Problem 1 (15 points) Find the explicit solution of the following initial-value problem.

$$2xyy' + y^2 + 2x = 0, \qquad y(1) = 1.$$

Let $u := y^2 \quad \hookrightarrow \quad xu' + u + 2x = 0$

$\Rightarrow \quad u' + \frac{u}{x} = -2$

$\Rightarrow \quad \underbrace{\mu u'}_{} + \frac{\mu}{x} u = -2\mu$

$\frac{d}{dx}(\mu u) + \underbrace{[-\mu' + \frac{\mu}{x}]}_{0} u = -2\mu$

$\frac{\mu'}{\mu} = \frac{1}{x} \Rightarrow \quad \mu = ax \quad$ take $a = 1 \Rightarrow \underline{\mu(x) = x}$

$\Rightarrow \quad \frac{d}{dx}(xu) = -2x$

$\Rightarrow \quad xu = -x^2 + c$

$\Rightarrow \quad u = -x + \frac{c}{x}$

$\Rightarrow \quad y^2 = -x + \frac{c}{x}$

$\Rightarrow \quad y = \pm\sqrt{-x + \frac{c}{x}}$

$y(1) = 1 \quad \hookrightarrow \quad 1 = \pm\sqrt{-1 + c}$

$\Rightarrow$ choose $+$ & $c = 2$

$\Rightarrow \quad \boxed{y = \sqrt{-x + \frac{2}{x}}}$

Problem 2 (10 points) Let $y(t)$ be a twice differentiable function with Laplace transform $Y(s)$ for $s > 0$. Show that for $s > 0$ the Laplace transform of $t^2 y''(t)$ is given by

$$\mathcal{L}\{t^2 y''(t)\} = s^2 Y''(s) + 4sY'(s) + 2Y(s).$$

$$\mathcal{L}\{t^2 y''(t)\} = \int_0^\infty e^{-st}\, t^2 y''(t)\, dt$$

$$= \int_0^\infty \left(\frac{\partial^2}{\partial s^2} e^{-st}\right) y''(t)\, dt$$

$$= \frac{d^2}{ds^2} \mathcal{L}\{y''(t)\}$$

$$= \frac{d^2}{ds^2} \left[s^2 Y(s) - y'(0) - sy(0) \right]$$

$$= \frac{d}{ds} \left[2s\, Y(s) + s^2 Y'(s) \right]$$

$$= 2\, Y(s) + 2s\, Y'(s) + 2s\, Y'(s) + s^2 Y''(s)$$

$$= s^2 Y''(s) + 4s\, Y'(s) + 2\, Y(s).$$

Problem 3 (20 points) Consider the equation $y''(t) + 2ay'(t) + by(t) = 0$ where a and b are real numbers.

3.a) (5 points) Reduce this equation to a system of first order equations.

$x_1 := y$

$x_2 := y' \quad \Rightarrow \quad x_2' = y'' = -(2ay' + by) = -2ax_2 - b$

$$\Rightarrow \begin{cases} x_1' = x_2 \\ x_2' = -bx_1 - 2ax_2 \end{cases} \qquad \vec{X} := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad \vec{X}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -b & -2a \end{bmatrix}}_{\mathbb{A}} \vec{X}$$

3.b) (15 points) Find conditions on a and b such that all solutions of the system of equations you find in part a of this problem tend to zero as $t \to +\infty$.

$$0 = \det(\mathbb{A} - r\mathbb{I}) = \det \begin{bmatrix} -r & 1 \\ -b & -2a-r \end{bmatrix} = r(r+2a) + b$$

$r^2 + 2ar + b = 0 \Rightarrow r = -a \pm \sqrt{a^2 - b}$

① If $a^2 = b \Rightarrow \vec{X}(t) = \begin{bmatrix} P_1(t) \\ P_2(t) \end{bmatrix} e^{-at} \to 0$ for $t \to +\infty$ if and only if $\boxed{a > 0}$

② If $a^2 \neq b \Rightarrow \vec{X}(t) = c_1 e^{(-a+\sqrt{a^2-b})t} \vec{\xi}^{(1)} + c_2 e^{(-a-\sqrt{a^2-b})t} \vec{\xi}^{(2)}$

