Challenging problems 2

Problem 1. Prove the following Lemma.

Lemma A Let $\Psi(t), t \geq 0$ be twice continuously differentiable function such that

$$\Psi(0) > 0, \quad \Psi'(0) > 0$$

and

$$\Psi''(t)\Psi(t) - 4\left[\Psi'(t)\right]^2 \geq 0, \ t > 0.$$

Then there exists T_0 such that

$$\Psi(t) \to \infty, \quad \text{as} \ \ t \to T_0.$$

Hint. Consider the function $\Phi(t) := [\Psi(t)]^{-\frac{1}{2}}$ and show that this function is concave.

Problem 2. Use the Lemma A to find conditions for the numbers y_0, y_1 for which solution of the prblem

$$y''(t) + y(t) = y^3(t), \quad y(0) = y_0, \ y'(0) = y_1$$

tend to infinity in a finite time.