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On the purely irregular fundamental group

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Based on the (not yet fully understood analogy) between irregular connections and wild ramification, we define a purely irregular fundamental group for complex algebraic varieties and prove some results about this fundamental group which are analogous to the p-adic étale fundamental group of algebraic varieties over fields of characteristic p.

1 Introduction

Let $X/\mathbb{C}$ be a smooth and connected algebraic variety over the field of complex numbers $\mathbb{C}$ and let $\mathrm{Mic}(X)$ denote the tannakian category of vector bundles with integrable connection on X. Note that by the Riemann-Hilbert correspondence [2] the *solution functor* defines an equivalence between the subcategory of $\mathrm{Mic}(X)$ consisting of regular connections and the category of $\mathbb{C}$-local systems on X_{an}.

However we will be interested in a subcategory of $\mathrm{Mic}(X)$ consisting of irregular connections which are iterated extensions of line bundles with connections. Our main interest will be the tannakian group of this category, which we will denote $\pi_{1,spi}(X)$.

Our reason for this is to state complex analogs of some of the results of Katz and Kedlaya on the *p-adic* étale fundamental groups of affine varieties over fields of *characteristic p* [4], [6]. Namely it is proven in Prop. 1.4.2 in [4] that if k is a field of characteristic p the natural pull-back functor from the category of finite etale coverings of $k[x]$ with monodromy group a p-group to the category of finite etale coverings of $k((x^{-1}))$ with monodromy group a p-group is an equivalence of categories. Our Theorem 2.0.7 below might be viewed as the complex analog of this fact. Similarly it is proven in [6, (2.6.12)] that $\pi_1^p(\mathbb{G}_m, *) \simeq \pi_1^p(\mathbb{A}^1, *) \times \pi_1^p(\mathbb{A}^1, *)$, when k is algebraically closed of characteristic p. Using Malgrange's theory of Stokes structures and the irregular Riemann-Hilbert correspondence in dimension one we prove a complex analog of this fact in Theorem 2.0.10 below. One expects such statements over $\mathbb{C}$ because of the analogy between the irregular phenomena in characteristic zero and the wild ramification in characteristic p [5].

2 Purely irregular connections

Definition 2.1 Let $\mathrm{Mic}_{pi}(X)$ denote the full subcategory of the category of modules with integrable connection $\mathrm{Mic}(X)$, consisting of those modules with connection (E, ∇) on X such that the associated analytic object (E_{an}, ∇_{an}) on X_{an} is trivial. We call such objects *purely irregular connections*.

Of course the name is slightly misleading since the category of purely irregular connections contains some regular connections as well, but these are the ones that are of the form $(\mathcal{O}, d)^{\oplus n}$. If $X/\mathbb{C}$ is proper then $\mathrm{Mic}_{pi}(X)$ is the trivial tannakian category.

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The following category is the one of main interest for us since it has the property that its objects are iterated extensions of rank one objects and hence is analogous to finite etale extensions with a p-power monodromy group which is the object of study in [6].

Definition 2.2 We call an object (E, ∇) in $\mathrm{Mic}_{pi}(X)$ *special* if it has a finite filtration, beginning with 0 and ending with E, by subbundles with connection such that the graded pieces are of rank at most one. We denote the category of special purely irregular connections by $\mathrm{Mic}_{spi}(X)$.

We call an extension of objects in $\mathrm{Mic}(X)$ *analytically split* if the extension splits over X_{an}. The special purely irregular connections are exactly those which are successive analytically split extensions of purely irregular line bundles with connection.

Remark 2.3 An example of a purely irregular connection which is not special is $(\mathcal{O}^{\oplus 2}, \nabla)$ on $\mathbb{A}^1$, where ∇ is given by $\nabla(f, g) := (df - f \cdot z \cdot dz, dg - f \cdot dz)$. This follows from the observation that if $f, g \in \mathbb{C}[z]$ and $\omega \in \Omega^1_{\mathbb{C}[z]/\mathbb{C}}$ satisfy $df - f \cdot z \cdot dz = -f \cdot \omega$ and $dg - f \cdot dz = -g \cdot \omega$ then $\omega = z \cdot dz$ and f and g are zero.