(2.1) If $a^2 > b$, $\vec{X}(t) \to 0$ as $t \to +\infty$ provided that

$-a \pm \sqrt{a^2 - b} < 0 \Rightarrow -a + \sqrt{a^2-b} < 0 \Rightarrow \sqrt{a^2-b} < a \Rightarrow$

$\Rightarrow a > 0$ and $a^2 - b < a^2 \Leftrightarrow b > 0$

So in this case $\boxed{a > 0 \text{ and } b > 0}$

(2.2) If $a^2 < b$, then $\vec{X}(t) \to 0$ for $t \to +\infty$ provided that

$-a < 0 \Rightarrow \boxed{a > 0}$

$\boxed{\text{So } a > 0 \text{ and } b > 0 \text{ if } a^2 > b.}$

Problem 4 (10 points) Calculate the coefficients of the Fourier series for the function $f : [-\pi, \pi] \to \mathbb{R}$ given by $f(x) = \pi - |x|$.

f is even $\Rightarrow$ coefficient of Sine Series are zero.

So the Fourier series for $f(x)$ is $\boxed{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)}$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} [\pi - |x|] \cos(nx)\, dx$.

$$\Rightarrow a_n = \underbrace{\int_{-\pi}^{\pi} \cos(nx)\, dx}_{I_0} + \frac{1}{\pi}\underbrace{\int_{-\pi}^{0} x \cos(nx)\, dx}_{I_1} + \frac{1}{\pi}\underbrace{\int_{0}^{\pi} -x \cos(nx)\, dx}_{I_2}$$

For $n = 0$: $I_0 = 2\pi$

For $n \geq 1$: $I_0 = \frac{\sin(nx)}{n}\Big|_{-\pi}^{\pi} = 0$

$$\int x \cos(\alpha x)\, dx = \frac{\partial}{\partial \alpha}\int \sin(\alpha x)\, dx = \frac{\partial}{\partial \alpha}\left[-\frac{\cos \alpha x}{\alpha}\right] + c$$

$$= \frac{x \sin(\alpha x)}{\alpha} + \frac{\cos \alpha x}{\alpha^2} + c \quad \text{for } \alpha \neq 0$$

$$\int x\, dx = \frac{x^2}{2} + c$$

For $n = 0$: $I_1 = \frac{x^2}{2}\Big|_{-\pi}^{0} = -\frac{\pi^2}{2}$, $I_2 = -\frac{x^2}{2}\Big|_{0}^{\pi} = -\frac{\pi^2}{2}$

$\Rightarrow a_0 = 2\pi + \frac{1}{\pi}(-\pi^2) = \pi \Rightarrow \boxed{a_0 = \pi}$

For $n > 0$: $I_1 = \frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\Big|_{-\pi}^{0} = \frac{1}{n^2} - \left(\frac{\cos(\pi n)}{n^2}\right) = \frac{1-(-1)^n}{n^2}$

& $I_2 = -\left(\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right)\Big|_{0}^{\pi} = -\left(\frac{\cos(\pi n)}{n^2} - \left(\frac{1}{n^2}\right)\right) = \frac{1-(-1)^n}{n^2}$

$\Rightarrow a_n = \frac{1}{\pi}\left[\frac{2(1-(-1)^n)}{n^2}\right] \Rightarrow \boxed{a_n = \frac{2[1-(-1)^n]}{\pi n^2} \text{ for } n \geq 1}$

Problem 5 Consider the boundary-value (eigenvalue) problem

$$y''(x) = \lambda\, y(x), \qquad x \in (0, 2\pi)$$
$$y'(0) = 0, \quad y'(2\pi) = 0.$$

where λ is a real number.

5.a (15 points) Find all values of λ for which this problem has a nonzero solution and determine these solutions, i.e., find the eigenvalues and eigenfunctions.

Warning: You must give a detailed justification of your response. Giving the values of λ without explaining how you obtain them and why there are no other possible values will not earn you any credit.

