

ON p -ADIC PERIODS FOR MIXED TATE MOTIVES OVER A NUMBER FIELD

ANDRE CHATZISTAMATIOU AND SINAN ÜNVER

ABSTRACT. For a number field, we have a Tannaka category of mixed Tate motives at our disposal. We construct p -adic points of the associated Tannaka group by using p -adic Hodge theory. Extensions of two Tate objects yield functions on the Tannaka group, and we show that evaluation at our p -adic points is essentially given by the inverse of the Bloch-Kato exponential map.

CONTENTS

Introduction	1
1. Filtered ϕ -modules and mixed Tate filtered ϕ -modules	2
1.1. Mixed Tate filtered ϕ -modules	2
1.2. The crystalline logarithmic point	6
1.3. The semistable logarithmic point	9
1.4. Tannaka group scheme of mixed Tate filtered ϕ -modules	14
2. Mixed Tate motives over a number field and logarithmic points	16
2.1. Mixed Tate motives	16
2.2. Crystalline characterization of unramified motives	18
2.3. Main theorem	20
2.4. Archimedean places	21
References	23

INTRODUCTION

For a number field E , one has an abelian category of mixed Tate motives $MT(E)$ [DG05]. A mixed Tate motive comes equipped with a weight filtration W , and the associated graded pieces are sums of Tate objects. There is a natural fibre functor ω defined by

$$\omega(M) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}_{-2n}^W(M));$$

we denote by G_ω the corresponding Tannaka group.

If \mathcal{O} denotes the ring of integers of E and $x \in \mathrm{Spec}(\mathcal{O})$ is a closed point, then Deligne and Goncharov construct a Tannaka subcategory $MT(\mathcal{O}_x)$ of $MT(E)$ consisting of motives which are unramified at x [DG05, 1.6]. We will denote its group of tensor automorphisms by G_x .

This work has been supported by the SFB/TR 45 “Periods, moduli spaces and arithmetic of algebraic varieties”.

To a mixed Tate motive M we can attach its p -adic realization M_p which is a representation of the Galois group of E with coefficients in \mathbb{Q}_p . If the point x lies over the prime p , then we can restrict in order to obtain a p -adic representation $M_{l,p}$ for the Galois group of the completion E_x at x . We will show that $M_{l,p}$ is always semistable. Furthermore, $M_{l,p}$ is crystalline if and only if M is unramified at x , i.e. $M \in MT(\mathcal{O}_x)$ (Theorem 2.2.3). In fact, p -adic representations attached to mixed Tate motives are contained in an abelian subcategory which admits a fibre functor τ similar to ω . Denoting by H_τ the corresponding Tannaka group over \mathbb{Q}_p , p -adic realization yields a group homomorphism

$$H_\tau \rightarrow G_\omega \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

The main purpose of this paper is to construct an $E_{x,st}$ -valued point η_{st} of H_τ , where $\text{Spec}(E_{x,st})$ is a 1-dimensional affine space over the field E_x . The E_x -valued points of $\text{Spec}(E_{x,st})$ correspond naturally to the extensions of the canonical logarithm $\log : \mathcal{O}_{E_x}^\times \rightarrow E$ to E^\times . Therefore, any choice of such an extension induces via η_{st} an E_x -valued point of H_τ and G_ω . For the Tannaka subcategory of crystalline representations the picture is simpler: if $H_{\tau,cris}$ denotes their Tannaka group and $\pi : H_\tau \rightarrow H_{\tau,cris}$ is the projection, then $\pi \circ \eta_{st}$ factors through $\text{Spec}(E_x)$ and we obtain an E_x -valued point η of $H_{\tau,cris}$. We denote by η_x^{ur} the image of η in G_x .

To state our main theorem, we need to recall how extensions M of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ in $MT(\mathcal{O}_x)$ give rise to functions on G_x for $n \geq 1$. The natural isomorphisms $\alpha : \mathbb{Q} \rightarrow \text{Hom}(\mathbb{Q}(n), \text{gr}_{-2n}^W M)$ and $\beta : \text{Hom}(\mathbb{Q}(0), \text{gr}_0^W M) \rightarrow \mathbb{Q}$ induce elements $\alpha^{-1} \in \omega(M)^\vee$ and $\beta^{-1} \in \omega(M)$; we set $M(\eta_x^{ur}) = \alpha^{-1}(\eta_x^{ur} \cdot \beta^{-1}(1))$.

Theorem (Theorem 2.3.3). *For all $n \geq 1$, the map*

$$\text{Ext}_{MT(\mathcal{O}_x)}^1(\mathbb{Q}(0), \mathbb{Q}(n)) \rightarrow E_x, \quad M \mapsto M(\eta_x^{ur}),$$

is the composition of the p -adic realization

$$\text{Ext}_{MT(\mathcal{O}_x)}^1(\mathbb{Q}(0), \mathbb{Q}(n)) \rightarrow \text{Ext}_{cris}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$$

and the inverse of the Bloch-Kato exponential map (2.3.1).

1. FILTERED ϕ -MODULES AND MIXED TATE FILTERED ϕ -MODULES

1.1. Mixed Tate filtered ϕ -modules.

1.1.1. Let K be a p -adic field with residue field k , i.e. $\text{char}(K) = 0$, K is complete with respect to a fixed discrete valuation and the residue field k is perfect of characteristic p . Let $W(k)$ be the ring of Witt vectors of k , $\sigma : W(k) \rightarrow W(k)$ the Frobenius lift and K_0 the field of fractions of $W(k)$.

1.1.2. We denote by MF_K^ϕ the category of filtered ϕ -modules, i.e. the objects are triples (M, ϕ, F) , where (M, ϕ) is an isocrystal over K_0 and F is a descending, exhaustive and separated filtration on $M_K = M \otimes_{K_0} K$. We denote by $MF_K^{\phi,N}$ the category of filtered (ϕ, N) -modules, i.e. objects are tuples (M, ϕ, N, F) with $(M, \phi, F) \in MF_K^\phi$ and $N : M \rightarrow M$ is a K_0 -linear endomorphism such that $N\phi = p\phi N$. We consider MF_K^ϕ as full subcategory of $MF_K^{\phi,N}$ via the functor $(M, \phi, F) \mapsto (M, \phi, 0, F)$.

The Dieudonné-Manin classification [Ma63, II, §4.1] implies, by descent, that every isocrystal (M, ϕ) over K_0 admits a slope decomposition

$$M = \bigoplus_{\lambda \in \mathbb{Q}} M_\lambda,$$

with $\phi(M_\lambda) = M_\lambda$ and $(M_\lambda, \phi|_{M_\lambda})$ is isoclynic of slope λ . From the relation $N\phi = p\phi N$, it follows that $N(M_\lambda) \subseteq M_{\lambda-1}$. In the following, we will use the notation:

$$M_{\leq \lambda} := \bigoplus_{\substack{\lambda' \in \mathbb{Q} \\ \lambda' \leq \lambda}} M_{\lambda'}, \quad M_{\geq \lambda} := \bigoplus_{\substack{\lambda' \in \mathbb{Q} \\ \lambda' \geq \lambda}} M_{\lambda'}.$$

Definition 1.1.3. We say that an object $(M, \phi, F) \in MF_K^\phi$ is a *mixed Tate filtered ϕ -module* if the following properties are satisfied:

- (1) There is an isomorphism of ϕ -modules

$$(M, \phi) \cong \bigoplus_{i \in I} (K_0, p^{n_i} \sigma),$$

for some index set I , and $n_i \in \mathbb{Z}$.

- (2) For all $i \in \mathbb{Z}$ the natural map

$$F^i M_K \rightarrow M_{\geq i} \otimes_{K_0} K$$

is an isomorphism.

We say that $(M, \phi, N, F) \in MF_K^{\phi, N}$ is a *mixed Tate filtered (ϕ, N) -module* if (M, ϕ, F) is a mixed Tate filtered ϕ -module.

We denote by MT_K^ϕ (resp. $MT_K^{\phi, N}$) the full subcategory of MF_K^ϕ (resp. $MF_K^{\phi, N}$) with mixed Tate filtered ϕ -modules (resp. (ϕ, N) -modules) as objects. The categories MT_K^ϕ and $MT_K^{\phi, N}$ are additive. Again, we consider MT_K^ϕ as full subcategory of $MT_K^{\phi, N}$.

For $(M, \phi, N, F) \in MT_K^{\phi, N}$, it follows from Property (1) that all the slopes of (M, ϕ) are integers. From Property (2) we conclude that the Hodge polygon of (M_K, F) equals the Newton polygon of (M, ϕ) .

Definition 1.1.4. (Tate objects) Let $n \in \mathbb{Z}$ be an integer. We define the *Tate object* $K(n) \in MT_K^\phi$ by

$$K(n) := (K_0, p^{-n} \sigma, F),$$

with F defined by

$$F^j = \begin{cases} K & \text{if } j \leq -n, \\ 0 & \text{if } j > -n. \end{cases}$$

Definition 1.1.5. (Weight filtration) Let $(M, \phi, N, F) \in MT_K^{\phi, N}$. Let $i \in \mathbb{Z}$ be an integer. We define an object $W_{2i}(M, \phi, N, F)$ in MF_K^ϕ by

$$W_{2i}(M, \phi, N, F) := (M_{\leq i}, \phi|_{M_{\leq i}}, N|_{M_{\leq i}}, F \cap M_{\leq i}).$$

We define an object $\text{gr}_{2i}^W(M, \phi, N, F)$ in MT_K^ϕ by

$$\text{gr}_{2i}^W(M, \phi, N, F) := (M_i, \phi|_{M_i}, \tilde{F}),$$

where \tilde{F} is defined as follows:

$$\tilde{F}^i M_i = M_i, \quad \tilde{F}^{i+1} M_i = 0.$$

Note that $N(M_{\leq i}) \subset M_{\leq i-1}$ and $N|_{M_{\leq i}}$ is well-defined.

Proposition 1.1.6. *Let $(M, \phi, N, F) \in MT_K^{\phi, N}$ and $i \in \mathbb{Z}$. The following statements hold.*

- (1) *The object $W_{2i}(M, \phi, N, F)$ is contained in $MT_K^{\phi, N}$.*
- (2) *There is an exact sequence*

$$(1.1.1) \quad 0 \rightarrow W_{2(i-1)}(M, \phi, N, F) \rightarrow W_{2i}(M, \phi, N, F) \rightarrow \mathrm{gr}_{2i}^W(M, \phi, N, F) \rightarrow 0.$$

Proof. It is sufficient to prove the statement for $(M, \phi, 0, F)$, i.e. for objects in MT_K^{ϕ} .