Definition 2.4 For $x \in X(\mathbb{C})$, the map $\omega(x) : \mathrm{Mic}_{pi}(X) \to Vec_{\mathbb{C}}$, that sends (E, ∇) to $E(x)$ is a fiber functor and we denote by $\pi_{1,pi}(X, x)$ the tannakian fundamental group of $\mathrm{Mic}_{pi}(X)$ at $\omega(x)$, that is, it is the affine group scheme over $\mathbb{C}$ that represents $\underline{Aut}_{\otimes}(\omega(x))$. Similarly, we define $\pi_{1,spi}(X, x)$.

Lemma 2.5 *For $x, y \in X(\mathbb{C})$ there is a canonical isomorphism between $\omega(x)$ and $\omega(y)$ as fiber functors on $\mathrm{Mic}_{pi}(X)$ and therefore there is a canonical isomorphism between $\pi_{1,pi}(X, x)$ and $\pi_{1,pi}(X, y)$. From now on we will omit the basepoint from the notation of the fundamental group and write $\pi_{1,pi}(X)$ and $\pi_{1,spi}(X)$ instead.*

P r o o f . A path in $X(\mathbb{C})$ from x to y determines an isomorphism between $\omega(x)$ and $\omega(y)$. The isomorphism is unique up to the action of $\pi_1(X_{an}(\mathbb{C}), y)$ on $\omega(y)$. But since by definition for every element in $\mathrm{Mic}_{pi}(X)$ the associated local system is trivial, this action is trivial as well. This shows that the isomorphism does not depend on the path. □

The following shows that, just like the topological fundamental group, the purely irregular fundamental group depends only on the codimension 1 information.

Lemma 2.6 *Let $\overline{X}/\mathbb{C}$ be a smooth, connected variety, let $Z \subseteq \overline{X}$ be a closed subvariety of codimension ≥ 2, and let $X := \overline{X} \setminus Z$. Then the natural map $\pi_{1,pi}(X) \to \pi_{1,pi}(\overline{X})$ is an isomorphism.*

P r o o f . Let $j : X \to \overline{X}$ be the inclusion. If (E, ∇) is a vector bundle with integrable connection on X, the proof of Proposition 4.6 in [1] on p. 24 shows that $(j_* E, j_* \nabla)$ is a vector bundle with integrable connection on $\overline{X}$. From this it follows immediately that restriction gives an equivalence of categories between $\mathrm{Mic}(\overline{X})$ and $\mathrm{Mic}(X)$. Finally note that a vector bundle with connection on $\overline{X}_{an}$ is trivial if and only if its restriction to X_{an} is, since the map $\pi_1(X_{an}, *) \to \pi_1(\overline{X}_{an}, *)$ is an isomorphism. From these remarks the statement follows immediately. □

Let $\mathbb{C}\{\{x\}\}$ denote the ring of *convergent* power series in x and $\mathbb{C}\{\{x, x^{-1}\}\}$ (resp. $\mathbb{C}\{(x)\}$) denote the ring of *convergent* Laurent series in x (resp. which are also meromorphic at zero). By *purely irregular connections* over $\mathbb{C}\{(x^{-1})\}$, we mean those which become trivial when considered over $\mathbb{C}\{\{x, x^{-1}\}\}$.

Theorem 2.7 *The natural functor*

$$\mathrm{Mic}_{spi}(\mathrm{Spec}(\mathbb{C}[x])) \to \mathrm{Mic}_{spi}(\mathrm{Spec}(\mathbb{C}\{(x^{-1})\}))$$

induced by the inclusion $\mathbb{C}[x] \to \mathbb{C}\{(x^{-1})\}$ is an equivalence of categories.