There are 3 possibilities:

(1) $\lambda > 0 \Rightarrow$ let $\mu := \sqrt{\lambda} > 0$ & $y'' - \mu^2 y = 0 \Rightarrow$

$r^2 - \mu^2 = 0 \Rightarrow r = \pm\mu \Rightarrow Y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x}$

$\Rightarrow Y'(x) = \mu\left(c_1 e^{\mu x} - c_2 e^{-\mu x}\right)$

$Y'(0) = 0 \Rightarrow \mu(c_1 - c_2) = 0 \underset{\mu \neq 0}{\Rightarrow} c_1 = c_2$

$Y'(2\pi) = 0 \Rightarrow \underset{\neq 0}{\mu}\left(c_1 e^{2\pi\mu} - \underset{c_1}{c_2} e^{-2\pi\mu}\right) = 0 \Rightarrow c_1\left(e^{2\pi\mu} - e^{-2\pi\mu}\right)$

$c_1 \neq 0$ because $c_1 = 0 \Rightarrow c_2 = 0 \Rightarrow y = 0$ which is not allowed

$\Rightarrow e^{2\pi\mu} = e^{-2\pi\mu} \Rightarrow e^{4\pi\mu} = 1 \Rightarrow \mu = 0$ but $\mu > 0$

No solution for $\lambda > 0$.

(2) $\lambda = 0 \Rightarrow Y'' = 0 \Rightarrow Y = ax + b \Rightarrow Y' = a$

$Y'(0) = 0 \Rightarrow a = 0 \Rightarrow Y'(2\pi) = 0$ ✓ $\Rightarrow \boxed{Y(x) = b}$

So $\lambda = 0$ is an eigenvalue
with eigenfunction $Y(x) = b$ when $b \in \mathbb{R}\setminus\{0\}$.

(3) $\lambda < 0 \Rightarrow$ let $\mu := \sqrt{-\lambda^2} > 0$ & $y'' + \mu^2 y = 0 \Rightarrow$

$r^2 = -\mu^2 \Rightarrow r = \pm i\mu \Rightarrow Y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x)$

$\Rightarrow Y'(x) = \mu\left[-c_1 \sin(\mu x) + c_2 \cos(\mu x)\right]$

$Y'(0) = 0 \Rightarrow \underset{\neq 0}{\mu} c_2 = 0 \Rightarrow c_2 = 0 \Rightarrow Y' = -\mu c_1 \sin(\mu x)$

$Y'(2\pi) = 0 \Rightarrow -\mu c_1 \sin(2\pi\mu) = 0 \underset{\mu \neq 0\ \&\ c_1 \neq 0}{\Rightarrow} 2\pi\mu = \pi n$ for $n \in \mathbb{Z}$

$\boxed{\mu = \frac{n}{2} \Rightarrow \lambda = -\frac{n^2}{4}}$

and $\boxed{Y(x) = c_1 \cos\left(\frac{nx}{2}\right)}$ $c_1 \in \mathbb{R}\setminus\{0\}$.

5.b (5 points) Show that the eigenfunctions with different eigenvalues are linearly independent.

let $\lambda_1 = -\frac{n_1^2}{4}$, $\lambda_2 = -\frac{n_2^2}{4}$, $n_1, n_2 \in \mathbb{N} = \{0,1,2,\dots\}$

be any pair of eigenvalues with eigenfunctions

$y_1 = a_1 \cos(\frac{n_1 x}{2})$ and $y_2 = a_2 \cos(\frac{n_2 x}{2})$, $a_1, a_2 =$

let $\lambda_1 \neq \lambda_2 \Rightarrow n_1 \neq n_2$.

If $\exists\, b_1, b_2 \in \mathbb{R}$ such that

$$b_1 y_1 + b_2 y_2 = 0 \quad (1)$$

$$\Rightarrow \quad b_1 y_1(x) + b_2 y_2(x) = 0, \quad \forall x \in (0, 2\pi)$$

Recall that $\int_0^{2\pi} \cos(\frac{n_1 x}{2}) \cos(\frac{n_2 x}{2})\, dx = 0$ because $n_1 \neq n_2$

$$\Rightarrow \int_0^{2\pi} y_1(x)\, y_2(x)\, dx = 0 \quad (2)$$

$$\Rightarrow \quad 0 = \int_0^{2\pi} \underbrace{[b_1 y_1(x) + b_2 y_2(x)]}_{0}\, y_1(x)\, dx =$$

$$= b_1 \underbrace{\int_0^{2\pi} y_1(x)^2 dx}_{\neq 0} + b_2 \underbrace{\int_0^{2\pi} y_1(x)\, y_2(x)\, dx}_{0 \text{ by } (2)}$$

because $y_1(x)^2 \geq 0$ & $y_1 \neq 0$.

$$\Downarrow$$

$$b_1 = 0 \overset{(1)}{\Rightarrow} \underset{y_2 \neq 0}{b_2 y_2 = 0} \Rightarrow b_2 = 0$$

