Problem 1a (*15 pts*) Solve the initial value problem for the Bernoulli equation

$$\begin{cases} y'(t) - y(t) = e^t y^2(t). \\ y(0) = 2. \end{cases}$$

Let $y = \frac{1}{v}$, so $y' = \frac{-v'}{v^2}$ $\Rightarrow$ $\frac{-v'}{v^2} - \frac{1}{v} = e^t \cdot \frac{1}{v^2}$

$\Rightarrow -v' - v = e^t \Rightarrow v' + v = -e^t$. Now the general solution of $v' + v = -e^t$

is $v(t) = \frac{1}{\mu(t)} \cdot \int \mu(t) \cdot (-e^t)\,dt$ where $\mu(t) = e^{\int 1 dt} = e^t$

$\Rightarrow v(t) = e^{-t} \cdot \int (-e^{2t})\,dt = e^{-t}\left(\frac{-e^{2t}}{2} + c\right) = \frac{-e^t}{2} + c \cdot e^{-t}$, $c \in \mathbb{R}$

So $y(t) = \left(\frac{-e^t}{2} + c \cdot e^{-t}\right)^{-1}$. Put $t = 0 \Rightarrow y(0) = \left(c - \frac{1}{2}\right)^{-1} = 2$

$\Rightarrow c = 1$. Thus the solution of the initial value problem

is $y(t) = \left(\frac{-e^t}{2} + e^{-t}\right)^{-1}$ //

1.b) (*5 pts.*) Is the solution of this problem unique? Why? (5 points)

Let $f(t, y) = y + e^t \cdot y^2$, so $\frac{\partial f}{\partial y} = 1 + 2e^t y$. We see that

both f and $\frac{\partial f}{\partial y}$ are continous on any rectangle containing the

point $(0, 2)$. Hence by the existence and uniqueness theorem the

initial value problem $y' = f(t, y)$, $y(0) = 2$ has a <u>unique solution</u>.

Problem 2 (15 points) Give definition of an exact equation. Show that the equation

$$y^3dx + 3xy^2dy = 0$$

is an exact equation and find its solution that satisfies the condition

$$y(1) = 2.$$

- Definition: The differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if there exists a function $\Psi(x,y)$ such that

$$\frac{\partial \Psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \Psi}{\partial y} = N(x,y).$$

- Theorem: Let M, N, M_y and N_x be continuous on a rectangle; $\alpha < x < \beta$, $\delta < y < \gamma$. Then $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if and only if $M_y(x,y) = N_x(x,y)$.

Now $M(x,y) = y^3$, $N(x,y) = 3xy^2$. So $M_y = 3y^2$, $N_x = 3y^2 \Rightarrow M_y = N_x$

So the equation is exact //

Let $\Psi(x,y)$ be a function such that $\Psi_x = M$, $\Psi_y = N$.

$\Psi_x = M = y^3 \Rightarrow \Psi(x,y) = \int y^3 dx = xy^3 + h(y)$

$\Rightarrow \Psi_y = 3xy^2 + h'(y) = N(x,y) = 3xy^2 \Rightarrow h'(y) = 0 \Rightarrow h(y)$ is constant.

Thus the solution of $y^3dx + 3xy^2dy = 0$ is $\Psi(x,y) = xy^3 = c$, $c \in \mathbb{R}$.

Since $y(1) = 2$, we've that $c = 1 \cdot 2^3 = 8$.

$\Rightarrow$ The solution satisfying $y(1) = 2$ is $xy^3 = 8$ //

Problem 4 (15 points) Use the method of variation of parameters to solve the problem

$$\begin{cases} y''(t) + 4y(t) = \sin(2t), & (1) \\ y(0) = 1, y'(0) = 1. \end{cases}$$

• Consider $y'' + 4y = 0$. The characteristic equation is $r^2 + 4 = 0$, so the roots are $r_{1,2} = \pm 2i$. Here the complementary solution of (1) is $y_c(t) = c_1 \cdot \cos(2t) + c_2 \cdot \sin(2t)$. Put $\underline{y_1 = \cos(2t)}$, $\underline{y_2 = \sin(2t)}$.