For (1). It is obvious that

$$W_{2i}(W_{2(i+1)}(M, \phi, F)) = W_{2i}(M, \phi, F),$$

for all (M, ϕ, F) . Therefore we may reduce to the case

$$W_{2(i+1)}(M, \phi, F) = (M, \phi, F).$$

In this case $M = M_{\leq i} \oplus M_{i+1}$, and we have to prove that for all $j \in \mathbb{Z}$ the map

$$F^j \cap (M_{\leq i} \otimes_{K_0} K) \rightarrow (M_{\leq i})_{\geq j} \otimes_{K_0} K$$

is an isomorphism. Since (M, ϕ, F) is an object in MT_K^{ϕ} , the map is injective. In particular, the map is an isomorphism for all $j \geq i+1$.

We need to show the surjectivity for $j \leq i$. By assumption, for every $m \in (M_{\leq i})_{\geq j} \otimes_{K_0} K$ there exists a preimage $m' \in F^j M_K$. By definition, the projection of m' to $M_{i+1} \otimes_{K_0} K$ vanishes, thus $m' \in F^j \cap (M_{\leq i} \otimes_{K_0} K)$.

For (2). There is an obvious morphism $W_{2(i-1)}(M, \phi, F) \rightarrow W_{2i}(M, \phi, F)$ in MT_K^{ϕ} . The morphism $W_{2i}(M, \phi, F) \rightarrow \mathrm{gr}_{2i}^W(M, \phi, F)$ is defined by the projection $M_{\leq i} \rightarrow M_i$. Since $F^{i+1} \cap (M_{\leq i} \otimes_{K_0} K) = 0$, the projection is compatible with the filtrations. Therefore the sequence (1.1.1) is well-defined.

In order to prove that the sequence is exact we need to show that it is an exact sequence of ϕ -modules and an exact sequence of filtered K -vector spaces. The first statement is obvious. For the second statement we note that all members in the sequence (1.1.1) are objects in MT_K^{ϕ} , thus the Hodge polygons equal the Newton polygons. In particular,

$$\dim(F^j \cap M_{\leq i}) = \dim(F^j \cap M_{\leq i-1}) + \dim \tilde{F}^j,$$

for all $j \in \mathbb{Z}$. This immediately implies the claim. \square

Corollary 1.1.7. *The category $MT_K^{\phi, N}$ is contained in the category of weakly admissible filtered (ϕ, N) -modules.*

Proof. We use the fact that weakly admissible filtered (ϕ, N) -modules are stable under extensions. Therefore the claim follows from Proposition 1.1.6 provided we prove that $\mathrm{gr}_{2i}^W(M, \phi, N, F)$ is weakly admissible for all $(M, \phi, N, F) \in MT_K^{\phi, N}$ and all $i \in \mathbb{Z}$. By Definition 1.1.5, $\mathrm{gr}_{2i}^W(M, \phi, N, F)$ is isomorphic to a direct sum of Tate objects $K(-i)$. Since Tate objects are (weakly) admissible, we are done. \square

In contrast to the category $MF_K^{\phi, N}$, the category of weakly admissible filtered (ϕ, N) -modules $MF_K^{\phi, N, wa}$ is an abelian category.

Proposition 1.1.8. *Let $f : (M, \phi_M, N_M, F_M) \rightarrow (M', \phi_{M'}, N_{M'}, F_{M'})$ be a morphism in $MT_K^{\phi, N}$. We denote by $\ker(f)$ and $\operatorname{coker}(f)$ the kernel of f and the cokernel of f in $MF_K^{\phi, N, wa}$, respectively. Then $\ker(f)$ and $\operatorname{coker}(f)$ are contained in $MT_K^{\phi, N}$. In particular, $MT_K^{\phi, N}$ is an abelian category.*

Proof. First, consider the full subcategory \mathcal{C} of isocrystals over K_0 with objects (M, ϕ) such that there exists an isomorphism

$$(M, \phi) \cong \bigoplus_{i \in I} (K_0, p^{n_i} \sigma).$$

It is easy to see that \mathcal{C} , as subcategory of the category of isocrystals, contains all the kernels and cokernels of morphisms in \mathcal{C} .

We denote by f_0 the induced morphism $(M, \phi_M) \rightarrow (M', \phi_{M'})$. Then

$$\ker(f) = (\ker(f_0), \phi|_{\ker(f_0)}, N|_{\ker(f_0)}, F \cap (\ker(f_0) \otimes_{K_0} K)).$$

We know that $\ker(f_0) \in \mathcal{C}$ and thus satisfies Property (1) of Definition 1.1.3. It remains to show that

$$F_M^i \cap (\ker(f_0) \otimes_{K_0} K) \rightarrow \ker(f_0)_{\geq i} \otimes_{K_0} K$$

is an isomorphism. We have a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & F_M^i \cap (\ker(f_0) \otimes_{K_0} K) & \longrightarrow & F_M^i & \longrightarrow & F_{M'}^i \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \ker(f_0)_{\geq i} \otimes_{K_0} K & \longrightarrow & M_{\geq i} \otimes_{K_0} K & \longrightarrow & M'_{\geq i} \otimes_{K_0} K. \end{array}$$

Moreover, both rows are exact, which implies Property (2) of Definition 1.1.3.

The claim for the cokernel follows dually. \square

1.1.9. The categories $MT_K^{\phi, N}$ and MT_K^{ϕ} are \mathbb{Q}_p -linear rigid \otimes -categories.

Lemma 1.1.10. *The functor*

$$(1.1.2) \quad \tilde{\omega} : MT_K^{\phi, N} \rightarrow (\mathbb{Q}_p\text{-vector spaces}), \quad (M, \phi, N, F) \mapsto \bigoplus_{n \in \mathbb{Z}} \tilde{\omega}_n(M, \phi, F),$$

with

$$\tilde{\omega}_n(M, \phi, F) = \operatorname{Hom}_{MT_K^{\phi}}(K(n), \operatorname{gr}_{-2n}^W(M, \phi, F)),$$

is a fibre functor. In particular, $(MT_K^{\phi, N}, \tilde{\omega})$ and $(MT_K^{\phi}, \tilde{\omega})$ are Tannaka categories.

Proof. It is easy to see that $\tilde{\omega}$ is a \otimes -functor. In order to see that $\tilde{\omega}$ is exact and faithful we will prove the existence of an isomorphism

$$(1.1.3) \quad \tilde{\omega}_{K_0} \xrightarrow{\cong} (\gamma : (M, \phi, N, F) \mapsto M),$$

where $\tilde{\omega}_{K_0}(M, \phi, N, F) = \tilde{\omega}(M, \phi, N, F) \otimes_{\mathbb{Q}_p} K_0$ and γ forgets about ϕ , N and F . Since γ is exact and faithful, this will imply the claim.

In order to construct (1.1.3), we observe that there is a functorial isomorphism

$$(1.1.4) \quad \operatorname{Hom}_{MT_K^{\phi}}(K(n), \operatorname{gr}_{-2n}^W(M, \phi, N, F)) \otimes_{\mathbb{Q}_p} K_0 \rightarrow M_{-n},$$

$$\phi \otimes a \mapsto a \cdot \phi(1).$$

\square

Proposition 1.1.11. *An object $(M, \phi, N, F) \in MF_K^{\phi, N, wa}$ belongs to $MT_K^{\phi, N}$ if and only if there exists an increasing exhaustive separated filtration W by subobjects of (M, ϕ, N, F) in $MF_K^{\phi, N, wa}$ such that W_i/W_{i-1} vanishes if i is odd, and is a sum of Tate objects $K(-\frac{i}{2})$ if i is even.*

Proof. For $(M, \phi, N, F) \in MT_K^{\phi, N}$, such a filtration exists by Definition 1.1.5, Proposition 1.1.6, and the fact that $\text{gr}_{2i}^W(M, \phi, N, F)$ is a sum of Tate objects $K(-i)$.

Suppose now that $(M, \phi, N, F) \in MF_K^{\phi, N, wa}$ admits a filtration W satisfying the assumptions. It is easy to see that (M, ϕ) satisfies Property (1) of Definition 1.1.3.

In general, if

$$0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$$

is an exact sequence in $MF_K^{\phi, N, wa}$, and M_1, M_2 satisfy Property (2), then M satisfies Property (2). By induction on i we conclude that $W_i \in MT_K^{\phi, N}$ for all i . \square

It is clear that any filtration as in Proposition 1.1.11 has to coincide with the weight filtration, and that any morphism between two objects in $MT_K^{\phi, N}$ has to be strict with respect to the weight filtrations on these objects.

1.2. The crystalline logarithmic point.

1.2.1. Recall from (1.1.2) that we have a fibre functor $\tilde{\omega}$ equipping MT_K^ϕ and $MT_K^{\phi, N}$ with the structure of Tannaka categories (Lemma 1.1.10). Let $G_{\tilde{\omega}}$ and $G_{\tilde{\omega}}^{st}$ denote the pro-algebraic groups which represent tensor automorphisms of $\tilde{\omega}$ on MT_K^ϕ and $MT_K^{\phi, N}$, respectively. In other words, we have $G_{\tilde{\omega}} = \underline{\text{Aut}}_{MT_K^\phi}^{\otimes} \tilde{\omega}$ and $G_{\tilde{\omega}}^{st} = \underline{\text{Aut}}_{MT_K^{\phi, N}}^{\otimes} \tilde{\omega}$. The goal of this section is to construct a non-trivial K -valued point η of $G_{\tilde{\omega}}$.

Definition 1.2.2. For $(M, \phi, F) \in MT_K^\phi$ we define

$$\eta(M, \phi, F) : M_K \rightarrow M_K$$

to be the unique endomorphism rendering the following diagram commutative:

$$(1.2.1) \quad \begin{array}{ccc} M_K = \bigoplus_{i \in \mathbb{Z}} M_i \otimes_{K_0} K & \xrightarrow{\bigoplus_i \iota_i \otimes_{K_0} K} & \bigoplus_{i \in \mathbb{Z}} M_{\geq i} \otimes_{K_0} K \\ \eta(M, \phi, F) \downarrow & & \downarrow (\bigoplus_{i \in \mathbb{Z}} \pi_i)^{-1} \\ M_K & \xleftarrow{\sum_{i \in \mathbb{Z}} \iota_i} & \bigoplus_{i \in \mathbb{Z}} F^i M_K, \end{array}$$

where $\iota_i : M_i \rightarrow M_{\geq i}$ is the obvious inclusion, $\pi_i : F^i M_K \rightarrow M_{\geq i} \otimes_{K_0} K$ is the projection and therefore by definition an isomorphism (Definition 1.1.3(2)), and $\sum_{i \in \mathbb{Z}} \iota_i$ is the sum over the obvious inclusions.

Lemma 1.2.3. *The morphisms η from Definition 1.2.2 define a tensor automorphism of the fibre functor $\tilde{\omega}_K = \tilde{\omega} \otimes_{\mathbb{Q}_p} K$.*

Proof. Via the \otimes -isomorphism (1.1.3) we may identify $\tilde{\omega} \otimes_{\mathbb{Q}_p} K_0$ with the forgetful functor $(M, \phi, F) \mapsto M$. After tensoring with K we obtain $\tilde{\omega}_K(M, \phi, F) = M_K$.

