Challenging problems 1

Problem 1. Let $a(t)$ and $b(t)$ be contiinous on $[0,\infty)$, $a(t)\geq a_0>0, \forall t\geq 0$ and

$$b(t)\to 0, \quad \text{as} \quad t\to 0.$$

Show that all solutions of the equation

$$y'(t)+a(t)y(t)=b(t) \tag{1}$$

tend to zero as $t\to\infty$.

Solution: First we mutiply (1) by $e^{\int_{t_0}^t a(s)ds}$ and obtain

$$\frac{d}{dt}\left(y(t)e^{\int_{t_0}^t a(s)ds}\right)=b(t)e^{\int_{t_0}^t a(s)ds}$$

Integrating this equality over the interval (t_0,t) we get

$$y(t)=y(t_0)e^{-\int_{t_0}^t a(s)ds}+e^{-\int_{t_0}^t a(s)ds}\int_{t_0}^t b(s)e^{\int_{t_0}^s a(\tau)d\tau}ds$$

Since $a(t)\geq a_0>0$ we have

$$e^{-\int_{t_0}^t a(s)ds}\leq e^{-c(t-t_0)}$$

Thus the first term on the right-hand side of (2) tends to zero as $t\to\infty$.

It is not difficult to show that the second integral also tends to zero as $t\to\infty$ employing the L'Hospital's rule.

Problem 2. Suppose that a function $f(y)$ is continuous on the interval $(\alpha,\beta]$

$$f(a)=f(b)=0, \quad \text{and} \quad f(y)>0, \quad \forall y\in(\alpha,\beta).$$

Show that if $y(t)$ is a solution of the problem

$$\begin{cases} y'(t) = f(y(t)), & t \in (a,b), \\ y(0) = y_0, & y_0 \in (a,b), \end{cases} \tag{A}$$

then

$$\lim_{t\to\infty} y(t) = b.$$

Solution: It is clear that the fucntions $y(t) = a,\ y(t) = b,\ \forall t \in (a,b)$ are solutions of the equation.

Since $f(y) > 0$ for all $y \in (a,b)$, a solution $u(x)$ of the equation satisfying the initial condition $u(0) = y_0,\ y_0 \in (a,b)$ is a momotonly increasing function for $t \geq 0$.. Because

$$u'(t) = f(u(t)) > 0, \quad \forall t \geq 0.$$

On the other hand the function $u(t)$ can't reach the value b. Otherwise the equation (A) whould have two different solutions under the intial condition $y(t_0) = b$ for some $t_1 > 0$. As a monotonly increasing and bounded above function, $u(t)$ has limit

$$\lim_{t\to\infty} u(t) = d_0.$$

Let us show that $b_0 = b$. Suppose that $b_0 \neq b$,then since $u(t) \to b_0$ as $t \to \infty$ we have

$$\lim_{t\to\infty} u'(t) = 0.$$

On the other hand

$$\lim_{t\to\infty} u'(t) = \lim_{t\to\infty} f(u(t)) = f(b_0) > 0.$$

Thus $b_0 = b$.