Problem 2 (*15*) Check that the function $y(x) = x^2$ is a solution of the equation

$$x^2y'' - 3xy' + 4y = 0$$

and find solution of this equation which satisfies the conditions

$$y(1) = 1, \quad y'(1) = 3.$$

This is Euler equation. We define a new variable $t = \ln x$, so that $x = e^t$. Then, by chain rule, $\frac{dy}{dx}$

$= \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x} = \frac{dy}{dt} \cdot \frac{1}{e^t}$. Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(\frac{dy}{dt} \cdot \frac{1}{e^t}\right) \underset{\text{chain rule}}{=} \frac{d}{dt}\left(\frac{dy}{dt} \cdot \frac{1}{e^t}\right) \cdot \underbrace{\frac{dt}{dx}}_{= \frac{1}{x} = \frac{1}{e^t}}$

$= \frac{d^2y}{dt^2} \cdot \frac{1}{e^t} \cdot \frac{1}{e^t} - \frac{dy}{dt} \cdot \frac{1}{e^t} \cdot \frac{1}{e^t} = \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \frac{1}{e^{2t}}$.

putting these in the equation,

$$(e^t)^2 \cdot \left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \cdot \frac{1}{e^{2t}} - 3 \cdot e^t \cdot \frac{dy}{dt} \cdot \frac{1}{e^t} + 4 \cdot y = 0$$

$\Rightarrow \frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 4y = 0$. Characteristic polyno-

-mial is equal to $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$.

Thus the general solution is of the form

$y(t) = c_1 \cdot e^{2t} + c_2 t e^{2t}$. Then the solution of the

original equation is $y(x) = c_1 e^{2 \cdot \ln x} + c_2 \ln x \cdot e^{2\ln x}$

$= c_1 x^2 + c_2 x^2 \ln x$. Using initial conditions,

$y(1) = c_1 = 1$, $\quad y'(1) = 2c_1 x + 2c_2 x \ln x + c_2 x^2 \frac{1}{x} \Big|_1$

$= 2c_1 + c_2 = 3 \Rightarrow c_2 = 1$.

Thus $y(x) = x^2 + x^2 \ln x$. //

Problem 5 (20 points) Consider the problem:

$$\begin{cases} u_t = 2u_{xx}, \ x \in (0,1), t > 0, \\ u(x,0) = \sin(\pi x) + 3\sin(2\pi x), \ x \in (0,1), \\ u(0,t) = u(1,t) = 0, \ t > 0. \end{cases} \qquad (A)$$

a) Solve the problem (A).

b) Show that the problem (A) has a unique solution.

a-) The solution of this heat equation is given by the formula $u(x,t) = \sum_{n=1}^{\infty} c_n \cdot e^{-n^2 \pi^2 \cdot 2 \cdot t} \sin(n\pi x)$,
where c_n's are coefficients of fourier sin series of $u(x,0) = \sin(\pi x) + 3\sin(2\pi x)$. So $c_1 = 1$, $c_2 = 3$ and rest are 0.

Thus $u(x,t) = e^{-2\pi^2 t} \sin(\pi x) + 3 \cdot e^{-8\pi^2 t} \sin(2\pi x)$.

b-) Assume two functions $u(x,t)$ and $v(x,t)$ satisfies the system. Lets show $u(x,t) = v(x,t)$. Since $u_t = 2u_{xx}$ and $v_t = 2v_{xx}$, substracting, $(u-v)_t = 2(u-v)_{xx}$.

Also $u(x,0) = v(x,0) = \sin(\pi x) + 3\sin(2\pi x)$, so $u-v(x,0) = 0$.

and similarly $u-v(0,t) = u-v(1,t) = 0$ for $t > 0$.

So the function $w = u - v$ satisfies the system

$w_t = 2w_{xx}$, (*) $w(x,0) = 0$, $w(0,t) = w(1,t) = 0$.

Our purpose is to show $w = 0$. Multiplying (*) by w and integrating, we get $\int_0^1 w w_t \, dx = 2\int_0^1 w_{xx} w \, dx$.

integration by parts for RHS

$$\Rightarrow \frac{d}{dt} \int_0^1 \frac{w^2}{2} dx = \underbrace{2 w_x w \Big|_0^1}_{0 \text{ by boundary conditions}} - 2\int_0^1 w_x^2 dx$$

$$\Rightarrow \frac{d}{dt} \int_0^1 \frac{w^2}{2} dx = -2\int_0^1 w_x^2 dx \le 0.$$ Thus $\int_0^1 w^2 dx$ is non increasing.

So $\int_0^1 w^2(x,t) dx \le \int_0^1 w^2(x,0) dx = 0$. But $w^2(x,t)$ is nonnegative, so $w^2(x,t) = 0 \Rightarrow w(x,t) = 0$ //

Problem 6 (20 points) Solve the problem

$$\begin{cases} u_{tt} = 9u_{xx}, \ x \in (0,\pi), t > 0, \\ u(x,0) = 0, \ \ u_t(x,0) = 3(\pi - x), \ \ x \in (0,\pi), \\ u(0,t) = u(\pi,t) = 0, \ t > 0. \end{cases}$$

The general solution of this wave equation is

$u(x,t) = \sum_{n=1}^{\infty} k_n \cdot \sin(nx) \cdot \sin(3nt)$ where $3n \cdot k_n =$

$\frac{2}{\pi} \int_0^{\pi} 3 \cdot (\pi - x) \sin(nx)\, dx$. (pages 510-511 of the textbook);

Lets calculate k_n's. $k_n = \frac{2}{\pi n} \int_0^{\pi} (\pi - x) \sin(nx)\, dx$.

$$= \underbrace{\frac{2}{n} \int_0^{\pi} \sin(nx)\, dx}_{I_1} - \underbrace{\frac{2}{\pi n} \int_0^{\pi} x \cdot \sin(nx)\, dx}_{I_2}.$$

$$I_1 = \frac{2}{n}\left(\frac{-\cos(nx)}{n}\right)\Big|_0^{\pi} = \frac{2}{n^2} \cdot (1 - \cos(n\pi)).$$

$$I_2 = \frac{2}{\pi n} \int_0^{\pi} x \sin(nx)\, dx = -\frac{2}{\pi n} x \cdot \frac{\cos(nx)}{n}\Big|_0^{\pi} + \frac{2}{\pi n} \int_0^{\pi} \frac{\cos(nx)}{n}$$

$$= -\frac{2}{n^2} \cdot \cos(n\pi) + \frac{2}{\pi n} \cdot \underbrace{\frac{\sin(nx)}{n^2}\Big|_0^{\pi}}_{= 0}$$

$$= -\frac{2}{n^2} \cos(n\pi).$$

Thus $k_n = I_1 - I_2 = \frac{2}{n^2}\left(1 - \cos(n\pi) + \cos(n\pi)\right) = \frac{2}{n^2}$.

Finally $u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n^2} \cdot \sin(nx) \cdot \sin(3nt)$.