First, let us prove that $\eta(M, \phi, F)$ is an automorphism. We denote by

$$\eta(M, \phi, F)[i, j] : M_j \otimes_{K_0} K \rightarrow M_i \otimes_{K_0} K$$

the the composition with the inclusion $M_j \otimes_{K_0} K \rightarrow M_K$ and the projection $M_K \rightarrow M_i \otimes_{K_0} K$. It is easy to see from the definitions that

$$(1.2.2) \quad \eta(M, \phi, F)[i, j] = \begin{cases} 0 & \text{if } i > j, \\ id_{M_i} & \text{if } i = j. \end{cases}$$

Therefore $\eta(M, \phi, F)$ is an automorphism.

Since the diagram (1.2.1) is functorial, η defines a natural transformation. The compatibility with the tensor product is obvious. \square

1.2.4. Let us explain the construction of η in the formalism of [Del94]. For $(M, \phi, F) \in MT_K^\phi$ there are three filtrations on M_K :

(1) the weight filtration:

$$W_i M_K = \begin{cases} M_{\leq \frac{i}{2}} \otimes_{K_0} K & \text{if } i \text{ is even} \\ M_{\leq \frac{i-1}{2}} \otimes_{K_0} K & \text{if } i \text{ is odd.} \end{cases}$$

(2) The Hodge filtration F .

(3) The filtration

$$\bar{F}^i M_K := M_{\geq i} \otimes_{K_0} K \quad \text{for all } i \in \mathbb{Z}.$$

The three filtration W, F, \bar{F} satisfy the condition

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\bar{F}}^q \mathrm{Gr}_n^W M_K = 0 \quad \text{for } n \neq p + q,$$

of [Del94, §1.1]. Induced by F, \bar{F} , we obtain maps

$$\begin{aligned} a_F : M_K &= \bigoplus_{i \in \mathbb{Z}} F^i \cap W_{2i} \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{2i}^W M_K, \\ a_{\bar{F}} : M_K &= \bigoplus_{i \in \mathbb{Z}} \bar{F}^i \cap W_{2i} \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{2i}^W M_K, \end{aligned}$$

where $F^i \cap W_{2i} \rightarrow \mathrm{Gr}_{2i}^W M_K$ is the natural map (and similarly for \bar{F}). We obtain a unipotent automorphism $d = a_{\bar{F}} a_F^{-1}$ of $\bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{2i}^W M_K$.

It is easy to see that we have the equality

$$\eta(M, \phi, F) = a_{\bar{F}}^{-1} \circ d \circ a_{\bar{F}}.$$

1.2.5. Let us see in explicit terms how η compares the crystalline structure with the Hodge filtration. For $(M, \phi, F) \in MT_K^\phi$ we say that $v_1, \dots, v_d \in \tilde{\omega}(M, \phi, F) \otimes_{\mathbb{Q}_p} K$ is a homogeneous basis if it is a basis of $\tilde{\omega}(M, \phi, F) \otimes_{\mathbb{Q}_p} K$, and for every v_i there is an integer n_i with $v_i \in \tilde{\omega}_{n_i}(M, \phi, F) \otimes_{\mathbb{Q}_p} K$; we set $\deg(v_i) = n_i$.

Recall that for all integers i we have isomorphisms

$$\begin{aligned} a_i : M_{-i} \otimes_{K_0} K &\xrightarrow{\cong} \tilde{\omega}_i(M, \phi, F) \otimes_{\mathbb{Q}_p} K, \\ b_i : F^{-i} \cap W_{-2i} M_K &\xrightarrow{\cong} \tilde{\omega}_i(M, \phi, F) \otimes_{\mathbb{Q}_p} K. \end{aligned}$$

The first map is the inverse of (1.1.4) and the second map is given by the composition

$$b_i : F^{-i} \cap W_{-2i} M_K \xrightarrow{\subset} M_K \xrightarrow{\text{projection}} M_{-i} \otimes_{K_0} K \xrightarrow{a_i} \tilde{\omega}_i(M, \phi, F) \otimes_{\mathbb{Q}_p} K.$$

For a homogeneous basis $\{v_j\}$ we set

$$v_j^{\mathrm{crys}} := a_{\deg(v_j)}^{-1}(v_j), \quad v_j^{\mathrm{Hodge}} := b_{\deg(v_j)}^{-1}(v_j).$$

We denote by $\{v_j^{\text{crys}, \vee}\}$ the basis dual to $\{v_j^{\text{crys}}\}$. By definition of η we have

$$v_i^{\vee}(\eta(v_j)) = v_i^{\text{crys}, \vee}(v_j^{\text{Hodge}}).$$

1.2.6. By Lemma 1.2.3, we obtain a K -valued point $\eta \in G_{\tilde{\omega}}(K)$; we call this point the *logarithmic point*. Let us check that η is not the identity.

Proposition 1.2.7. *Let $n \in \mathbb{Z}$ be an integer. We have*

$$(1.2.3) \quad \text{Ext}_{MT_K^\phi}^1(K(0), K(n)) \cong \begin{cases} K & \text{if } n > 0 \\ 0 & \text{if } n \leq 0. \end{cases}$$

Let

$$(1.2.4) \quad 0 \rightarrow K(n) \xrightarrow{\iota} (E, \phi, F) \xrightarrow{\pi} K(0) \rightarrow 0$$

be an extension. For $n \neq 0$, there are unique sections $f : E \rightarrow K_0$ and $v : K_0 \rightarrow E$ of the underlying maps of K_0 -isocrystals of ι and π , respectively. The isomorphism (1.2.3), for $n \neq 0$, is given by the formula

$$E \mapsto f(\eta(E, \phi, F)(v(1))).$$

Proof. First, we consider the case $n = 0$. Let (E, ϕ, F) be as in (1.2.4). We have $F^0(E_K) = E_K$ and $F^1(E_K) = 0$ by Definition 1.1.3(2). In view of Definition 1.1.3(2) there is an isomorphism $(E, \phi) \cong (K_0, \sigma) \oplus (K_0, \sigma)$, thus there is a section of π in MT_K^ϕ .

For $n \neq 0$: From the slope decomposition we obtain natural sections f, v as ϕ -modules. If $n < 0$ then $F^1 E_K = \iota(K)$ which means $(E, \phi, F) = K(0) \oplus K(n)$.

For $n > 0$, we can uniquely write $F^{n+1} E_K = K\langle a \cdot \iota(1) + v(1) \rangle$ with $a \in K$. Obviously,

$$f(\eta(E, \phi, F)(v(1))) = a.$$

It is clear that $F^{n+1} E_K$ is the only invariant for extensions. \square

1.2.8. Recall that we have a fibre functor $\tilde{\omega}$ (1.1.2) to the category of \mathbb{Q}_p -vector spaces. In the obvious way $\tilde{\omega}$ factors through the category of graded \mathbb{Q}_p -vector spaces. Furthermore, we have an automorphism η of $\tilde{\omega}_K$ (Lemma 1.2.3).

Definition 1.2.9. We define \mathcal{C}_η to be the category of pairs (V, η) , where V is a finite dimensional graded \mathbb{Q}_p -vector space and $\eta : V \otimes K \rightarrow V \otimes K$ is a K -linear map such that for all $n \in \mathbb{Z}$:

$$(1.2.5) \quad (\eta - id)(V_n \otimes K) \subset \bigoplus_{i>n} V_i \otimes K.$$

Morphisms $(V_1, \eta_1) \rightarrow (V_2, \eta_2)$ are \mathbb{Q}_p -linear morphisms $\tau : V_1 \rightarrow V_2$ which respect the grading and commute with the endomorphisms η_i , i.e. $\eta_2 \circ (\tau \otimes id_K) = (\tau \otimes id_K) \circ \eta_1$.

The category \mathcal{C}_η is a \otimes -category with

$$(V_1, \eta_1) \otimes (V_2, \eta_2) = (V_1 \otimes V_2, \eta_1 \otimes \eta_2).$$

Proposition 1.2.10. *The functor*

$$\Psi : MT_K^\phi \rightarrow \mathcal{C}_\eta$$

$$(M, \phi, F) \mapsto \left(\bigoplus_{n \in \mathbb{Z}} \tilde{\omega}_n(M, \phi, F), \eta(M, \phi, F) \right)$$

is an equivalence of \otimes -categories.

Proof. By Lemma 1.2.3, η is functorial and Ψ is a \otimes -functor. It follows from (1.2.2) that

$$(\eta - id)(\tilde{\omega}_n \otimes K) \subset \bigoplus_{i>n} \tilde{\omega}_i \otimes K.$$

We define a functor

$$\begin{aligned} \Phi : \mathcal{C}_\eta &\rightarrow MT_K^\phi \\ (\bigoplus_{n \in \mathbb{Z}} V_n, \eta) &\mapsto (\bigoplus_{n \in \mathbb{Z}} (V_{-n} \otimes_{\mathbb{Q}_p} K_0, p^{-n} \otimes \sigma), F), \end{aligned}$$

with the following filtration:

$$F^i := \eta \left(\bigoplus_{j \geq i} V_{-j} \otimes_{\mathbb{Q}_p} K \right),$$

for all i . Property (1.2.5) implies that Φ is well-defined. From Definition 1.2.2 it easily follows that $\Psi \circ \Phi = id_{\mathcal{C}_\eta}$.

On the other hand, we have $\Phi \circ \Psi \xrightarrow{\cong} id_{MT_K^\phi}$ via

$$\begin{aligned} \Phi \circ \Psi(M, \phi, F) &\rightarrow (M, \phi, F) \\ \bigoplus_{n \in \mathbb{Z}} \tilde{\omega}_{-n}(M, \phi, F) \otimes_{\mathbb{Q}_p} K_0 &\xrightarrow{(1.1.4)} M. \end{aligned}$$

□

Remark 1.2.11. Via the dictionary of Section 1.2.4, Proposition 1.2.10 is a variant of [Del94, Proposition 1.2].

1.3. The semistable logarithmic point.

1.3.1. Let K be as in §1.1.1 with residue field k . We denote by ν_K the valuation of K .

1.3.2. Recall that we have a homomorphism

$$[\cdot] : k^\times \rightarrow \mathcal{O}_K^\times, \quad x \mapsto [x],$$

by taking the Teichmüller lift. Denoting by $U_K := \{x \in \mathcal{O}_K^\times; x \in 1 + m_K\}$ the 1-units, we obtain a decomposition

$$\mathcal{O}_K^\times = k^\times \times U_K.$$

The logarithm

$$(1.3.1) \quad \log : \mathcal{O}_K^\times \rightarrow \mathcal{O}_K$$

is by definition trivial on the factor k^\times and is given by

$$\log(u) = \sum_{n \geq 1} (-1)^{n+1} \frac{(u-1)^n}{n} \quad \text{for all } u \in U_K.$$

1.3.3. We consider $\mathcal{O}_{K,\mathbb{Q}}^\times := \mathcal{O}_K^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ and $K_{\mathbb{Q}}^\times := K^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ as \mathbb{Q} -vector spaces, therefore we may form the symmetric algebras $\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times)$ and $\mathrm{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times)$. The exact sequence

$$0 \rightarrow \mathcal{O}_{K,\mathbb{Q}}^\times \rightarrow K_{\mathbb{Q}}^\times \xrightarrow{\nu_K} \mathbb{Q} \rightarrow 0$$

implies that $\mathrm{Spec}(\mathrm{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times))$ is a 1-dimensional affine space over the scheme $\mathrm{Spec}(\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times))$. In other words, for $x \in K^\times$ with $\nu_K(x) \neq 0$, the map

$$\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times)[X] \rightarrow \mathrm{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times), \quad X \mapsto x,$$

is an isomorphism.

