Deformations of Bloch groups and Aomoto dilogarithms in characteristic p

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Abstract. In this paper, we study the Bloch group $B_2(\mathbb{F}[\varepsilon]_2)$ over the ring of dual numbers of the algebraic closure of the field with p elements, for a prime $p \geq 5$. We show that a slight modification of Kontsevich's $1\frac{1}{2}$-logarithm defines a function on $B_2(\mathbb{F}[\varepsilon]_2)$. Using this function and the characteristic p version of the additive dilogarithm function that we previously defined, we determine the structure of the infinitesimal part of $B_2(\mathbb{F}[\varepsilon]_2)$ completely. This enables us to define invariants on the group of deformations of Aomoto dilogarithms and determine its structure. This final result might be viewed as the analog of Hilbert's third problem in characteristic p.

1. Introduction

1.1. For an abelian group M, let $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$. The general conjectures on motives produce certain complexes of sheaves $\Gamma(n)$, for each $n \geq 1$ [2, §5.10 D], on the Zariski site of S, which are concentrated in the degrees $[1, n]$. For S regular, these complexes would have the property that the cohomology groups $H^i(S, \Gamma(n))_{\mathbb{Q}}$ are isomorphic to the n-th graded piece $K_{2n-i}(S)^{(n)}_{\mathbb{Q}}$ of the K-theory of S, with respect to the γ-filtration [2, 5.10 B].

A candidate for such a complex was constructed unconditionally by Bloch [4], [19], when $n = 2$ and $S = \operatorname{Spec} k$, where k is an infinite field. We start with describing the analog of this complex over an artin local ring A, which we denote by $\gamma_A(2)$. This complex plays a central role below.

Definition 1.1.1. For A any artinian local ring with infinite residue field, the Bloch group $B_2(A)$ is the free abelian group generated by the symbols $[x]$ such that $x(1-x) \in A^{\times}$, modulo the subgroup generated by elements of the form

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)],$$

for all $x, y \in A^{\times}$ such that $(1-x)(1-y)(1-x/y) \in A^{\times}$.

Definition 1.1.2. The Bloch complex over A is the complex $\gamma_A(2)$:

$$B_2(A) \xrightarrow{\delta_A} \Lambda^2 A^{\times} \tag{1.1.1}$$

concentrated in degrees 1 and 2, where δ_A is defined by $\delta_A([x]) = x \wedge (1-x)$.

Suslin [19, Theorem 5.2] proves that there is a natural surjection

$$K_3(k) \to H^1(\gamma_k(2)), \tag{1.1.2}$$

which induces an isomorphism $K_3(k)^{(2)}_{\mathbb{Q}} \xrightarrow{\sim} H^1(\gamma_k(2)_{\mathbb{Q}})$. For the degree two part, one has $K_2(k)^{(2)}_{\mathbb{Q}} = K_2^M(k)_{\mathbb{Q}} \xrightarrow{\sim} H^2(\gamma_k(2)_{\mathbb{Q}})$.

Therefore, $\gamma_k(2)_{\mathbb{Q}}$ can be thought of as a complex, with an explicit description, that computes the weight two motivic cohomology over a field with $\mathbb{Q}$-coefficients.

1.2. For a field k, we let $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$. For a functor $F : (alg/k) \to (Ab)$, from k-algebras to abelian groups, we let

$$F(k[\varepsilon]_2)^\circ := \ker(F(k[\varepsilon]_2) \to F(k)),$$

denote the *infinitesimal part* of $F(k[\varepsilon]_2)$. Note that $F(k[\varepsilon]_2)^\circ$ is naturally a direct summand of $F(k[\varepsilon]_2)$.