• Now let $y(t) = u_1(t) \cdot y_1(t) + u_2(t) \cdot y_2(t)$ be the solution of (1).
By method of variations of parameters we have

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin(2t) \end{bmatrix} \Rightarrow \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & +2\cos(2t) \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sin(2t) \end{bmatrix}$$

$$\Rightarrow u_1' = \frac{-\sin^2(2t)}{2} = \frac{\cos(4t) - 1}{4}, \quad u_2' = \frac{\sin(2t) \cdot \cos(2t)}{2} = \frac{\sin(4t)}{4}$$

$$\Rightarrow u_1 = \int \frac{\cos 4t - 1}{4}\, dt = \frac{\sin(4t)}{16} - \frac{t}{4} + c_1,$$

$$u_2 = \int \frac{\sin(4t)}{4}\, dt = \frac{-\cos(4t)}{16} + c_2$$

Note: $\cos(2t) \cdot \sin(4t) - \sin(2t) \cdot \cos(4t) = \sin(2t)$, so we can eliminate in $y(t)$ as $\sin(2t)$ is a complementary solution

$$\Rightarrow y(t) = u_1 \cdot y_1 + u_2 \cdot y_2$$

$$\Rightarrow y(t) = c_1 \cdot \cos(2t) + c_2 \cdot \sin(2t) - \frac{t \cdot \cos(2t)}{4}, \quad y(0) = c_1 = 1$$

$$y'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t) - \frac{\cos(2t)}{4} + \frac{t \cdot \sin(2t)}{2}, \quad y'(0) = 2c_2 - \frac{1}{4} = 1, \text{ so}$$

$$c_2 = \frac{5}{8}$$

Here the solution of the IVP is

$$y(t) = \cos(2t) + \frac{5}{8}\sin(2t) - \frac{t \cdot \cos(2t)}{4} \quad //$$

Problem 3 (15 points) Given a differential equation with constant coefficients

$$y'' + ay' + by = 0. \qquad (1)$$

Find conditions on the numbers a and b for which all solutions of this equation are tending to zero as $t \to \infty$ and the conditions for which all solutions are bounded on $[0, \infty)$.

The characteristic equation of (1) is $r^2 + ar + b = 0$.

Now $\Delta = a^2 - 4b$, so the roots are $r_{1,2} = \dfrac{-a \pm \sqrt{a^2-4b}}{2}$

$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$,	$y(t) = c_1 e^{rt} + c_2 t e^{rt}$,	$y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)$,
$r_{1,2} \in \mathbb{R},\ r_1 \neq r_2$	$r = r_1 = r_2$	$r_{1,2} = \lambda \pm i\mu$

Since $\lim\limits_{t\to\infty} y(t) = 0$, we must have $\begin{cases} r_1, r_2 < 0 & \text{if } \Delta > 0 \\ r < 0 & \text{if } \Delta = 0 \\ \lambda < 0 & \text{if } \Delta < 0 \end{cases}$

So; if $\Delta > 0$, then $-a \pm \sqrt{a^2-4b} < 0 \Rightarrow -a < \sqrt{a^2-4b} < a \Rightarrow a, b > 0,\ a^2 > 4b$

if $\Delta = 0$, then $r = \frac{-a}{2} < 0 \Rightarrow a > 0,\ a^2 = 4b \Rightarrow a, b > 0,\ a^2 = 4b$

if $\Delta < 0$, then $\lambda = \frac{-a}{2} < 0 \Rightarrow a > 0,\ a^2 < 4b \Rightarrow a, b > 0,\ a^2 < 4b$

Here we must have $\boxed{a > 0,\ b > 0}$ in order to have $\lim\limits_{t\to\infty} y(t) = 0$ for all $c_1, c_2 \in \mathbb{R}$.

- To have bounded $y(t)$ on $[0, \infty)$ we may additionally have the following cases;

if $\Delta > 0$; then $r_1 = 0,\ r_2 < 0 \Rightarrow \boxed{a > 0,\ b = 0}$

if $\Delta < 0$; then $\lambda = \frac{-a}{2} = 0 \Rightarrow \boxed{a = 0,\ b > 0}$

if $\Delta = 0$, then $r < 0$ (for $r = 0$, $c_1 + c_2 t$ is not bounded)

Problem 5

5.a) (*5 pts.*) Give the statement of the existence and uniqueness theorem for the initial value problem for second order linear ODE's.