The logarithm (1.3.1) induces a ring homomorphism

$$(1.3.2) \quad \mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times) \rightarrow K,$$

because K is torsion free.

Definition 1.3.4. We define the K -algebra K_{st} by

$$K_{st} := \mathrm{Sym}_{\mathbb{Q}}(K_{\mathbb{Q}}^\times) \otimes_{\mathrm{Sym}_{\mathbb{Q}}(\mathcal{O}_{K,\mathbb{Q}}^\times)} K.$$

By base change, $\mathrm{Spec}(K_{st})$ is a 1-dimensional affine space over K . We have a natural logarithm

$$(1.3.3) \quad \log_{st} : K^\times \rightarrow K_{st}, \quad x \mapsto x \otimes 1.$$

The K -valued points of $\mathrm{Spec}(K_{st})$ admit the following description:

$$(1.3.4) \quad \mathrm{Spec}(K_{st})(K) = \{\text{extensions } \log : K^\times \rightarrow K \text{ of (1.3.1)}\} \\ f \mapsto f^* \circ \log_{st}.$$

By an extension $\log : K^\times \rightarrow K$ we mean a homomorphism such that the restriction to \mathcal{O}_K^\times equals (1.3.1).

1.3.5. The p -adic Hodge theory for K (and fixed valuation ν_K) depends for semistable representations on the choice of a logarithm

$$\log : K^\times \rightarrow K.$$

It will be important for us that our constructions do not depend on a particular choice, and for this we have to recall the basic constructions of p -adic Hodge theory.

We denote by R the ring

$$R := \varprojlim \mathcal{O}_{\bar{K}}/p\mathcal{O}_{\bar{K}},$$

where the maps are given by rising to the p -th power $x \mapsto x^p$. Denoting by $C_K = \widehat{\bar{K}}$ the p -adic completion of \bar{K} we have a multiplicative bijection

$$\varprojlim \mathcal{O}_{C_K} \rightarrow R,$$

where the projective system is defined by rising to the p -th power again. In other words, we can represent every element x in R by $(x^{(0)}, x^{(1)}, \dots)$ with $x^{(n)} \in \mathcal{O}_{C_K}$ and $x^{(n-1)} = (x^{(n)})^p$.

Let $\nu_{\bar{K}}$ (resp. ν_{C_K}) be the extension of ν_K (resp. $\nu_{\bar{K}}$) to \bar{K} (resp. C_K). The map

$$\nu_R : R \setminus \{0\} \rightarrow \mathbb{Q}, \quad x \mapsto \nu_{C_K}(x^{(0)})$$

can be extended to a valuation

$$\nu_R : \mathrm{Frac}(R)^\times \rightarrow \mathbb{Q}$$

with valuation ring R .

Let B_{cris} be the crystalline period ring; we define

$$B_{st} = \mathrm{Sym}_{\mathbb{Q}}(\mathrm{Frac}(R)^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathrm{Sym}_{\mathbb{Q}}(R^{\times} \otimes_{\mathbb{Z}} \mathbb{Q})} B_{cris},$$

where $\mathrm{Sym}_{\mathbb{Q}}(R^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}) \rightarrow B_{cris}$ is induced by the crystalline logarithm

$$\log_{cris} : R^{\times} \rightarrow B_{cris}.$$

Again, $R^{\times} = \bar{k}^{\times} \times (1 + m_R)$; \log_{cris} is trivial on \bar{k}^{\times} and given by

$$\log_{cris}(u) = \sum_{n \geq 1} (-1)^{n+1} \frac{([u] - 1)^n}{n}$$

for $u \in 1 + m_R$, where $[u]$ denotes the Teichmüller lift of u in the Witt ring $W(R)$ of R .

By construction we have a natural logarithm

$$\log_{st} : \mathrm{Frac}(R)^{\times} \rightarrow B_{st}, \quad x \mapsto x \otimes 1.$$

The ring B_{st} has the following properties.

- (1) We have a $\mathrm{Gal}(\bar{K}/K)$ -action on B_{st} extending the action on B_{cris} .
- (2) We have a Frobenius map $\phi : B_{st} \rightarrow B_{st}$ extending the Frobenius map on B_{cris} . Moreover,

$$\phi \circ \log_{st} = p \log_{st}.$$

- (3) We have a B_{cris} -linear derivation $N : B_{st} \rightarrow B_{st}$ such that

$$N(\log_{st}(x)) = \nu_R(x) \quad \text{for all } x \in \mathrm{Frac}(R)^{\times}.$$

After choosing a logarithm

$$\log : K^{\times} \rightarrow K,$$

which extends (1.3.1), we obtain a morphism of B_{cris} -algebras

$$\gamma_{\log} : B_{st} \rightarrow B_{dR}.$$

The morphism depends on the choice of \log , and the filtration on B_{st} induced by the filtration on B_{dR} via γ_{\log} depends on \log . In order to simplify the comparison between different logarithms we will restrict ourselves to logarithms \log such that $\log(K_0^{\times}) \subset K_0$. In other words, we will only consider K_0 -valued points of $\mathrm{Spec}(K_{0,st})$.

Proposition 1.3.6. *For $\log, \log' \in \mathrm{Spec}(K_{0,st})(K_0)$ there is a unique ring homomorphism*

$$\delta_{\log, \log'} : B_{st} \rightarrow B_{st}$$

such that $\gamma_{\log'} \circ \delta_{\log, \log'} = \gamma_{\log}$. The map $\delta_{\log, \log'}$ is given by

$$(1.3.5) \quad \delta_{\log, \log'} = \exp \left(\frac{\log(x) - \log'(x)}{\nu_K(x)} N \right)$$

for every $x \in K_0 \setminus \mathcal{O}_{K_0}^{\times}$.

Proof. Uniqueness follows from the fact that γ_{\log} is injective.

Choose $\tilde{p} \in R$ with $\tilde{p}^{(0)} = p$. By definition we have

$$\gamma_{\log}(\log_{st}(\tilde{p})) = \log_{dR}([\tilde{p}]/p) + \log(p),$$

where \log_{dR} is defined by the usual series since $[\tilde{p}]/p$ is a 1-unit in B_{dR} . Since $\text{Spec}(B_{st})$ is a 1-dimensional affine space over $\text{Spec}(B_{cris})$, there exists a unique morphism of B_{cris} -algebras $\delta_{\log, \log'}$ such that

$$\delta_{\log, \log'}(\log_{st}(\tilde{p})) = \log_{st}(\tilde{p}) + \log(p) - \log'(p).$$

Obviously, $\delta_{\log, \log'}$ satisfies $\gamma_{\log'} \circ \delta_{\log, \log'} = \gamma_{\log}$ and the equality (1.3.5). \square

By using γ_{\log} we obtain a filtration on B_{st} . The p -adic Hodge theory [CF00, Thm. A] asserts that the functor

$$(1.3.6) \quad D_{st, \log} : (\text{semistable } \mathbb{Q}_p\text{-representations of } \text{Gal}(\bar{K}/K)) \rightarrow MF_K^{\phi, N, w.a.}$$

$$V \mapsto (B_{st} \otimes_{\mathbb{Q}_p} V)^{\text{Gal}(\bar{K}/K)}$$

is an equivalence of categories. We will use the subscript \log in $D_{st, \log}$ to emphasize the dependence on \log .

Denoting by forget_F the functor $\text{forget}_F(M, \phi, N, F) = (M, \phi, N)$, we get

$$\text{forget}_F \circ D_{st, \log} = \text{forget}_F \circ D_{st, \log'},$$

because only the filtration depends on the embedding to B_{dR} . Proposition 1.3.6 implies that for the filtrations we have the following comparison:

$$(1.3.7) \quad F_{D_{st, \log'}^i}^i(V) = \exp\left(\frac{\log(x) - \log'(x)}{\nu_K(x)} N\right) F_{D_{st, \log}^i}^i(V),$$

for all $i \in \mathbb{Z}$ and all $x \in K_0 \setminus \mathcal{O}_{K_0}^\times$.

Definition 1.3.7. Let K be a p -adic field. We denote by MT_{G_K} the full subcategory of p -adic representations V of $\text{Gal}(\bar{K}/K)$ which admit an increasing exhaustive separated filtration W by subrepresentations of V such that W_i/W_{i-1} vanishes if i is odd, and is a sum of Tate objects $\mathbb{Q}_p(-\frac{i}{2})$ if i is even. We call an object of MT_{G_K} a *mixed Tate representation* of $\text{Gal}(\bar{K}/K)$.

Proposition 1.3.8. *Let $\log \in \text{Spec}(K_{0, st})(K_0)$. Then*

$$MT_{G_K} = D_{st, \log}^{-1}(MT_K^{\phi, N}).$$

In particular, every mixed Tate representation is semistable.

Proof. From Proposition 1.1.11 it follows that every object in $D_{st, \log}^{-1}(MT_K^{\phi, N})$ admits a filtration W satisfying the properties of Definition 1.3.7.

Now, suppose that V is a p -adic representation of $\text{Gal}(\bar{K}/K)$ which admits a filtration W as in Definition 1.3.7. If we know that V is semistable then clearly $D_{st, \log}(V) \in MT_K^{\phi, N}$ by Proposition 1.1.11, again. Therefore it suffices to prove that V is semistable.

We use induction on the length of the filtration W of V . If the filtration W has length ≤ 1 , semistability of V follows from those of $\mathbb{Q}_p(n)$. In general, let n be the smallest integer such that $W_{2n}V = V$. Then we have an exact sequence

$$0 \rightarrow (W_{2n-2}V) \otimes \mathbb{Q}_p(n) \rightarrow V \otimes \mathbb{Q}_p(n) \rightarrow (V/W_{2n-2}V) \otimes \mathbb{Q}_p(n) \rightarrow 0.$$

By the induction hypothesis the terms on the left and right are semistable. Moreover, since the weights of the term on the left are ≤ -2 and the term on the right

has weight 0, we have

$$\begin{aligned} F^0 D_{dR}((W_{2n-2}V) \otimes \mathbb{Q}_p(n)) &= 0 \\ F^0(D_{dR}((V/W_{2n-2}V) \otimes \mathbb{Q}_p(n))) &= D_{dR}((V/W_{2n-2}V) \otimes \mathbb{Q}_p(n)). \end{aligned}$$

Therefore [Nek93, Proposition 1.28] shows that the middle term is also semistable. \square

Obviously,

$$(1.3.8) \quad \tau = \tilde{\omega} \circ D_{st, \log}$$

is independent of \log , and (MT_{G_K}, τ) is a Tannaka category (by Lemma 1.1.10).