Bloch and Esnault constructed a dilogarithm in the infinitesimal case and proved that it gives a regulator-like map for an algebraically closed field k of characteristic 0. More precisely, for such a field k, they construct a complex $T\mathbb{Q}(2)(k)$ [5, (2.7)]:

$$TB_2(k)_{\mathbb{Q}} \xrightarrow{\partial} k \otimes k^\times$$

concentrated in degrees 1 and 2, such that

$$H^1(T\mathbb{Q}(2)(k)) \simeq K_3(k[t]_2,(t))^{(2)}_{\mathbb{Q}} \text{ and } H^2(T\mathbb{Q}(2)(k)) \simeq K_2(k[t]_2,(t))^{(2)}_{\mathbb{Q}},$$

exactly as in the isomorphism relating motivic cohomology to K-theory [5, Proposition 2.1]. To prove this, they construct an additive dilogarithm map

$$\rho : TB_2(k) \to k,$$

and show that when composed with the injection $H^1(T\mathbb{Q}(2)(k)) \to TB_2(k)_{\mathbb{Q}}$, ρ induces an isomorphism $H^1(T\mathbb{Q}(2)(k)) \xrightarrow{\sim} k$.

In [5], the construction of $T\mathbb{Q}(2)(k)$ is based on the localization sequence in K-theory. In [21], it is shown that, if k is a field of characteristic 0, the infinitesimal part $\gamma_{k[\varepsilon]_2}(2)^\circ_{\mathbb{Q}}$ of the Bloch complex $\gamma_{k[\varepsilon]_2}(2)_{\mathbb{Q}}$ also has the expected cohomology groups [21, Theorem 1.3.1]: there is an exact sequence

(1.2.1) $$0 \to K_3(k[\varepsilon]_2,(\varepsilon))^{(2)}_{\mathbb{Q}} \to B_2(k[\varepsilon]_2)^\circ_{\mathbb{Q}} \xrightarrow{\delta} (\Lambda^2 k[\varepsilon]_2^\times)^\circ \to K_2(k[\varepsilon]_2,(\varepsilon))^{(2)}_{\mathbb{Q}} \to 0.$$

This was proved by first constructing an additive dilogarithm:

$$li_2 : B_2(k[\varepsilon]_2) \to k,$$

defined as

$$li_2([s+\alpha\varepsilon]) := -\frac{\alpha^3}{2s^2(1-s)^2}$$

for $s+\alpha\varepsilon \in \mathbb{F}[\varepsilon]_2$, with $s \in \mathbb{F}\backslash\{0,1\}$. This map induces an isomorphism [21, Theorem 1.3.2]:

$$K_3(k[\varepsilon]_2,(\varepsilon))^{(2)}_{\mathbb{Q}} \xrightarrow{\sim} k.$$

1.3. The main aim of the present note is to understand the structure of $B_2(\mathbb{F}[\varepsilon]_2)$ in characteristic p. Namely, we fix a prime $p \geq 5$ and let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$, the field with p elements.

Kontsevich defined and studied a function $\pounds_1$ [14], [8, Definition 4.1], (§2.1), which he called the $1\frac{1}{2}$-logarithm since it satisfies a functional equation with four terms. In [8], this finite logarithm was generalized to higher weights which were shown to satisfy functional equations that are infinitesimal versions of the functional equations satisfied by the ordinary polylogarithms.

In this note, we make this more precise in the weight two case. Namely, we show that a slight modification $\mathfrak{Li}_2$ (Definition 2.1.1) of Kontsevich's function defines a function on the Bloch group over $\mathbb{F}[\varepsilon]_2$ (Theorem 2.2.2):

$$\mathfrak{Li}_2 : B_2(\mathbb{F}[\varepsilon]_2)^\circ \to \mathbb{F}. \tag{1.3.1}$$

Using $\mathfrak{Li}_2$ and li_2, we show that $\gamma_{\mathbb{F}[\varepsilon]_2}(2)^\circ$ computes the infinitesimal part of the motivic cohomology of $\mathbb{F}[\varepsilon]_2$ of weight two.

Theorem 1.3.1. *The complex*

$$0 \to K_3(\mathbb{F}[\varepsilon]_2, (\varepsilon)) \to B_2(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\delta} (\Lambda^2 \mathbb{F}[\varepsilon]_2^\times)^\circ \to 0,$$

which extends the infinitesimal part of the Bloch complex, is exact.

Note that $K_2^M(\mathbb{F}[\varepsilon]_2) = K_2(\mathbb{F}[\varepsilon]_2) = 0$ (Lemma 3.1.5) and hence the decomposable part $K_2(\mathbb{F}[\varepsilon]_2) \cdot K_1(\mathbb{F}[\varepsilon]_2)$ of $K_3(\mathbb{F}[\varepsilon]_2)$ is also 0. Therefore the above exact sequence is the precise analog of (1.2.1) in characteristic p.