Theorem: Let $p(t)$, $q(t)$ and $g(t)$ be continous on an open interval I. Suppose $t_0 \in I$. Then the initial value problem

$$y'' + p(t) \cdot y' + q(t) \cdot y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'$$

has a unique solution on I.

5.b) (*15 pts.*) Suppose that $p(t), q(t)$ be continuous functions on (a, b),$y_1(t)$ is a nonzero solution of the equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

on (a,b) and $f(t)$ is continuously differentiable function on (a, b). Show that if $f(t)$ has a local extremum on (a, b), then $\{y_1(t), f(t)y_1(t)\}$ can not be a fundamental set of solutions of this equation on (a, b).

The Wronskian of y_1 and $f \cdot y_1$ is

$$W[y_1, f \cdot y_1](t) = \begin{vmatrix} y_1 & f \cdot y_1 \\ y_1' & (f' \cdot y_1 + f \cdot y_1') \end{vmatrix} = f' \cdot y_1^2 .$$

Suppose $f(t)$ has a local extremum at $c \in (a, b)$. Since $f(t)$ is differentiable on (a, b), we must have that $f'(c) = 0$.

But then $W[y_1, f \cdot y_1](c) = f'(c) \cdot y_1(c)^2 = 0$, and so

$\{y_1, f \cdot y_1\}$ can not be a fundamental set of solutions.

Problem 6 (*15 pts.*) Find a series solution in powers of x of the problem

$$\begin{cases} y''(x) + 2xy'(x) + 2y(x) = 0, \\ y(0) = 2, \ y'(0) = 1. \end{cases}$$

Let $y(x) = \sum_{n=0}^{\infty} a_n . x^n$, so $y'(x) = \sum_{n=1}^{\infty} n . a_n . x^{n-1}$, $y''(x) = \sum_{n=2}^{\infty} n.(n-1).a_n.x^{n-2}$

$\Rightarrow \ x.y'(x) = \sum_{n=1}^{\infty} n.a_n.x^n$, $y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1).a_{n+2}.x^n$

$$\Rightarrow \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + 2na_n + 2a_n \right] x^n + a_0 + 2.1.a_2 = 0$$

$\Rightarrow \ a_0 + 2a_2 = 0$ and $(n+2)a_{n+2} + 2a_n = 0$ for $n \geq 1$.

$a_2 = \dfrac{-a_0}{2}$ $\qquad a_3 = -\dfrac{2a_1}{3}$

$a_4 = \dfrac{-a_2}{2} = \dfrac{a_0}{2^2}$ $\qquad a_5 = \dfrac{-2a_3}{5} = \dfrac{2^2 . a_1}{3.5}$

$a_6 = \dfrac{-a_4}{3} = \dfrac{-a_0}{2^2.3}$ $\qquad a_7 = \dfrac{-2a_5}{7} = \dfrac{-2^3.a_1}{3.5.7}$

$a_8 = \dfrac{-a_6}{4} = \dfrac{a_0}{2^2.3.4}$

$\vdots$ $\qquad \vdots$

$a_{2n+1} = \dfrac{(-2)^n . 1}{1.3.5.\ldots.(2n+1)}$

$a_{2n} = \dfrac{(-1)^n . a_0}{2.n!}$

$$\Rightarrow y(x) = a_0 . \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{2.n!} x^{2n} + 1 \right) + a_1 . \left(\sum_{n=0}^{\infty} \frac{(-2)^n . x^{2n+1}}{1.3.\ldots.(2n+1)} \right)$$

$$\Rightarrow y'(x) = a_0 . \left(\sum_{n=1}^{\infty} \frac{(-1)^n . x^{2n-1}}{(n-1)!} \right) + a_1 . \left(1 + \sum_{n=1}^{\infty} \frac{(-2)^n . x^{2n}}{1.3.\ldots.(2n-1)} \right)$$

$\Rightarrow \left.\begin{array}{l} y(0) = a_0 = 2 \\ y'(0) = a_1 = 1 \end{array}\right\} \Rightarrow$ the solution is $\boxed{y(x) = 2.\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n . x^{2n}}{2.n!}\right) + \sum_{n=0}^{\infty} \frac{(-2)^n . x^{2n+1}}{1.3.\ldots.(2n+1)}}$