1.3.9. Recall from Lemma 1.1.10 that $(MT_K^{\phi, N}, \tilde{\omega})$ is a Tannaka category. We will use the ring K_{st} (Definition 1.3.4) and \log_{st} (1.3.3).

Definition 1.3.10. For a logarithm $\log \in \text{Spec}(K_{0, st})(K_0)$ and $(M, \phi, N, F) \in MT_K^{\phi, N}$ we define $\eta_{st, \log}(M, \phi, N, F) \in \text{End}_{K_{st}}(M \otimes_{K_0} K_{st})$ by

$$\eta_{st, \log}(M, \phi, N, F) := \exp\left(\frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N\right) \eta(M, \phi, F),$$

for $x \in K_0^\times \setminus \mathcal{O}_{K_0}^\times$. For the definition of $\eta(M, \phi, F)$ we refer to Definition 1.2.2.

Obviously, $\eta_{st, \log}$ does not depend on the choice $x \in K_0^\times \setminus \mathcal{O}_{K_0}^\times$, but it depends on \log .

Lemma 1.3.11. *Let $\log \in \text{Spec}(K_{0, st})(K_0)$. The morphisms $\eta_{st, \log}$ from Definition 1.3.10 define a tensor automorphism of the fibre functor $\tilde{\omega}_{K_{st}} = \tilde{\omega} \otimes_{\mathbb{Q}_p} K_{st}$. In other words, $\eta_{st, \log} \in G_{\tilde{\omega}}^{st}(K_{st})$ with $G_{\tilde{\omega}}^{st} = \underline{\text{Aut}}_{MT_K^{\phi, N}}^\otimes \tilde{\omega}$.*

Proof. Via the \otimes -isomorphism (1.1.3) we may identify $\tilde{\omega} \otimes_{\mathbb{Q}_p} K_0$ with the forgetful functor $(M, \phi, N, F) \mapsto M$. After tensoring with K_{st} we obtain $\tilde{\omega}_{K_{st}}(M, \phi, N, F) = M \otimes_{K_0} K_{st}$. Lemma 1.2.3 implies that $\eta(M, \phi, F)$ is a tensor automorphism, thus it suffices to prove the statement for $\exp\left(\frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N\right)$. The functoriality follows immediately. The compatibility with the \otimes -structure follows from

$$N_{M_1 \otimes M_2} = N_{M_1} \otimes 1 + 1 \otimes N_{M_2}.$$

\square

Lemma 1.3.12. *The K_{st} -valued point*

$$\eta_{st} = \eta_{st, \log} \circ D_{st, \log}$$

of $\underline{\text{Aut}}_{MT_{G_K}}^\otimes \tau$ is independent of the choice of $\log \in \text{Spec}(K_{0, st})(K_0)$.

Proof. Let $\log, \log' \in \text{Spec}(K_{0, st})(K_0)$ and $V \in MT_{G_K}$. In view of (1.3.7) we get

$$(1.3.9) \quad \eta(\text{forget}_N D_{st, \log'}(V)) = \exp\left(\frac{\log(x) - \log'(x)}{\nu_K(x)} N\right) \eta(\text{forget}_N D_{st, \log}(V)),$$

for every $x \in K_0 \setminus \mathcal{O}_{K_0}^\times$, and $\text{forget}_N(M, \phi, N, F) = (M, \phi, F)$. Thus

$$\begin{aligned} \eta_{st, \log'} D_{st, \log'}(V) &= \exp\left(\frac{\log'(x) - \log_{st}(x)}{\nu_K(x)} N\right) \eta(\text{forget}_N D_{st, \log'}(V)) \\ &= \exp\left(\frac{\log(x) - \log_{st}(x)}{\nu_K(x)} N\right) \eta(\text{forget}_N D_{st, \log}(V)) \quad \text{by (1.3.9),} \\ &= \eta_{st, \log} D_{st, \log}(V). \end{aligned}$$

□

Example 1.3.13. By Kummer theory any $q \in K^\times$ defines an extension V of the $\text{Gal}(\bar{K}/K)$ -representation $\mathbb{Q}_p(0)$ by $\mathbb{Q}_p(1)$. This in turn gives via $D_{st, \log}$ an extension of $K(0)$ by $K(1)$ in $MT_K^{\phi, N}$:

$$0 \rightarrow K(1) \rightarrow M \rightarrow K(0) \rightarrow 0,$$

which may be described as follows. The underlying K_0 -space of M has a basis e_0, e_1 such that the following conditions are satisfied:

- (1) the action of ϕ is given by $\phi(e_i) = p^{-i}e_i$ for $i = 0, 1$,
- (2) e_1 is the image of $1 \in K(1)$,
- (3) e_0 maps to $1 \in K(0)$.

The filtration is given by $F^{-1}M_K = M_K$, $F^0M_K = K \cdot \langle \log(q)e_1 + e_0 \rangle$ and $F^1M_K = 0$. Finally N is given by $Ne_0 = -\nu_K(q) \cdot e_1$ and $Ne_1 = 0$. Then we easily compute

$$\eta_{st}(V) = \begin{pmatrix} 1 & 0 \\ -\frac{\nu_K(q)(\log(x) - \log_{st}(x))}{\nu_K(x)} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ \log(q) & 1 \end{pmatrix}$$

for the obvious basis of

$$\tau(V) = \text{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) \oplus \text{Hom}(\mathbb{Q}(1), \mathbb{Q}(1)),$$

and every $x \in K_0 \setminus \mathcal{O}_{K_0}^\times$, or, equivalently, for every $x \in K \setminus \mathcal{O}_K^\times$. If $\nu_K(q) \neq 0$ then we can take $q = x$ in order to see that

$$\eta_{st}(V) = \begin{pmatrix} 1 & 0 \\ \log_{st}(q) & 1 \end{pmatrix}$$

holds for all $q \in K^\times$.

1.4. Tannaka group scheme of mixed Tate filtered ϕ -modules.

Definition 1.4.1. We define \mathcal{L} to be the graded \mathbb{Q}_p -Lie algebra freely generated by K^\vee (i.e. the dual of K , where K is considered as \mathbb{Q}_p -vector space) in each degree $i > 0$. In other words, \mathcal{L} is defined by

(1.4.1)

$$\text{Hom}_{(\text{graded } \mathbb{Q}_p\text{-Lie algebras})}(\mathcal{L}, \mathcal{T}) = \text{Hom}_{(\text{graded } \mathbb{Q}_p\text{-vector spaces})}(\bigoplus_{i \in \mathbb{Z}_{>0}} K^\vee, \mathcal{T}),$$

for every graded \mathbb{Q}_p -Lie algebra \mathcal{T} . Via (1.4.1) we get \mathbb{Q}_p -linear maps

$$a_i : K^\vee \rightarrow \mathcal{L}_i, \quad \text{for all } i > 0.$$

Obviously, \mathcal{L} is concentrated in positive degrees and each \mathcal{L}_i is a finite dimensional \mathbb{Q}_p -vector space.

Definition 1.4.2. For all $n > 0$, we define a \mathbb{Q}_p -Lie algebra $\mathcal{L}_{\leq n}$ by

$$\mathcal{L}_{\leq n} = \mathcal{L} / (\oplus_{i>n} \mathcal{L}_i).$$

We set $\hat{\mathcal{L}} = \varprojlim_n \mathcal{L}_{\leq n}$. For every field extension $K \supset \mathbb{Q}_p$ we define $\hat{\mathcal{L}}_K := \varprojlim_n (\mathcal{L}_{\leq n} \otimes_{\mathbb{Q}_p} K)$.

We will be only interested in the finite dimensional graded representations of \mathcal{L} , which can be identified with the finite dimensional graded representations of $\hat{\mathcal{L}}$.

1.4.3. There is natural element $\epsilon \in \hat{\mathcal{L}}_K$ defined as follows:

$$(1.4.2) \quad \epsilon := \sum_{i>0} (a_i \otimes id_K)(id),$$

with $id \in K^\vee \otimes_{\mathbb{Q}_p} K$ the canonical element. After choosing a \mathbb{Q}_p -basis v_1, \dots, v_d of K , we see that

$$\epsilon = \sum_{i>0} \sum_{j=1}^d a_i(v_j^\vee) \otimes v_j.$$

Let V be a finite dimensional graded representation of \mathcal{L} then $\exp(\epsilon)$ is a unipotent automorphism of $V \otimes_{\mathbb{Q}_p} K$.

Proposition 1.4.4. *The \otimes -functor*

$$\begin{aligned} \Psi : (\text{finite dim. graded } \mathcal{L}\text{-modules}) &\rightarrow \mathcal{C}_\eta \\ V &\mapsto (V, \exp(\epsilon)) \end{aligned}$$

is an equivalence of categories.

Proof. Note first that Ψ is well-defined, because $(\exp(\epsilon) - id)(V_n) \subset \oplus_{i>n} V_i$, for all n , thus (1.2.5) is satisfied.

Let us prove that Ψ is essentially surjective. Let $(\oplus_n V_n, \eta)$ be an object of \mathcal{C}_η . Since $1 - \eta$ is nilpotent we can define

$$\tilde{\epsilon} = \log(\eta) = \log(1 - (1 - \eta)).$$

For every $i > 0$ we define a \mathbb{Q}_p -linear map $\beta_i : K^\vee \rightarrow \text{End}(V)_i$ by

$$\beta_i(f) = \sum_n (id_{V_{n+i}} \otimes f) \circ \text{proj}_{V_{n+i} \otimes K} \circ \tilde{\epsilon} \circ \text{incl}_{V_n},$$

where $\text{incl}_{V_n} : V_n \rightarrow V$ and $\text{proj}_{V_{n+i} \otimes K} : V \otimes K \rightarrow V_{n+i} \otimes K$ is the inclusion and the projection, respectively. Via (1.4.1) we obtain a graded representation $\rho : \mathcal{L} \rightarrow \text{End}(V)$. We need to show that $\exp(\rho(\epsilon)) = \eta$, or equivalently $\rho(\epsilon) = \tilde{\epsilon}$. This is a straight forward computation which we leave to the reader.