1.4. For k an infinite field, a graded Hopf algebra $A.(k)$, called the Hopf algebra of Aomoto polylogarithms, is constructed in [3]. The importance of this Hopf algebra rests on the expectation that the graded pieces of $A.(k)_\mathbb{Q}$ of degrees less than or equal to two coincide with those of the conjectured graded Hopf algebra associated to the category of mixed Tate motives $\mathcal{MTM}_k(\mathbb{Q})$ [7, §2] over $\operatorname{Spec} k$ with coefficients in $\mathbb{Q}$. Therefore, of particular relevance is the group $A_2(k)$, and part of the comultiplication map $\nu_{1,1} : A_2(k) \to A_1(k) \otimes A_1(k)$, defined in [3, §2.1]. The elements of $A_2(k)$ are called the Aomoto dilogarithms.

We use Theorem 1.3.1 to describe $A_2(\mathbb{F}[\varepsilon]_2)^\circ$ (the definition is recalled in §4.1), which can be thought of as the group of deformations of Aomoto dilogarithms:

Theorem 1.4.1. *The maps $\mathfrak{Li}_2$, li_2 and $\nu_{1,1}$ induce an isomorphism*

$$A_2(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} \mathbb{F} \oplus \mathbb{F} \oplus (\mathbb{F} \otimes \mathbb{F}).$$

Here, Theorem 1.4.1 can be seen as a characteristic p analogue of Sydler's Theorem [20], [11, §6.3], [22, §2.2], on Hilbert's 3rd problem, with one more invariant $\mathfrak{Li}_2$ in addition to the Dehn invariant and the volume invariant.

For the approach to infinitesimal motivic cohomology, based on additive Chow groups, we refer the reader to the works of Park [17] and Rülling [18].

1.5. The paper is organized as follows.

In §2, we show that the modification $\mathfrak{Li}_2$ of Kontsevich's $1\frac{1}{2}$-logarithm defines a function from $B_2(\mathbb{F}[\varepsilon]_2)$ to $\mathbb{F}$.

In §3, we use a result of Suslin [19] and our functions $\mathfrak{Li}_2$ and li_2 to compute the cohomology of the Bloch complex over $\mathbb{F}[\varepsilon]_2$ (Theorem 3.2.4).

In §4, we prove Theorem 4.4.1, which describes the deformations of Aomoto polylogarithms in characteristic p.

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Notation. The notations $B_2(A), \delta_A$, and $\gamma_A(2)$ are defined in Definition 1.1.1 and Definition 1.1.2. We omit the subscript when it is clear from the context.

Unless otherwise stated, all tensor products are over $\mathbb{Z}$. For an abelian group E, we let $E_{\mathbb{Q}} := E \otimes \mathbb{Q}$.

For $n \in \mathbb{N}$, let $\mathbb{F}(n)$ denote the abelian group $\mathbb{F}$, together with the $\mathbb{F}^\times$ action given by $\lambda \star \alpha := \lambda^n \alpha$, for $\lambda \in \mathbb{F}^\times$ and $\alpha \in \mathbb{F}$.

For a set X, we let $\mathbb{Z}[X]$ denote the free abelian group with basis X. For $x \in X$, we denote by $[x]$, the corresponding element in $\mathbb{Z}[X]$.

For a ring A, we let $A^\flat := \{a \in A |\, a(1-a) \in A^\times\}$, and $\mathbb{P}^n_A := \mathrm{Proj}(A[z_0, \cdots, z_n])$.

2. A Regulator on the Infinitesimal Bloch Group

2.1. Let $\mathbb{F}_p$ be the field with p elements, where $p \geq 5$, and $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. In this section, we will prove that the modification $\mathfrak{Li}_2$ of Kontsevich's $1\frac{1}{2}$-logarithm $\pounds_1$ [14] defines a map from the infinitesimal Bloch group $B_2(\mathbb{F}[\varepsilon]_2)^\circ$ (§1.1) to $\mathbb{F}$.

First we define $\mathfrak{Li}_2$.

