Problem 3 Let $\mathbf{K}$ and $\mathbf{L}$ be invertible $n \times n$ matrices. Show that the following statements are true.

3.a) $\mathbf{KL}$ is also invertible. (5 points)

Since K and L are invertible $\det(K) \neq 0$, $\det(L) \neq 0$.

Therefore $\det(KL) = \det(K) \cdot \det(L) \neq 0$.

So KL is not a singular matrix. Thus KL is invertible

3.b) $(\mathbf{KL})^{-1} = \mathbf{L}^{-1}\mathbf{K}^{-1}$. (5 points)

$$(L^{-1}K^{-1})(KL) = L^{-1}(K^{-1}K)L = L^{-1} \cdot I\,L$$

$$= L^{-1}L = I \Rightarrow (KL)^{-1} = L^{-1}K^{-1}$$

3.c) If $\mathbf{K}$ is a symmetric matrix, i.e, $K_{ij} = K_{ji}$, then $\mathbf{K}^{-1}$ is also a symmetric matrix. (5 points)

$$K^{-1} = [C_{ij}], \text{ where } C_{ij} = (-1)^{i+j} \frac{\det(M^{(ij)})}{\det(K)}$$

Since K is a symmetric matrix

$$\det(M^{(ij)}) = \det(M^{(ij)}).$$

Hence $$C_{ij} = (-1)^{i+j} \frac{\det(M^{(ij)})}{\det(K)} = C_{ji} \Rightarrow$$

K^{-1} is a symmetric matrix.

Problem 4

4.a) Give the definition of the rank of an $m \times n$ matrix. (2 points)

Rank of an $m \times n$ matrix is the maximum number of its linearly-independent rows (or columns).

4.b) Find the rank of $\begin{bmatrix} 1+i & 2 & i \\ 3-i & 2-2i & 2+i \end{bmatrix}$. Justify your response. (3 points)

$$\begin{vmatrix} 1+i & 2 \\ 3-i & 2-2i \end{vmatrix} = (1+i)(2-2i) - 6 + 2i$$

$$= 4 - 6 + 2i = -2 + 2i \neq 0.$$

Rank of this matrix is $\underline{2}$.

Problem 5 Find all real numbers α for which the solution of following initial-value problem tends to zero as $t \to \infty$. (10 points)

$$y'(t) + \alpha\, y(t) = e^{-t}, \qquad (1)$$
$$y(0) = 1. \qquad (2)$$

Multiplying the equation (1) by $e^{\alpha t}$ we obtain:

$$\left(y(t)e^{\alpha t}\right)' = e^{(\alpha-1)t}$$

Integrating this equality and using the initial condition (2) we get (assuming $\alpha \neq 1$):

$$y(t)e^{\alpha t} - 1 = \frac{1}{\alpha-1}\left[e^{(\alpha-1)t} - 1\right]$$

$$\Rightarrow \quad y(t) = e^{-\alpha t} + \frac{1}{\alpha-1}\left[e^{-t} - e^{-\alpha t}\right].$$

Hence if $\alpha > 0$ and $\alpha \neq 1$ solution of (1),(2) thends to $\underline{0}$ as $t \to \infty$. If $\alpha \leq 0$ then $y(t) \not\to 0$ as $t \to \infty$

When $\alpha = 1$ we multiply the equation $y' + y = e^{-t}$ by e^{t} and get

$$\left(y(t)e^{t}\right)' = 1$$

$$\Longrightarrow \quad y(t) = e^{-t} + te^{-t} \longrightarrow 0 \text{ as } t \to \infty.$$

Conclusion: if $\alpha > 0$ then $y(t) \longrightarrow 0$ as $t \to \infty$.

Problem 6

6.a) Give the statement of Abel's Theorem on homogeneous second order linear ODEs. (3 points)

If $y_1(t)$, $y_2(t)$ are solutions of the equation

$$y''(t) + p(t)y'(t) + q(t)y(t) = 0$$

where p and q are continuous on some interval (a,b), then

$$(A) \quad W[y_1,y_2](t) = Ce^{-\int p(t)\,dt}, \quad t \in (a,b)$$

$C = \text{Const}.$

6.b) Let p and q be real-valued functions defined on $\mathbb{R}$. Show that if p is differentiable and $p(t) > 0$ for all $t \in \mathbb{R}$, then the Wronskian of any two solutions of the differential equation:

$$[p(t)y']' + q(t)y = 0, \qquad (*)$$

is given by $W(t) = \dfrac{c}{p(t)}$, where c is a constant. (7 points)

$$(*) \Rightarrow \quad p(t)y'' + p'(t)y' + q(t)y = 0$$

$$\Rightarrow \quad y'' + \frac{p'(t)}{p(t)}y' + \frac{q(t)}{p(t)}y = 0.$$

Due to the Abel's formula (A) we have

$$W[y_1,y_2] = C\,e^{-\int \frac{p'(t)}{p(t)}dt} = Ce^{-\ln p(t)} =$$

$$= \frac{C}{p(t)}$$

Problem 7 Let $f : [0, \infty) \to \mathbb{R}$ be the function defined by:

$$f(t) := \begin{cases} 3 & \text{for} \quad 0 \le t < 2, \\ 3\, e^{2-t} & \text{for} \quad t \ge 2. \end{cases}$$