Next, we need to prove that Ψ is fully faithful. Clearly, Ψ is faithful. Let V, U be two graded representations of \mathcal{L} , and suppose $\tau : V \rightarrow U$ is a morphism which respects the grading and commutes with $\exp(\epsilon)$. Then τ commutes with ϵ . Fix a

basis v_1, \dots, v_d of K . For $v \in V$ we get

$$\begin{aligned}\tau\epsilon(v) &= \tau\left(\sum_{i>0} \sum_j a_i(v_j^\vee)(v) \otimes v_j\right) \\ &= \sum_{i>0} \sum_j \tau(a_i(v_j^\vee)(v)) \otimes v_j, \\ \epsilon\tau(v) &= \sum_{i>0} \sum_j a_i(v^\vee)_j(\tau(v)) \otimes v_j.\end{aligned}$$

Therefore $\tau \circ a_i(v_j^\vee) = a_i(v_j^\vee) \circ \tau$ for all i, j . Since \mathcal{L} is generated by the elements $\{a_i(v_j^\vee)\}_{i,j}$ we see that τ is a morphism of \mathcal{L} -representations. \square

Corollary 1.4.5. *There is an equivalence of \otimes -categories*

$$\Theta : MT_K^\phi \rightarrow (\text{finite dim. graded } \mathcal{L}\text{-modules})$$

such that

- $\text{forg} \circ \Theta = \tilde{\omega}$, where forg forgets about the \mathcal{L} -action.
- $\exp(\epsilon) |_{\Theta(M, \phi, F)} = \eta(M, \phi, F)$.

Proof. Follows immediately from Proposition 1.2.10 and Proposition 1.4.4. \square

Corollary 1.4.6. *Let U be the pro-algebraic group $U = \varprojlim_n \exp(\mathcal{L}/(\oplus_{i>n} \mathcal{L}_i))$. Let $G_{\tilde{\omega}}$ be the Tannaka group attached to the fibre functor $\tilde{\omega}$ (Lemma 1.1.10).*

There is an isomorphism

$$G_{\tilde{\omega}} \cong \mathbb{G}_m \times U$$

such that $\eta \in G_{\tilde{\omega}}(K)$ corresponds to $\exp(\epsilon) \in U(K)$.

Proof. In view of Corollary 1.4.5, the statement follows from the fact that $\mathbb{G}_m \times U$ is the Tannaka group of the fibre functor:

$$\text{forg} : (\text{graded finite dimensional } \mathcal{L}\text{-modules}) \rightarrow (\mathbb{Q}_p\text{-vector spaces}).$$

The action of \mathbb{G}_m on U is induced by the action of \mathbb{G}_m on \mathcal{L} given by the grading:

$$\mathbb{G}_m \times \mathcal{L}_i \rightarrow \mathcal{L}_i; \quad (t, x) \mapsto t^i \cdot x.$$

\square

2. MIXED TATE MOTIVES OVER A NUMBER FIELD AND LOGARITHMIC POINTS

2.1. Mixed Tate motives.

2.1.1. Let E be a number field and S a set of finite places. Let \mathcal{O} be the ring of integers of E , and $|\text{Spec}(\mathcal{O})|$ the maximal spectrum of \mathcal{O} . We denote by

$$\mathcal{O}_S := \bigcap_{x \in \text{Spec}(\mathcal{O}) \setminus S} \mathcal{O}_x$$

the ring of S -integers of E ; the elements of \mathcal{O}_S are integral outside of S . We will be mainly interested in two cases for S . In the first case, we have $S = |\text{Spec}(\mathcal{O})|$ and $\mathcal{O}_S = E$. In the second case, we have $S = |\text{Spec}(\mathcal{O})| \setminus \{x\}$, for a point $x \in |\text{Spec}(\mathcal{O})|$, and $\mathcal{O}_S = \mathcal{O}_x$ is the local ring at x .

2.1.2. Deligne and Goncharov defined in [DG05, 1.6] an abelian category of mixed Tate motives $MT(\mathcal{O}_S)$. By definition it is the full subcategory of $MT(E)$ consisting of objects which are unramified outside S in the following sense. Let $x \in |\mathrm{Spec}(\mathcal{O})|$ be a point lying over a prime p ; then we say that $M \in MT(E)$ is unramified at x if for all primes $\ell \neq p$ the corresponding Galois representation M_ℓ is unramified at x , i.e. the inertia subgroup I_x (which is only well-defined up to conjugation) acts trivially at M_ℓ [DG05, Proposition 1.8].

2.1.3. For extensions of Tate objects we know that:

$$\begin{aligned} \mathrm{Ext}_{MT(\mathcal{O}_S)}^1(\mathbb{Q}(0), \mathbb{Q}(1)) &= \mathcal{O}_S^\times \otimes \mathbb{Q}, \\ \mathrm{Ext}_{MT(\mathcal{O}_S)}^1(\mathbb{Q}(0), \mathbb{Q}(n)) &= \begin{cases} 0 & \text{if } n \leq 0, \\ \mathrm{Ext}_{MT(E)}^1(\mathbb{Q}(0), \mathbb{Q}(n)) & \text{if } n \neq 1, \end{cases} \\ \mathrm{Ext}_{MT(\mathcal{O}_S)}^2(\mathbb{Q}(0), \mathbb{Q}(n)) &= 0 \quad \text{for all } n \in \mathbb{Z}, \end{aligned}$$

(see [DG05, Proposition 1.9]).

2.1.4. Every object of $MT(\mathcal{O}_S)$ comes equipped with a finite increasing functorial weight filtration, indexed by even integers. For all $n \in \mathbb{Z}$ the graded pieces $\mathrm{gr}_{2n}^W(M)$ are sums of copies of $\mathbb{Q}(-n)$.

In view of [DG05, 1.1] the \otimes -functor

$$(2.1.1) \quad \begin{aligned} \omega : MT(\mathcal{O}_S) &\rightarrow (\mathbb{Q}\text{-vector spaces}), \\ \omega(M) &:= \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}(\mathbb{Q}(n), \mathrm{gr}_{-2n}^W(M)), \end{aligned}$$

is a fibre functor, therefore $MT(\mathcal{O}_S)$ is a Tannaka category. We denote by $G_{S,\omega}$ the group scheme of \otimes -automorphisms of ω . By [DG05, 2.1] we can write $G_{S,\omega}$ as a semi-direct product:

$$G_{S,\omega} = \mathbb{G}_m \ltimes U_{S,\omega},$$

where $U_{S,\omega}$ is a unipotent group and $G_{S,\omega} \rightarrow \mathbb{G}_m$ is induced by the obvious grading of ω . If $S = |\mathrm{Spec}(\mathcal{O})|$ then we simply write $G_\omega = G_{S,\omega}$.

2.1.5. *Functor to p -adic representations.* Let $x \in |\mathrm{Spec}(\mathcal{O})|$ be a point lying over a prime p . Let $K = E_x$ be the completion of E at the place x . Choose algebraic closures \bar{E} , \bar{K} , and an embedding $\iota : \bar{E} \rightarrow \bar{K}$.

To $M \in MT(E)$ we can attach a Galois representation M_p of $\mathrm{Gal}(\bar{E}/E)$ with coefficients in \mathbb{Q}_p , which is called the p -adic realization of M . By using ι , we get a continuous homomorphism

$$\mathrm{Gal}(\bar{K}/K) \rightarrow \mathrm{Gal}(\bar{E}/E),$$

and we can restrict M_p in order to obtain a p -adic representation $M_{\iota,p}$ of $\mathrm{Gal}(\bar{K}/K)$.

Proposition 2.1.6. *The assignment $M \mapsto M_{\iota,p}$ defines a functor*

$$(\cdot)_{\iota,p} : MT(E) \rightarrow MT_{G_K}.$$

See Definition 1.3.7 for MT_{G_K} .

Proof. The p -adic realization is functorial. Thus we only need to show that $M_{\iota,p} \in MT_{G_K}$, which follows immediately from the existence of the weight filtration of M and Definition 1.3.7. \square

The set $\{\iota : \bar{E} \rightarrow \bar{E}_x\}$ of embeddings over E is a torsor under the Galois group $\text{Gal}(\bar{E}/E)$, and for every $g \in \text{Gal}(\bar{E}/E)$ there is a natural transformation:

$$(2.1.2) \quad \alpha_g : (\cdot)_{\iota,p} \xrightarrow{\cong} (\cdot)_{\iota \circ g,p}.$$

Lemma 2.1.7. *For the fibre functor τ (defined in (1.3.8)) and the fibre functor ω defined in (2.1.1) we have a canonical isomorphism*

$$\tau \circ (\cdot)_{\iota,p} \cong \omega \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

For every $g \in \text{Gal}(\bar{E}/E)$, the diagram

$$(2.1.3) \quad \begin{array}{ccc} \tau \circ (\cdot)_{\iota,p} & \xrightarrow{(2.1.2)} & \tau \circ (\cdot)_{\iota \circ g,p} \\ & \searrow \cong & \swarrow \cong \\ & \omega \otimes_{\mathbb{Q}} \mathbb{Q}_p & \end{array}$$

is commutative.

Proof. Straightforward. \square

2.1.8. Recall that we have constructed a K_{st} -valued η_{st} of $\underline{\text{Aut}}^{\otimes} \tau$ (Lemma 1.3.12).

Proposition 2.1.9. *For every embedding ι , $\eta_x := \eta_{st} \circ (\cdot)_{\iota,p}$ defines a K_{st} -valued point of $\underline{\text{Aut}}_{MT(E)}^{\otimes} \omega$ which is independent of the choice of ι .*

Proof. Since $\tau \circ (\cdot)_{\iota,p} = \omega \otimes_{\mathbb{Q}} \mathbb{Q}_p$ by Lemma 2.1.7, $\eta_{st} \circ (\cdot)_{\iota,p}$ is a K_{st} -valued point of $\underline{\text{Aut}}^{\otimes} \omega$.

The independence of the choice of ι follows from the commutative diagram

$$\begin{array}{ccccc} (\tau \circ (\cdot)_{\iota,p}) \otimes_{\mathbb{Q}_p} K_{st} & \xrightarrow{\eta_{st}} & & \xrightarrow{\eta_{st}} & (\tau \circ (\cdot)_{\iota \circ g,p}) \otimes_{\mathbb{Q}_p} K_{st} \\ \downarrow \tau(2.1.2) \otimes K_{st} & \searrow & \omega \otimes_{\mathbb{Q}} K_{st} & \swarrow & \downarrow \tau(2.1.2) \otimes K_{st} \\ & & \omega \otimes_{\mathbb{Q}} K_{st} & & \\ & \swarrow & & \searrow & \\ (\tau \circ (\cdot)_{\iota \circ g,p}) \otimes_{\mathbb{Q}_p} K_{st} & \xrightarrow{\eta_{st}} & & \xrightarrow{\eta_{st}} & (\tau \circ (\cdot)_{\iota \circ g,p}) \otimes_{\mathbb{Q}_p} K_{st} \end{array}$$

where the triangles are commutative by Lemma 2.1.7, and the square is commutative because η_{st} is functorial. \square

2.2. Crystalline characterization of unramified motives.

2.2.1. Let E be a number field, and let M be a mixed Tate motive over E , i.e. an object in $MT(E)$. Let ν be a finite place of E , M is unramified at ν [DG05, Definition 1.4, §1.7] if the coaction [DG05, (1.2.2)]

$$e_M : \omega(M) \rightarrow \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \otimes \omega(M)$$

of $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) = E^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$ on $\omega(M)$ factors through a coaction of $\mathcal{O}_{\nu}^{\times} \otimes_{\mathbb{Z}} \mathbb{Q}$.