Definition 2.1.1. For $s + \alpha\varepsilon \in \mathbb{F}[\varepsilon]_2^\flat$, let

$$\mathfrak{Li}_2([s+\alpha\varepsilon]) := \frac{\alpha}{s(1-s)} \sum_{1 \leq k \leq p-1} \frac{s^{k/p}}{k}.$$

Note that for s in $\mathbb{F}$, $s^{1/p}$ is uniquely defined.

Definition 2.1.2. For $s \in \mathbb{F}^\flat$, let $\{s\} := s + s(1-s)\varepsilon$ and let $\langle s \rangle = [\{s\}] \in \mathbb{Z}[\mathbb{F}[\varepsilon]_2^\flat]$.

A finite version of the logarithm was defined in [8, Definition 4.1] and [14]:

$$\pounds_1(s) = \sum_{1 \leq k \leq p-1} \frac{s^k}{k},$$

for $s \in \mathbb{F}$.

The following functional equations are satisfied by $\pounds_1$ [8, Proposition 4.7, Proposition 4.9]:

$$\pounds_1(x) = -x^p \pounds_1(\frac{1}{x}) \tag{2.1.1}$$

$$\pounds_1(x) = \pounds_1(1-x) \tag{2.1.2}$$

and

$$\pounds_1(x) - \pounds_1(y) + x^p \pounds_1(\frac{y}{x}) + (1-x)^p \pounds_1(\frac{1-y}{1-x}) = 0. \tag{2.1.3}$$

Let $\mathbb{F}^\times$ act on the $\mathbb{F}$-algebra $\mathbb{F}[\varepsilon]_2$ by dilation, i.e. $\lambda \in \mathbb{F}^\times$ acts by sending ε to $\lambda\varepsilon$. By the functoriality of $B_2(\cdot)$ for ring homomorphisms, this gives an action of $\mathbb{F}^\times$ on $B_2(\mathbb{F}[\varepsilon]_2)$. Namely,

$$\lambda \star \sum_{1 \leq i \leq n} k_i [s_i + \alpha_i \varepsilon] := \sum_{1 \leq i \leq n} k_i [s_i + \alpha_i \lambda \varepsilon],$$

for $\lambda \in \mathbb{F}^\times$, $s_i + \alpha_i\varepsilon \in \mathbb{F}[\varepsilon]_2^\flat$ and $k_i \in \mathbb{Z}$, for $1 \le i \le n$.

Note that
$$\frac{\{t\}}{\{s\}} = s \star \{\frac{t}{s}\}, \quad \frac{1-\{s\}}{1-\{t\}} = (t-1) \star \{\frac{1-s}{1-t}\}, \text{ and } 1-\{s\}^{-1} = (1-s^{-1})(1-\varepsilon).$$
Therefore
$$\frac{1-\{s\}^{-1}}{1-\{t\}^{-1}} = \frac{1-s^{-1}}{1-t^{-1}},$$
and hence
$$\mathfrak{Li}_2([\frac{1-\{s\}^{-1}}{1-\{t\}^{-1}}]) = 0$$
and
$$[\frac{1-\{s\}^{-1}}{1-\{t\}^{-1}}] = 0 \in B_2(\mathbb{F}[\varepsilon]_2)^\circ.$$

By the notations above, we have
$$\mathfrak{Li}_2(\lambda \star \langle s\rangle) = \lambda \pounds_1(s)^{1/p}. \tag{2.1.4}$$

The functional equations (2.1.1), (2.1.2) and (2.1.3) then take the form:

$$\mathfrak{Li}_2(\langle s\rangle) = \mathfrak{Li}_2(\langle 1-s\rangle), \tag{2.1.5}$$

$$\mathfrak{Li}_2(\langle s\rangle) = -s\mathfrak{Li}_2(\langle \frac{1}{s}\rangle), \tag{2.1.6}$$

and
$$\mathfrak{Li}_2(\langle s\rangle) - \mathfrak{Li}_2(\langle t\rangle) + s\mathfrak{Li}_2(\langle \frac{t}{s}\rangle) + (1-s)\mathfrak{Li}_2(\langle \frac{1-t}{1-s}\rangle) = 0. \tag{2.1.7}$$

2.2. Next we show that $\mathfrak{Li}_2$ defines a map on the infinitesimal Bloch group.

Lemma 2.2.1. *$\mathfrak{Li}_2$ maps the element*
$$[\{s\}] - [\{t\}] + [\frac{\{t\}}{\{s\}}] - [\frac{1-\{s\}^{-1}}{1-\{t\}^{-1}}] + [\frac{1-\{s\}}{1-\{t\}}]$$
to 0.

