-4s}\left[\mathcal{L}\{t^2\} + 8L\{t\} + 16\mathcal{L}\{1\}\right] = e^{-4s}\left[\frac{2}{s^3} + \frac{8}{s^2} + \frac{16}{s}\right]$$

Problem 4b. Find the inverse Laplace transform of

$$\frac{1}{s^2(s^2+4)}.$$

Solution.

$$\frac{1}{s^2(s^2+4)} = \frac{1}{2}\frac{1}{s^2}\frac{2}{s^2+4} = \frac{1}{2}\mathcal{L}\{t\}\mathcal{L}\{\sin(2t)\}.$$

Thus we have

$$\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+4)}\} = \frac{1}{2}\int_0^t (t-\tau)\sin(2\tau)d\tau.$$

$$\int_0^t (t-\tau)\sin(2\tau)d\tau = t\int_0^t \sin(2\tau)d\tau - \int_0^t \tau\sin(2\tau)d\tau =$$

$$-\frac{1}{2}t\left[\cos(2\tau)\right]_0^t + \frac{1}{2}\left[\tau\cos(2\tau)\right]_0^t - \frac{1}{2}\int_0^t \cos(2\tau)d\tau = \frac{1}{2}t - \frac{1}{4}\sin(2t)$$

Thus

$$\mathcal{L}^{-1}\{\frac{1}{s^2(s^2+4)}\} = \frac{1}{4}t - \frac{1}{8}\sin(2t).$$

Problem 5. Using the Laplace transform, find the solution of the initial value problem

$$y'' + 4y = f(t), \qquad (A)$$

$$y(0) = 0,\ y'(0) = 1, \qquad (1)$$

where

$$f(t) = \begin{cases} t^2 \text{ if } 0 \leqslant t < 1, \\ 1 \text{ if } t \geqslant 1. \end{cases}$$

Solution. Since

$$f(t) = t^2 - u_1(t)t^2 + u_1(t).$$

we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} - e^{-s}\mathcal{L}\{(t+1)^2\} + \mathcal{L}\{u_1(t)\} = \frac{1}{s^3} - \frac{2e^{-s}}{s^3} - \frac{2e^{-s}}{s^2}.$$

Thus we obtain from (A) and (B):

$$s^2Y(s) + 4Y(s) = 1 + \frac{2}{s^3} - \frac{2e^{-s}}{s^3} - \frac{2e^{-s}}{s^2}$$

and

$$Y(s) = \frac{1}{s^2+4} + \frac{2}{s^3(s^2+4)} - \frac{2e^{-s}}{s^3(s^2+4)} - \frac{2e^{-s}}{s^2(s^2+4)}$$

Therefore

$$y(t) = \frac{1}{2}\sin(2t) + \frac{1}{2}f(t) - \frac{1}{2}u_1(t)f(t-1) - u_1(t)g(t-1),$$

where

$$f(t) = t^2 * \sin(2t), \quad g(t) = t * \sin(2t)$$