7.a) Express f in terms of the unit step function. (5 points)

$$f(t) = 3(1 - u_2(t)) + 3 u_2(t) e^{2-t}$$

$$= 3 + u_2(t)\left(e^{2-t} - 1\right)$$

7.b) Find the Laplace transform of f and simplify it as much as possible. (5 points)

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{3\} - 3\mathcal{L}\{u_2(t)\}$$

$$+ 3\mathcal{L}\left\{u_2(t) e^{-(t-2)}\right\}$$

$$= \frac{3}{s} - 3\frac{e^{-2s}}{s} + 3e^{-2s} \cdot \frac{1}{s+1}$$

$$= 3\left[\frac{1 - e^{-2s}}{s}\right] + 3\frac{e^{-2s}}{s+1}$$

Problem 8 Find the general (real) solution of the following system of ODEs. (10 points)

$$\begin{aligned} x_1' &= 2x_1 + 3x_2, \\ x_2' &= -3x_1 + 2x_2. \end{aligned}$$

The characteristic equation

$$\begin{vmatrix} 2-r & 3 \\ -3 & 2-r \end{vmatrix} = 0 \text{ or } (r-2)^2 + 9 = 0$$

has complex-conjugate roots $r_{1,2} = 2 \mp 3i$.

Inserting $r_1 = 2-3i$ into the system

$$\begin{aligned} (2-r)w_1 + 3w_2 &= 0 \\ -3w_1 + (2-r)w_2 &= 0 \end{aligned}$$

we find the eigenvector

$$\vec{v}^{(1)} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Thus the general solution of the system has the form

$$\vec{X}(t) = e^{2t} C_1 \left(\cos(3t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(3t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$+ e^{2t} C_2 \left(\sin(3t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \cos(3t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} C_1 e^{2t} \cos(3t) + C_2 e^{2t} \sin(3t) \\ C_1 e^{2t} \sin(3t) - C_2 e^{2t} \cos(3t) \end{bmatrix}.$$

9.b) Solve the following initial-value problem. (10 points)

$$x_1' = 3x_1 + 2x_2 + 3e^{3t},$$
$$x_2' = x_1 + 2x_2 + e^{2t},$$
$$x_1(0) = x_2(0) = 0.$$

$$\vec{x}(t) = \Psi(t)\vec{x}(0) + \Psi(t)\int_0^t \Psi^{-1}(s)\vec{g}(s)\,ds,$$

where $\Psi(t)$ is the fundamental matrix (2).

$$\vec{x}(t) = \Psi(t)\int_0^t \Psi^{-1}(s)\vec{g}(s)\,ds,$$ since $x_1(0)=x_2(0)=0$.

$$\Psi^{-1}(s) = \frac{1}{3e^{5s}}\begin{bmatrix} e^{4s} & -2e^{4s} \\ e^{s} & e^{s} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} e^{-s} & -2e^{-s} \\ e^{-4s} & e^{-4s} \end{bmatrix}$$

$$\int_0^t \Psi^{-1}(s)\begin{bmatrix} 3e^{3s} \\ e^{2s} \end{bmatrix} ds = \frac{1}{3}\int_0^t \begin{bmatrix} 3e^{2s} - 2e^{s} \\ 3e^{-s} + e^{-2s} \end{bmatrix} ds$$

$$= \begin{bmatrix} \frac{1}{2}e^{2t} - \frac{2}{3}e^{t} + \frac{1}{6} \\ -e^{t} - \frac{1}{6}e^{-2t} + \frac{7}{6} \end{bmatrix}.$$

$$\vec{x}(t) = \begin{bmatrix} e^{t} & 2e^{4t} \\ -e^{t} & e^{4t} \end{bmatrix}\begin{bmatrix} \frac{1}{2}e^{2t} - \frac{2}{3}e^{t} + \frac{1}{6} \\ -e^{t} - \frac{1}{6}e^{-2t} + \frac{7}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -2e^{5t} + \frac{7}{3}e^{4t} + \frac{1}{3}e^{3t} - e^{2t} + \frac{1}{6}e^{t} \\ -e^{5t} + \frac{7}{6}e^{4t} - \frac{1}{2}e^{3t} + \frac{1}{2}e^{2t} - \frac{1}{6}e^{t} \end{bmatrix}.$$