Proof. The expression above is equal to
$$\langle s\rangle - \langle t\rangle + s \star \langle \frac{t}{s}\rangle - [\frac{1-s^{-1}}{1-t^{-1}}] + (t-1) \star \langle \frac{1-s}{1-t}\rangle.$$
The image of the second to last element under $\mathfrak{Li}_2$ is 0. The last element is mapped to
$$(t-1)\mathfrak{Li}_2(\langle \frac{1-s}{1-t}\rangle) = (1-s)\mathfrak{Li}_2(\langle \frac{1-t}{1-s}\rangle)$$
by (2.1.6). Then (2.1.7) proves the claim. □

Theorem 2.2.2. *$\mathfrak{Li}_2$ descends to give a map*
$$\mathfrak{Li}_2 : B_2(\mathbb{F}[\varepsilon]_2)^\circ \to \mathbb{F}.$$

Proof. Let $x := \alpha \star \{s\}$ and $y := \beta \star \{t\}$. Let $F(x, y)$ denote the image of $[x] - [y] + [y/x] - [(1 - x^{-1})/(1 - y^{-1})] + [(1 - x)/(1 - y)]$ under $\mathfrak{Li}_2$.

For any $x \in \mathbb{F}[\varepsilon]_2^\flat$, there exists $s \in \mathbb{F}^\flat$ and $\alpha \in \mathbb{F}$ such that $x = \alpha \star \{s\}$. Then by (2.1.6),

$$\mathfrak{Li}_2([x]) = \alpha\mathfrak{Li}_2(\langle s\rangle) = -\alpha s\mathfrak{Li}_2(\langle \frac{1}{s}\rangle) = -\alpha\mathfrak{Li}_2(s \star \langle \frac{1}{s}\rangle) = -\alpha\mathfrak{Li}_2([\frac{1}{\{s\}}]) = -\mathfrak{Li}_2([\frac{1}{x}]).$$

Hence $F(x, y) = -F(y, x)$. Clearly, if $\alpha = \beta = 0$ then $F(x, y) = 0$. Also if $\alpha = \beta$ then $F(x, y) = \alpha F(\{s\}, \{t\}) = 0$, by Lemma 2.2.1.

Without loss of generality, we assume that $\beta = 1$. Then $F(x, y) = F(x, \{t\}) = F(\{s\}, \{t\}) + (\alpha - 1)G(s, t)$, where $G(s, t) =$

$$\mathfrak{Li}_2(\langle s\rangle) + \frac{s(s-1)}{s-t}\mathfrak{Li}_2(\langle \frac{t}{s}\rangle) + \frac{1-t^{-1}}{s^{-1}-t^{-1}}\mathfrak{Li}_2(\langle \frac{1-s^{-1}}{1-t^{-1}}\rangle) - \frac{s(1-t)}{s-t}\mathfrak{Li}_2(\langle \frac{1-s}{1-t}\rangle).$$

Therefore, it suffices to show that

$$G(s, t) = 0.$$

Using the inversion formula, $\mathfrak{Li}_2(\langle a\rangle) = -a\mathfrak{Li}_2(\langle \frac{1}{a}\rangle)$, for $a = \frac{1-s^{-1}}{1-t^{-1}}$ and $a = \frac{1-s}{1-t}$, $G(s, t)$ can be rewritten as

$$\frac{s(s-1)}{s-t}(\mathfrak{Li}_2(\langle \frac{t}{s}\rangle) - \mathfrak{Li}_2(\langle \frac{1-t}{1-s}\rangle) + \frac{t}{s}\mathfrak{Li}_2(\langle \frac{1-t^{-1}}{1-s^{-1}}\rangle) + (1 - \frac{t}{s})\frac{1}{s-1}\mathfrak{Li}_2(\langle s\rangle)).$$