2.2.2. Recall from Proposition 2.1.6 that $M_{\iota,p}$ is a mixed Tate Galois representation of $G_K = \text{Gal}(\bar{K}/K)$ for the completion $\bar{K} = E_\nu$ at ν . In particular, $M_{\iota,p}$ is semistable (Proposition 1.3.8). In the following we will simply write $M_p = M_{p,\iota}$. We call M_p *crystalline* if the monodromy operator N of $D_{st}(M_p)$ is trivial, or equivalently if

$$(B_{cris} \otimes_{\mathbb{Q}_p} M_p)^{G_K} \rightarrow (B_{st} \otimes_{\mathbb{Q}_p} M_p)^{G_K}$$

is an isomorphism.

Theorem 2.2.3. *Let M be a mixed Tate motive over E and ν a finite place of E . Then M is unramified at ν if and only if M_p is crystalline.*

Proof. First note that the statement that M is unramified at ν is equivalent to the statement that for every subquotient N of M which is of the form

$$0 \rightarrow \mathbb{Q}(n+1) \rightarrow N \rightarrow \mathbb{Q}(n) \rightarrow 0,$$

for some n , the extension class $\text{Ext}^1(\mathbb{Q}(n), \mathbb{Q}(n+1)) = \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) = E^\times \otimes \mathbb{Q}$ lies in $\mathcal{O}_\nu^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ [DG05, §1.4].

Also in the category of p -adic representations of a p -adic field K , a representation in $\text{Ext}_{G_K}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ that is associated to some $q \in K^\times \otimes \mathbb{Q} \subseteq \varprojlim_n (K^\times / (K^\times)^{p^n}) \otimes \mathbb{Q} = \text{Ext}_{G_K}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(1))$ is crystalline if and only if $q \in \mathcal{O}_K^\times \otimes \mathbb{Q}$ [Tsu02, Example 2.3.2].

First, suppose that M_p is crystalline, then every subquotient of M_p is crystalline. So in order to prove that M is unramified at ν we may assume that $M = N$, where N is as above with $n = 0$ (after Tate twist). Therefore we have an extension in $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$, defined by some $q \in E^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, whose p -adic realization is crystalline at ν . Then the above remark implies that the image of q in $E_\nu^\times \otimes \mathbb{Q}$ lies in $\hat{\mathcal{O}}_\nu^\times \otimes \mathbb{Q}$, hence $q \in \mathcal{O}_\nu^\times \otimes \mathbb{Q}$ and M is unramified at ν .

Suppose conversely that M is unramified at ν . We have to show that the monodromy operator N on $D_{st}(M_p) =: D(M)$ vanishes. Note that N maps the slope λ piece of $D(M)$ to the slope $\lambda - 1$ piece. Therefore, if N is nonzero on $D(M)$ then there exists an n such that N is nonzero on $D(W_{2n}M_p/W_{2n-4}M_p) = D((W_{2n}M/W_{2n-4}M)_p)$. Replacing M by $(W_{2n}M/W_{2n-4}M) \otimes \mathbb{Q}(n)$ we may assume that M is defined by a class in $\text{Ext}^1(\mathbb{Q}(0)^{\oplus r}, \mathbb{Q}(1)^{\oplus s}) = \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))^{\oplus rs}$, M is unramified, and N is nonzero on $D(M)$. By passing to a subquotient we may further assume that $r = s = 1$. This gives an extension in $\text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1))$ which is unramified at ν (and hence defined by some $q \in \mathcal{O}_\nu^\times \otimes \mathbb{Q}$) and whose p -adic realization is not crystalline at ν . This is a contradiction. \square

2.2.4. Recall the notation of Section 2.1.1. Let $x \in |\text{Spec}(\mathcal{O})|$ be a point; in the following we will work with $S = |\text{Spec}(\mathcal{O})| \setminus \{x\}$, thus $\mathcal{O}_S = \mathcal{O}_x$.

Let p be the prime lying under x . In view of Theorem 2.2.3, we know that $MT(\mathcal{O}_x)$ is the full subcategory of $MT(E)$ consisting of motives M such that the p -adic realization M_p is crystalline at x .

We denote by G_x the group scheme of \otimes -automorphisms of the fibre functor (see (2.1.1))

$$(2.2.1) \quad \omega : MT(\mathcal{O}_x) \rightarrow (\mathbb{Q}\text{-vector spaces}).$$

The group scheme G_x is a quotient of $G_\omega = \text{Aut}_{MT(E)}^\otimes \omega$.

Lemma 2.2.5. *The morphism $\mathrm{Spec}(E_{x,st}) \xrightarrow{\eta_x} G_\omega \rightarrow G_x$ factors through the structure morphism $\mathrm{Spec}(E_{x,st}) \rightarrow \mathrm{Spec}(E_x)$ and thus defines a point $\eta_x^{ur} \in G_x(E_x)$.*

Proof. The point η_x was defined in Proposition 2.1.9. If $M \in MT(\mathcal{O}_x)$ then $D_{st}(M_{l,p})$ has vanishing monodromy operator N and $\eta_{st}(M_{l,p}) = \eta_{st,\log} D_{st}(M_{l,p})$ takes values in E_x by Definition 1.3.10. \square

2.3. Main theorem.

2.3.1. Let $x \in |\mathrm{Spec}(\mathcal{O})|$ and let E_x be the completion of E at x . Bloch and Kato [BK90, Definition 3.10] define an exponential map

$$(2.3.1) \quad \exp : E_x \rightarrow \mathrm{Ext}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n)), \quad \text{for all } n \geq 1,$$

where Ext^1 is computed in the category of p -adic representation of $\mathrm{Gal}(\bar{E}_x/E_x)$. Note that, in fact, the image of the exponential map lies among the crystalline representations $\mathrm{Ext}_{crys}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$ [BK90, Example 3.9]. Via p -adic Hodge theory, we obtain a map

$$(2.3.2) \quad E_x \rightarrow \mathrm{Ext}_{crys}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n)) \cong \mathrm{Ext}_{MT_{E_x}^\phi}^1(E_x(0), E_x(n)),$$

which, by abuse of notation, will also be called the Bloch-Kato exponential map.

2.3.2. For an extension $M \in \mathrm{Ext}_{MT(\mathcal{O}_x)}^1(\mathbb{Q}(0), \mathbb{Q}(n))$ with $n \geq 1$, there are natural maps $v_0 : \mathbb{Q} \rightarrow \omega(M)$ and $f_n : \omega(M) \rightarrow \mathbb{Q}$ defined as follows. By definition, there are isomorphisms $\alpha : \mathbb{Q}(n) \rightarrow \mathrm{gr}_{-2n}^W M$ and $\beta : \mathrm{gr}_0^W M \rightarrow \mathbb{Q}(0)$; we define

$$\begin{aligned} v_0 : \mathbb{Q} &= \mathrm{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) \xrightarrow{\beta^{-1}} \omega_0(M) \rightarrow \omega(M), \\ f_n : \omega(M) &\rightarrow \omega_n(M) \xrightarrow{\alpha^{-1}} \mathrm{Hom}(\mathbb{Q}(n), \mathbb{Q}(n)) = \mathbb{Q}. \end{aligned}$$

Therefore, we can attach to M a function in $\mathbb{A}^1(G_x)$, defined by

$$M(t) := f_n(t \cdot v_0),$$

for every point $t : T \rightarrow G_x$.

Theorem 2.3.3. *Let E be a number field and \mathcal{O} be the ring of integers. Let $x \in |\mathrm{Spec}(\mathcal{O})|$ be a closed point over a prime p . For the Tannaka category $(MT(\mathcal{O}_x), \omega)$ of mixed Tate motives we denote by G_x the group scheme of \otimes -automorphisms of ω . For all $n \geq 1$, the map*

$$\mathrm{Ext}_{MT(\mathcal{O}_x)}^1(\mathbb{Q}(0), \mathbb{Q}(n)) \rightarrow E_x, \quad M \mapsto M(\eta_x^{ur}),$$

induced by $\eta_x^{ur} \in G_x(E_x)$ (see Lemma 2.2.5), is the composition of the p -adic realization

$$\mathrm{Ext}_{MT(\mathcal{O}_x)}^1(\mathbb{Q}(0), \mathbb{Q}(n)) \rightarrow \mathrm{Ext}_{crys}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n))$$

and the inverse of the Bloch-Kato exponential map (2.3.1).

Proof. Let us prove that evaluation at the point η_x^{ur} has the desired compatibility with the Bloch-Kato exponential map (see (2.3.2))

$$\exp : E_x \rightarrow \mathrm{Ext}_{crys}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n)) \xrightarrow{\cong} \mathrm{Ext}_{MT_{E_x}}^1(E_x(0), E_x(n)).$$

For this we need to recall the construction of the exponential map. For the rest of the proof let $K := E_x$. First there is an exact sequence [BK90, Proposition 1.17]:

$$(2.3.3) \quad 0 \rightarrow \mathbb{Q}_p \rightarrow B_{crys}^{\varphi=1} \oplus B_{dR}^+ \rightarrow B_{dR} \rightarrow 0,$$

where the first map sends x to (x, x) and the second one sends (x, y) to $x - y$.

For $n \geq 1$, the Bloch-Kato construction gives a map

$$K = (\mathbb{Q}_p(n) \otimes B_{dR})^{G_K} \rightarrow \text{Ext}_{\text{crys}}^1(\mathbb{Q}_p(0), \mathbb{Q}_p(n)).$$

This map is obtained as follows. First tensor the above exact sequence with $\mathbb{Q}_p(n)$:

$$0 \rightarrow \mathbb{Q}_p(n) \rightarrow (\mathbb{Q}_p(n) \otimes B_{\text{crys}}^{\varphi=1}) \oplus (\mathbb{Q}_p(n) \otimes B_{dR}^+) \rightarrow \mathbb{Q}_p(n) \otimes B_{dR} \rightarrow 0.$$

Then an element a in $K(n) = (\mathbb{Q}_p(n) \otimes B_{dR})^{G_K}$ gives a map $\mathbb{Q}_p \rightarrow \mathbb{Q}_p(n) \otimes B_{dR}$, pulling back the above exact sequence via this map gives the extension we were looking for.