So (1) $\Rightarrow b_1 = b_2 = 0 \Rightarrow \{y_1, y_2\}$ is linearly-indepen

Problem 6 (25 points) Solve the following problem.

$$u_{tt} + u_t = u_{xx}, \qquad x \in (0, \pi), \quad t > 0$$
$$u(0,t) = 0, \quad u(\pi.t) = 0, \qquad t > 0$$
$$u(x,0) = 0, \quad u_t(x,0) = \sin(2x) + 3\sin(4x), \quad x \in (0.\pi).$$

$u(x,t) = X(x)\,T(t) \Rightarrow \quad X T'' + X T' = X'' T$

$\dfrac{T'' + T'}{T} = \dfrac{X''}{X} = \lambda \Rightarrow \begin{cases} X'' = \lambda X & ① \\ T'' + T' - \lambda T = 0 & ② \end{cases}$

$u(0,t) = 0 \Rightarrow X(0) = 0$ ③
$u(\pi,t) = 0 \Rightarrow X(\pi) = 0$ ④

①, ③, ④ $\Rightarrow \quad \lambda = -n^2, \; n \in \mathbb{Z}^+, \quad X = a \sin(nx)$

② $\Rightarrow \quad T'' + T' + n^2 T = 0$
$r^2 + r + n^2 = 0 \Rightarrow r = \dfrac{-1 \pm \sqrt{1-4n^2}}{2}$

Becaus $n \geq 1 \quad 1-4n^2 < 0 \Rightarrow r = -\frac{1}{2} \pm \frac{i}{2}\sqrt{4n^2-1} = -\frac{1}{2} \pm i\sqrt{n^2-}$

$\Downarrow$

$T(t) = e^{-\frac{t}{2}}\left[c_1 \sin\left(\sqrt{n^2-\tfrac{1}{4}}\, t\right) + c_2 \cos\left(\sqrt{n^2-\tfrac{1}{4}}\, t\right)\right]$

$\Downarrow$

$u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{t}{2}}\left[a_n \cos\left(\sqrt{n^2-\tfrac{1}{4}}\, t\right) + b_n \sin\left(\sqrt{n^2-\tfrac{1}{4}}\, t\right)\right]\sin(nx)$

$u(x,0) = 0 \Rightarrow \sum_{n=1}^{\infty} a_n \sin(nx) = 0 \quad \Rightarrow \boxed{a_n = 0}$

$u_t(x,t) = \sum_{n=1}^{\infty}\Big\{\left(-\tfrac{1}{2}\right)\left[\overset{0}{a_n}\cos\left(\sqrt{n^2-\tfrac{1}{4}}\, t\right) + b_n \sin\left(\sqrt{n^2-\tfrac{1}{4}}\, t\right)\right] +$
$\sqrt{n^2-\tfrac{1}{4}}\left[-\overset{0}{a_n}\sin\left(\sqrt{n^2-\tfrac{1}{4}}\, t\right) + b_n \cos\left(\sqrt{n^2-\tfrac{1}{4}}\, t\right)\right]\Big\} e^{-\frac{t}{2}} \sin$

$\Downarrow$

$u_t(x,0) = \sum_{n=1}^{\infty} \sqrt{n^2-\tfrac{1}{4}}\, b_n \sin(nx) \Rightarrow \sum_{n=1}^{\infty} \sqrt{n^2-\tfrac{1}{4}}\, b_n \sin(nx) = \sin(2x$
$+ 3\sin$

$\Rightarrow \boxed{\begin{array}{l} b_n = 0 \quad \text{for} \quad n \neq 2 \text{ and } n \neq 4 \\ b_2 = \dfrac{1}{\sqrt{4-\frac{1}{4}}} = \dfrac{2}{\sqrt{15}}, \quad b_4 = \dfrac{3}{\sqrt{16-\frac{1}{4}}} = \dfrac{6}{\sqrt{63}} \end{array}}$

$\Rightarrow \boxed{u(x,t) = e^{-\frac{t}{2}}\left[\dfrac{2}{\sqrt{15}} \sin\left(\dfrac{\sqrt{15}\,t}{2}\right)\sin(2x) + \dfrac{6}{\sqrt{63}} \sin\left(\dfrac{\sqrt{63}\,t}{2}\right)\sin(4x)\right.}$