If in the last expression we use the identity

$$\frac{1}{s-1}\mathfrak{Li}_2(\langle s\rangle) = \frac{1}{s-1}\mathfrak{Li}_2(\langle 1-s\rangle) = \mathfrak{Li}_2(\langle \frac{1}{1-s}\rangle) = \mathfrak{Li}_2(\langle \frac{s}{s-1}\rangle),$$

all we need to show reduces to:

$$\mathfrak{Li}_2(\langle \frac{t}{s}\rangle) - \mathfrak{Li}_2(\langle \frac{1-t}{1-s}\rangle) + \frac{t}{s}\mathfrak{Li}_2(\langle \frac{1-t^{-1}}{1-s^{-1}}\rangle) + (1 - \frac{t}{s})\mathfrak{Li}_2(\langle \frac{s}{s-1}\rangle) = 0.$$

But this is nothing but the main functional equation (2.1.7):

$$\mathfrak{Li}_2(\langle a\rangle) - \mathfrak{Li}_2(\langle b\rangle) + a\mathfrak{Li}_2(\langle \frac{b}{a}\rangle) + (1-a)\mathfrak{Li}_2(\langle \frac{1-b}{1-a}\rangle) = 0,$$

with $a = \frac{t}{s}$ and $b = \frac{1-t}{1-s}$. □

3. Cohomology of the Bloch Complex of Weight Two

In this section, we compute the infinitesimal part of the cohomology of $\gamma_{\mathbb{F}[\varepsilon]_2}(2)$, the Bloch complex of weight two over $\mathbb{F}[\varepsilon]_2$. Recall that $\gamma_{\mathbb{F}[\varepsilon]_2}(2)$ is defined in (1.1.1) as the complex

$$B_2(\mathbb{F}[\varepsilon]_2) \xrightarrow{\delta} \Lambda^2\mathbb{F}[\varepsilon]_2^\times, \tag{3.0.1}$$

with δ defined by $\delta([x]) = x \wedge (1 - x)$. In the following, for a group G, $H_\cdot(G, \mathbb{Z})$ denotes the discrete group homology of G with coefficients in $\mathbb{Z}$.

3.1. Our aim in this section is to show that there is a surjection

$$H_3(\mathrm{SL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \to \ker(\delta).$$

3.1.1. We begin by relating $H_3(\mathrm{GL}(\mathbb{F}[\varepsilon]_2),\mathbb{Z})$ to $\ker(\delta)$. Let us first recall the construction of the natural map

$$H_3(\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2),\mathbb{Z}) \to \ker(\delta)$$

in [19, §2].

Let us call an n-tuple of points $(x_0,\cdots,x_{n-1})$ with x_i in $\mathbb{P}^1(\mathbb{F}[\varepsilon]_2)$, for $0 \le i \le n-1$, *in generic position*, if its reduction to an n-tuple of points $(\underline{x}_0,\cdots,\underline{x}_{n-1})$ with $\underline{x}_i$ in $\mathbb{P}^1(\mathbb{F})$, has the property that $\underline{x}_i \neq \underline{x}_j$ for $i \neq j$. Let $\tilde{C}_n(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))$ denote the free abelian group generated by the $(n+1)$-tuple of points $(x_0,\cdots,x_n)$, with x_i in $\mathbb{P}^1(\mathbb{F}[\varepsilon]_2)$, in generic position. The differentials $d:\tilde{C}_{n+1}(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2)) \to \tilde{C}_n(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))$ defined by

$$d(x_0,\cdots,x_{n+1}) := \sum_{0\le i\le n+1} (-1)^i (x_0,\cdots,\hat{x}_i,\cdots,x_{n+1})$$

and the augmentation map $e:\tilde{C}_0(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2)) \to \mathbb{Z}$ defined by $e((x_0)) = 1$, give an acyclic complex [19, Lemma 2.1]:

$$\tilde{C}_*(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2)) \to \mathbb{Z} \to 0.$$

The last complex is naturally a complex of $\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2)$-modules, where, for $n \ge 0$, the group $\tilde{C}_n(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))$ is endowed with the natural $\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2)$ action and $\mathbb{Z}$ with the trivial $\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2)$ action. Therefore, if $C_n(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))$ denotes the $\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2)$-coinvariants of $\tilde{C}_n(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))$, we have natural maps

$$H_i(\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2),\mathbb{Z}) \to H_i(C_*(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))),$$

for every $i \ge 0$. In particular, we have a natural map:

$$H_3(\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2),\mathbb{Z}) \to H_3(C_*(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))). \tag{3.1.1}$$