More explicitly, for $a \in K$ the extension constructed above is:

$$0 \rightarrow V_n \rightarrow V \rightarrow V_0 \rightarrow 0,$$

where $V_0 = \mathbb{Q}_p \cdot t^n \otimes at^{-n}$, $V_n = \mathbb{Q}_p \cdot t^n$, and V is a 2-dimensional representation of G_K with basis which can be described as follows. By the exact sequence (2.3.3), there exists $x \in B_{\text{crys}}^{\varphi=1}$ and $y \in B_{dR}^+$ such that $at^{-n} = x - y$. Then V has basis $\{(t^n \otimes x, t^n \otimes y), (t^n \otimes 1, t^n \otimes 1)\}$. For $\sigma \in G_K$,

$$\sigma(t^n \otimes x, t^n \otimes y) = (t^n \otimes x, t^n \otimes y) + \gamma(\sigma)(t^n \otimes 1, t^n \otimes 1),$$

for some $\gamma(\sigma) \in \mathbb{Q}_p$. Therefore

$$\chi_{\text{cyc}}(\sigma)^n \sigma(x) = x + \gamma(\sigma)$$

and

$$\chi_{\text{cyc}}(\sigma)^n \sigma(y) = y + \gamma(\sigma)$$

Let us now try to find what this extension corresponds to after we apply the functor $(\cdot \otimes B_{\text{crys}})^{G_K}$. First note that $(V \otimes B_{\text{crys}})^{G_K}$ has basis

$$e_n := (t^n \otimes 1, t^n \otimes 1) \otimes t^{-n}$$

and

$$e_0 := (t^n \otimes x, t^n \otimes y) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes x.$$

That e_n is invariant under the Galois action is clear. In order to see that e_0 is G_K invariant let $\sigma \in G_K$. Then

$$\sigma(e_0) = (t^n \otimes (x + \gamma(\sigma)), t^n \otimes (y + \gamma(\sigma))) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes (x + \gamma(\sigma)) = e_0.$$

Now note that $\varphi(e_n) = p^{-n}e_n$ and $\varphi(e_0) = e_0$. Furthermore e_n is the image of $1 \in K(n)$ and e_0 maps to $1 \in K(0)$ in the exact sequence (note that $\mathbb{Q}_p(0)$ is identified with V_0 via the map that sends 1 to $t^n \otimes at^{-n}$):

$$0 \rightarrow K(n) \rightarrow (V \otimes B_{\text{crys}})^{G_K} \rightarrow K(0) \rightarrow 0.$$

Therefore in order to compare Bloch-Kato's construction we need only compute the filtration on $(V \otimes B_{\text{crys}})^{G_K} \otimes_{K_0} K$. So we need to compute the 0-th piece of the filtration on $(V \otimes B_{dR})^{G_K}$.

We claim that $ae_n + e_0 \in \text{Fil}^0(V \otimes B_{dR})^{G_K}$. This follows immediately from

$$ae_n + e_0 = (t^n \otimes x, t^n \otimes y) \otimes 1 - (t^n \otimes 1, t^n \otimes 1) \otimes y,$$

and the fact that $y \in B_{dR}^+$. Now Proposition 1.2.7 implies the claim. \square

2.4. Archimedean places. In this section we recall the story for archimedean places; our reference is [Del94] and [BD94].

2.4.1. Let E be a number field and $\sigma : E \rightarrow \mathbb{C}$ an embedding. To $M \in MT(E)$ we can attach a *real* mixed Tate Hodge structure M_σ . Recall that a real mixed Tate Hodge structure (H, W, F) consists of an \mathbb{R} -vector space H , an increasing filtration W of H , and a decreasing filtration F of $H \otimes_{\mathbb{R}} \mathbb{C}$ such that

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\bar{F}}^q \mathrm{Gr}_n^W (H \otimes \mathbb{C}) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even and } (p, q) \neq (\frac{n}{2}, \frac{n}{2}). \end{cases}$$

Induced by F, \bar{F} , we obtain maps

$$\begin{aligned} a_F : H \otimes \mathbb{C} &= \bigoplus_{i \in \mathbb{Z}} F^{-i} \cap W_{-2i} \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{-2i}^W H \otimes \mathbb{C}, \\ a_{\bar{F}} : H \otimes \mathbb{C} &= \bigoplus_{i \in \mathbb{Z}} \bar{F}^{-i} \cap W_{-2i} \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{-2i}^W H \otimes \mathbb{C}, \end{aligned}$$

where $F^i \cap W_{2i} \rightarrow \mathrm{Gr}_{2i}^W H \otimes \mathbb{C}$ is the natural map (and similarly for \bar{F}). For the automorphism $d = a_{\bar{F}} a_F^{-1}$ of $\bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{-2i}^W H \otimes \mathbb{C}$ we know that

$$(d - 1)(\mathrm{Gr}_{-2i}^W H \otimes \mathbb{C}) \subset \bigoplus_{j > i} \mathrm{Gr}_{-2j}^W H \otimes \mathbb{C},$$

by [Del94, p.510], and $\bar{d} = d^{-1}$ [Del94, p.513].

2.4.2. Let \mathcal{C} be the category of pairs $(\bigoplus_i H_i, d)$ where $\bigoplus_i H_i$ is a graded \mathbb{R} -vector space and $d : \bigoplus_i H_i \otimes \mathbb{C} \rightarrow \bigoplus_i H_i \otimes \mathbb{C}$ is an automorphism satisfying the conditions $\bar{d} = d^{-1}$ and $(d - 1)(H_i) \subset \bigoplus_{j > i} H_j$, for all i .

Proposition 2.4.3. [Del94, p.514] *The functor*

$$\begin{aligned} &(\text{Real mixed Tate Hodge structures}) \rightarrow \mathcal{C} \\ &(H, W, F) \mapsto (\bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{-2i}^W H, d) \end{aligned}$$

is an equivalence of categories.

The maps d define a \mathbb{C} -valued \otimes -automorphism for the fibre functor

$$\begin{aligned} \tilde{\omega} : (\text{Real mixed Tate Hodge structures}) &\rightarrow (\mathbb{R}\text{-vector spaces}), \\ \tilde{\omega}(H, W, F) &= \bigoplus_{i \in \mathbb{Z}} \mathrm{Gr}_{-2i}^W H. \end{aligned}$$

2.4.4. Recall the definition of ω in (2.1.1). For the functor

$$\mathcal{R}_\sigma : MT(E) \rightarrow (\text{Real mixed Tate Hodge structures}), \quad M \mapsto M_\sigma,$$

we have an isomorphism

$$(2.4.1) \quad \omega \otimes_{\mathbb{Q}} \mathbb{R} \cong \tilde{\omega} \circ \mathcal{R},$$

depending on the choice $(2\pi i)^n$ as a generator for the real vector space underlying $\mathbb{R}(n)$, in other words we have to choose a square root of -1 in \mathbb{C} . In order to avoid this choice one can define

$$\tilde{\omega}(H, W, F) = \bigoplus_{n \in \mathbb{Z}} i^n \cdot \mathrm{Gr}_{-2n}^W H,$$

as in [BD94, p.111], but we won't do that.

Via Equation (2.4.1), d defines a \mathbb{C} -valued point of G_ω , the \otimes -automorphisms of the fibre functor ω . We define $\epsilon = \log(d)$, ϵ defines a \mathbb{C} -valued point of $\mathrm{Lie}(G_\omega)$.

The dictionary for the notation of [BD94, p.111] is

$$d = b^{-1}, \quad \epsilon = -2 \cdot N,$$

and N is purely imaginary.

2.4.5. For $z \in E \setminus \{0, 1\}$ there is a polylogarithm motive $\{z\} \in MT(E)$ (strictly speaking it is a pro-object). The motive $\{z\}$ is defined as a subquotient of the motivic paths from the tangent vector $t_0 = z$, in the tangent space at 0, to z (see [DG05, Theorem 4.4]). The \mathbb{Q} -Hodge realization of $\{z\}$ is described in [BD94, p.98] and uniquely determines $\{z\}$.

For every $k \in \mathbb{Z}_{\geq 0}$ we have natural isomorphisms

$$\alpha_k : \mathrm{gr}_{-2k}^W \{z\} \xrightarrow{\cong} \mathbb{Q}(k);$$

we define $v_0 \in \omega(\{z\})$ and $f_k \in \omega(\{z\})^\vee$ by

$$\begin{aligned} v_0 : \mathbb{Q} &= \mathrm{Hom}(\mathbb{Q}(0), \mathbb{Q}(0)) \xrightarrow{\alpha_0^{-1}} \omega_0(\{z\}) \\ f_k : \omega(\{z\}) &\rightarrow \omega_k(\{z\}) \xrightarrow{\alpha_k} \mathrm{Hom}(\mathbb{Q}(k), \mathbb{Q}(k)) = \mathbb{Q}. \end{aligned}$$

We denote by $(v_0, \{z\}, f_k) \in \mathbb{A}^1(\mathrm{Lie} G_\omega)$ the function

$$X \mapsto f_k(X \cdot v_0).$$

By [BD94, Proposition 2.7] we have

$$(v_0, \{z\}, f_k)(\epsilon) = \begin{cases} 2i \sum_{\ell} b_\ell \frac{\log(z\bar{z})}{\ell!} \mathrm{Im}(\mathrm{Li}_{k-\ell}(z)) & \text{if } k \text{ is even,} \\ 2i \sum_{\ell} b_\ell \frac{\log(z\bar{z})}{\ell!} \mathrm{Re}(\mathrm{Li}_{k-\ell}(z)) & \text{if } k \text{ is odd.} \end{cases}$$

Here, $\{b_\ell\}$ are the Bernoulli numbers and Li is the polylogarithm. For k even, the result does not depend on the choice of the square root of -1 ; for k odd it is independent up to a sign.

REFERENCES

- [BD94] Alexander Beilinson and Pierre Deligne. Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 97–121. Amer. Math. Soc., Providence, RI, 1994.
- [BK90] Spencer Bloch and Kazuya Kato. L -functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.
- [CF00] Pierre Colmez and Jean-Marc Fontaine. Construction des représentations p -adiques semi-stables. *Invent. Math.*, 140(1):1–43, 2000.
- [Del94] Pierre Deligne. Structures de Hodge mixtes réelles. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 509–514. Amer. Math. Soc., Providence, RI, 1994.
- [DG05] Pierre Deligne and Alexander B. Goncharov. Groupes fondamentaux motiviques de Tate mixte. *Ann. Sci. École Norm. Sup. (4)*, 38(1):1–56, 2005.
- [Ma63] Yuri Manin. The theory of commutative formal groups over fields of finite characteristic. *Russian Math. Surveys*, 18:1–83, 1963.
- [Nek93] Jan Nekovář. On p -adic height pairings. In *Séminaire de Théorie des Nombres, Paris, 1990–91*, volume 108 of *Progr. Math.*, pages 127–202. Birkhäuser Boston, Boston, MA, 1993.
- [Tsu02] Takeshi Tsuji. Semi-stable conjecture of Fontaine-Jannsen: a survey. *Astérisque*, (279):323–370, 2002. Cohomologies p -adiques et applications arithmétiques, II.

FACHBEREICH MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, 45117 ESSEN, GERMANY
E-mail address: a.chatzistamatiou@uni-due.de

MATHEMATICS DEPARTMENT, KOÇ UNIVERSITY, 34450, ISTANBUL, TURKEY
E-mail address: sunver@ku.edu.tr