P r o o f . Since the complex line has trivial fundamental group every object in $\mathrm{Mic}(\mathrm{Spec}(\mathbb{C}[x]))$ is purely irregular and every extension is analytically split. The rank one purely irregular connections over $\mathbb{C}[x]$ are in bijection with $x \cdot \mathbb{C}[x]$. The bijection associates to $F(x) \in x \cdot \mathbb{C}[x]$ the connection $(\mathcal{O}, \nabla)$ with $\nabla(1) = -F'(x) \cdot dx$, in other words the connection whose solution is the function $e^{F(x)}$.

Over $\mathbb{C}\{(x^{-1})\}$ a connection $(\mathcal{O}, \nabla)$ is purely irregular if and only if $\mathrm{Res}_{\infty} \nabla(1) \in \mathbb{Z}$. If $G(x^{-1}) \in \mathbb{C}\{\{x^{-1}\}\}$ and $n \in \mathbb{Z}$ then $x^n \cdot e^{G(x^{-1})} \in \mathbb{C}\{(x^{-1})\}$. Therefore if $\nabla(1) \in \mathbb{C}\{\{x^{-1}\}\}$ then the connection is trivial on $\mathrm{Spec}(\mathbb{C}\{(x^{-1})\})$. These imply that there is a bijection between the isomorphism classes of purely irregular connections of rank one over $\mathbb{C}\{(x^{-1})\}$ and the set $\mathbb{C}\{(x^{-1})\}/\mathbb{C}\{\{x^{-1}\}\} \simeq x \cdot \mathbb{C}[x]$. This shows that the functor in the statement of the theorem establishes an equivalence between rank one objects.

If $(\mathcal{O}, \nabla)$ and $(\mathcal{O}, \nabla')$ are two connections of rank one

$$Ext^1_{spi}((\mathcal{O}, \nabla), (\mathcal{O}, \nabla')) \simeq H^1_{dR}(\cdot, (\mathcal{O}, \nabla') \otimes (\mathcal{O}, \nabla)^{-1})$$

where the dot is $\mathbb{A}^1$ or $\mathrm{Spec}(\mathbb{C}\{(x^{-1})\})$. Since

$$H^i_{dR}(\cdot, (M, \nabla)) = 0$$

for $i \geq 2$ to finish the proof it is sufficient to show that the map induces isomorphisms on $H^1_{dR}(\cdot, (\mathcal{O}, \nabla))$. Let ∇ be such a connection that corresponds to $F(x) \in x \cdot \mathbb{C}[x]$. Assume that $g(x) \cdot dx$, where $g(x) \in \mathbb{C}[x]$ becomes a coboundary after it is pulled back to a form on $\mathrm{Spec}(\mathbb{C}\{(x^{-1})\})$. Therefore there exists an $f \in \mathbb{C}\{(x^{-1})\}$ such that $g = f' - F' \cdot f$. Since f is a convergent Laurent series after multiplying this equality with e^{-F} and integrating we get $f = e^F \int e^{-F} \cdot g + C \cdot e^F$ for some $C \in \mathbb{C}$. In particular, $f \in \mathbb{C}\{\{x\}\} \cap \mathbb{C}\{(x^{-1})\} = \mathbb{C}[x]$. This proves injectivity on $H^1_{dR}(\cdot, (\mathcal{O}, \nabla))$.

