Challenging problems 1

Problem 1. Let $a(t)$ and $b(t)$ be contiinous on $[0, \infty)$, $a(t) \geq a_0 > 0, \forall t \geq 0$ and

$$b(t) \to 0, \quad \text{as} \quad t \to 0.$$

Show that all solutions of the equation

$$y'(t) + a(t)y(t) = b(t)$$

tend to zero as $t \to \infty$.

Problem 2. Suppose that a function $f(y)$ is continuous on the interval $[a, b]$

$$f(a) = f(b) = 0, \quad \text{and} \quad f(y) > 0, \quad \forall y \in (a, b).$$

Show that if $y(t)$ is a solution of the problem

$$\begin{cases} y'(t) = f(y(t)), \quad t \in (a, b), \\ y(0) = y_0, \quad y_0 \in (a, b), \end{cases}$$

then

$$\lim_{t \to \infty} y(t) = b.$$