To see the surjectivity of the map let $g(x) \in \mathbb{C}\{(x^{-1})\}$. Then $g(x) \cdot dx$ is the boundary of $f = e^F \int e^{-F} g$ and $f = f_0 + f_\infty$, with $f_0 \in \mathbb{C}\{\{x\}\}$ and $f_\infty \in \mathbb{C}\{\{x^{-1}\}\}$. Note that since $g = f' - F' \cdot f$, with $F(x) \in \mathbb{C}[x]$ and $g(x) \in \mathbb{C}\{(x^{-1})\}$, $f_0' - F' \cdot f_0$ is a polynomial $p(x)$. This shows that $p(x) \cdot dx$ maps to $g(x) \cdot dx$ in the first cohomology group. $\square$

Remark 2.8 The theorem above can be used to define a fiber functor on the tensor category

$$\mathrm{Mic}_{spi}(\mathrm{Spec}(\mathbb{C}\{(x^{-1})\})),$$

by using a fiber functor on $\mathrm{Mic}_{spi}(\mathrm{Spec}(\mathbb{C}[x]))$.

Remark 2.9 Note that Kedlaya's theorem shows that if k is of characteristic p then the inclusion $k[x] \rightarrow k((x^{-1}))$ induces an isomorphism of the p-adic fundamental groups (2.6.11 in [6]), where $k((x^{-1}))$ denotes the *formal* Laurent series in x^{-1} which are meromorphic at infinity. An analogous statement cannot be true over $\mathbb{C}$: in fact the composition $\mathrm{Mic}_{spi}(\mathbb{C}[x]) \rightarrow \mathrm{Mic}(\mathbb{C}[x]) \rightarrow \mathrm{Mic}(\mathbb{C}((x^{-1})))$ induced by the inclusion $\mathbb{C}[x] \rightarrow \mathbb{C}((x^{-1}))$ is not even fully-faithful. In order to see this note the following. Let $(\mathcal{O}, \nabla)$ denote the line bundle with connection given by $\nabla(1) = -x \cdot dx$. The isomorphism classes of extensions of $(\mathcal{O}, d)$ by $(\mathcal{O}, \nabla)$ are given by $H^1_{dR}(\mathbb{A}^1, (\mathcal{O}, \nabla))$ and $H^1_{dR}(\mathrm{Spec}(\mathbb{C}((x^{-1}))), (\mathcal{O}, \nabla))$ on $\mathbb{A}^1$ and $\mathrm{Spec}(\mathbb{C}((t^{-1})))$ respectively. It is easy to see that in the first cohomology group the cocycle $1 \cdot dx$ is non-zero, whereas in the second cohomology group it is the coboundary of

$$-x^{-1} + \sum_{1 \leq n} (-1)^{n+1} \cdot \frac{(2n)!}{2^n \cdot n!} \cdot x^{-(2n+1)}.$$

This shows that non-split extensions in $\mathrm{Mic}_{spi}(\mathbb{C}[x])$ can split in $\mathrm{Mic}(\mathbb{C}((x^{-1})))$, and hence the functor above cannot be fully-faithful.

Let G_1 and G_2 be two affine group schemes over $\mathbb{C}$. Consider the category consisting of finite dimensional vector spaces V over $\mathbb{C}$ together with two (possibly non-commuting) representations of G_1 and G_2 over V. This is clearly a neutral tannakian category over $\mathbb{C}$ together with the forgetful functor to $Vec_\mathbb{C}$ as a fiber functor. The fundamental group of this category is an affine group scheme over $\mathbb{C}$ which we denote by $G_1 * G_2$, and call the *free product* of G_1 and G_2. The extension of this concept to finitely many groups $G_1, \ldots, G_n$ is clear and it also shows the associativity of the product. By tannaka duality [3], it is clear that the free product of G_1 and G_2 is nothing other then their coproduct in the category of affine group schemes over $\mathbb{C}$. Therefore this concept is the exact analog of the concept of free product of discrete groups.

Theorem 2.10 *Let $\overline{X}/\mathbb{C}$ be a smooth, projective curve then*

$$\pi_{1,pi}(\overline{X} \setminus \{x_1, \ldots, x_n\}) \simeq \pi_{1,pi}(\overline{X} \setminus \{x_1\}) * \cdots * \pi_{1,pi}(\overline{X} \setminus \{x_n\}) \simeq \pi_{1,pi}(\mathbb{A}^1)^{*n}.$$

Similarly for pi replaced with spi.

P r o o f.   Note that if (E, ∇) is a purely irregular connection on $X := \overline{X} \setminus \{x_1, \ldots, x_n\}$, then the corresponding local system E^∇ of (E, ∇) is trivial. By the irregular Rimann-Hilbert correspondence (Théorème (2.2)

in [7]) (E, ∇) is completely determined by the Stokes structure on E^{∇}. Hence the category of purely irregular connections on X is equivalent to the category of Stokes structures on trivial local systems on X. But a Stokes structure on a trivial local system L is a collection of Stokes structures on $L|_{D_{x_i}}$, for $1 \leq i \leq n$, where D_{x_i} is a puctured disc around x_i. From this and the definition of the free product the statement follows immediately.   □

Remark 2.11 The above result shows that $\pi_{1,pi}(\mathbb{G}_m) \simeq \pi_{1,pi}(\mathbb{A}^1) * \pi_{1,pi}(\mathbb{A}^1)$, which might be viewed as a complex analog of (2.6.12) in [6].

Corollary 2.12 *Let* $X := \mathbb{P}^1 \setminus \{x_1, \ldots, x_n\}$ *then* $\pi_{1,spi}(X) \simeq \pi_{1,spi}(\mathbb{C}\{(x^{-1})\})^{*n}$.

Theorem 2.13 *Let X and Y be smooth varieties over $\mathbb{C}$. If X is complete then the map* $p_{2,*} : \pi_{1,spi}(X \times Y) \to \pi_{1,spi}(Y)$, *induced by the second projection, is an isomorphism.*

P r o o f.   First let us show that the pull-back functor $\mathrm{Mic}_{spi}(Y) \to \mathrm{Mic}_{spi}(X \times Y)$ is an essentially surjective if we restrict ourselves to rank one objects.

Assume that we are given a purely irregular line bundle with connection (L, ∇) on $X \times Y$. Since a purely irregular vector bundle with connection on a complete variety is trivial we see that $L|_{X \times y} \simeq \mathcal{O}_X$ for all $y \in Y$. Therefore, by the Seesaw Principle (Corollary 6, p. 54 [9]) we will assume without loss of generality that $L = p_2^* L_0$ for some line bundle L_0 on Y. Choose $x_0 \in X$ and let $(L_0, \nabla_0) := (L, \nabla)|_{x_0 \times Y}$. We claim that $p_2^*(L_0, \nabla_0) = (L, \nabla)$. Since this question is local on Y, we will assume without loss of generality that $L_0 = \mathcal{O}_Y$. The connection on $\mathcal{O}_{X \times Y}$ is given by an algebraic 1-form $\nabla(1) = -df/f$ for some analytic function f on $X \times Y$. Since, X being complete, f is constant on all $X \times y$, we see that $df/f = p_2^*(df/f)|_{x_0 \times Y}$. This in turn implies that $(L, \nabla) = p_2^*(L_0, \nabla_0)$. Hence we have the essential surjectivity on rank one objects.

Next we would like to see the essential surjectivity in general. We would like to see that for any $(E, \nabla) \in \mathrm{Mic}_{spi}(Y)$ the map

$$\mathrm{Ext}^1_{Y,spi}((\mathcal{O}, d), (E, \nabla)) \to \mathrm{Ext}^1_{X \times Y, spi}((\mathcal{O}, d), p_2^*(E, \nabla))$$

is an isomorphism. For $(F, \nabla) \in \mathrm{Mic}_{spi}(Z)$ let $H^1_{dR,spi}(Z, (F, \nabla))$ be the kernel of the map $H^1_{dR}(Z, (F, \nabla)) \to H^1_{dR}(Z_{an}, (F, \nabla)_{an})$. Then we need to show that

$$H^1_{dR,spi}(Y, (E, \nabla)) \to H^1_{dR,spi}(X \times Y, p_2^*(E, \nabla))$$

is an isomorphism. Note that by the Kunneth formula

$$H^1_{dR}(X \times Y, p_2^*(E, \nabla)) \simeq (H^1_{dR}(X, (\mathcal{O}, d)) \otimes H^0_{dR}(Y, (E, \nabla))) \oplus (H^0_{dR}(X, (\mathcal{O}, d)) \otimes H^1_{dR}(Y, (E, \nabla))).$$

Similarly for the cohomology of the analytic space $(X \times Y)_{an}$. Next note that since

$$H^1_{dR}(X, (\mathcal{O}, d)) \simeq H^1_{dR}(X_{an}, (\mathcal{O}, d))$$

and $H^0_{dR}(Y, (E, \nabla)) \to H^0_{dR}(Y_{an}, (E, \nabla)_{an})$ is injective we see that

$$H^1_{dR,spi}(X \times Y, p_2^*(E, \nabla)) \subseteq H^0_{dR}(X, (\mathcal{O}, d)) \otimes H^1_{dR}(Y, (E, \nabla)) \simeq H^1_{dR}(Y, (E, \nabla)).$$

This immediately implies the bijectivity. This in turn implies the essential surjectivity of $p_2^* : \mathrm{Mic}_{spi}(Y) \to \mathrm{Mic}_{spi}(X \times Y)$, by induction on the rank of vector bundles with connection.

To see full-faithfulness, noting that

$$\mathrm{Hom}_Z((M_1, \nabla_1), (M_2, \nabla_2)) = \mathrm{H}^0_{dR}(Z, (M_2, \nabla_2) \otimes (M_1, \nabla_1)^{\vee}),$$

we need to show that the map

$$p_2^* : H^0_{dR}(Y, (E, \nabla)) \to H^0_{dR}(X \times Y, p_2^*(E, \nabla))$$

is an isomorphism, which is immediate from the Kunneth formula and the connnectedness of X.   □

Remark 2.14 Note that the above is the analog of the fact that $\pi_1(X \times Y, *) \simeq \pi_1(X, *) \times \pi_1(Y, *)$ for the ordinary topological fundamental group or the same fact for Nori's fundamental group scheme in case X and Y are complete [8]. The completeness assumption in the above statement is necessary. Choose 0 as the basepoint of $\mathbb{A}^n$. One can see that the natural map $\pi_{1,spi}(\mathbb{A}^2) \to \pi_{1,spi}(\mathbb{A}^1) \times \pi_{1,spi}(\mathbb{A}^1)$ is not an isomorphism as follows. First note that the map $\pi_{1,spi}(\mathbb{A}^2)^{ab} \to \pi_{1,spi}(\mathbb{A}^1)^{ab} \times \pi_{1,spi}(\mathbb{A}^1)^{ab}$ after abelianization has a section induced by the imbeddings $\mathbb{A}^1 \to \mathbb{A}^2$ using the maps $x \to (x, 0)$ and $x \to (0, x)$. Therefore if the map were an isomorphism then for any purely irregular line bundle with connnection (L, ∇) on $\mathbb{A}^2$,

$$(L, \nabla) \simeq p_1^*(L, \nabla)|_{\mathbb{A}^1 \times 0} \otimes p_2^*(L, \nabla)|_{0 \times \mathbb{A}^1}$$

Consider the connection on $\mathcal{O}_{\mathbb{A}^2}$ corresponding to the function e^{xy}, i.e. it is the connnection that sends 1 to $-(x \cdot dy + y \cdot dx)$. Then both $(\mathcal{O}, \nabla)|_{\mathbb{A}^1 \times 0}$ and $(\mathcal{O}, \nabla)|_{0 \times \mathbb{A}^1}$ are trivial, however $(\mathcal{O}, \nabla)$ is not.

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