There is an isomorphism

$$\alpha : C_3(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))/d(C_4(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))) \xrightarrow{\sim} B_2(\mathbb{F}[\varepsilon]_2) \tag{3.1.2}$$

such that $\alpha^{-1}([x]) = (0,x,1,\infty)$ [21, Remark 3.8.2]. Moreover, this isomorphism fits into a commutative diagram [21, (3.8.3)]:

$$\begin{array}{ccc} C_3(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))/d(C_4(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2))) & \xrightarrow{d} & C_2(\mathbb{P}^1(\mathbb{F}[\varepsilon]_2)) \\ \downarrow & & \downarrow \\ B_2(\mathbb{F}[\varepsilon]_2) & \xrightarrow{\delta} & \Lambda^2\mathbb{F}[\varepsilon]_2^\times. \end{array}$$

This immediately implies that the map in (3.1.1) composed with α has image in $\ker(\delta)$ and hence gives the map

$$H_3(\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2),\mathbb{Z}) \to \ker(\delta)$$

we were looking for.

The following statement is exactly the same as in [21, Proposition 5.1.1], but with $\mathbb{Z}$-coefficients instead of $\mathbb{Q}$-coefficients. However, exactly the same proof works in this case.

Proposition 3.1.1. *The natural map*

$$H_3(\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \to \ker(\delta)$$

above, is a surjection. Moreover, $H_3(\mathrm{T}_2(\mathbb{F}[\varepsilon]_2))$ lies in the kernel of this map, where

$$\mathrm{T}_2(\mathbb{F}[\varepsilon]_2) \subseteq \mathrm{GL}_2(\mathbb{F}[\varepsilon]_2)$$

denotes the subgroup of diagonal matrices.

Next we extend this map using Suslin's stabilization theorem.

Proposition 3.1.2. *There is a natural map*

$$H_3(\mathrm{GL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \to \ker(\delta)$$

whose restriction to $H_3(\mathrm{GL}_2(\mathbb{F}[\varepsilon]_2), \mathbb{Z})$ is the map in Proposition 3.1.1.

Proof. First we would like to extend this map to a map:

$$H_3(\mathrm{GL}_3(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \to \ker(\delta). \tag{3.1.3}$$

Such a map is constructed in [21, §3.8], but with $\mathbb{Q}$-coefficients instead of $\mathbb{Z}$-coefficients. The only place where $\mathbb{Q}$-coefficients is used in *loc. cit.* is in the proof of [21, Claim 3.8.7]. Therefore, replacing that statement with Claim 3.1.3 below, [21, §3.8] gives the map (3.1.3), extending the map in Proposition 3.1.1.

Claim 3.1.3. *With the identification (3.1.2), we have*

$$(x_1, x_2, x_3, x_4) = \mathrm{sign}(\sigma)(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$$

in $B_2(\mathbb{F}[\varepsilon]_2)$, for any $\sigma \in S_4$, and any 4-tuple of points (x_1, x_2, x_3, x_4), with x_i in $\mathbb{P}^1(\mathbb{F}[\varepsilon]_2)$, for $1 \leq i \leq 4$, in general position.

Proof. The proof is exactly the same as that of [21, Claim 3.8.7]; except we need to check that, for every $x \in \mathbb{F}[\varepsilon]_2^\flat$, we have

$$[x] = -[x^{-1}], \text{ and } [1-x] = -[x]$$

in $B_2(\mathbb{F}[\varepsilon]_2)$.

If we let $\langle x \rangle := [x] + [x^{-1}]$, then exactly as in [19, Lemma 1.2], $2\langle y \rangle = 0$, and $\langle y \rangle + \langle x/y \rangle = \langle x \rangle$, for x, y, and $x/y \in \mathbb{F}[\varepsilon]_2^\flat$. Given $x \in \mathbb{F}[\varepsilon]_2^\flat$, since $\mathbb{F}$ is algebraically closed and $p > 2$, there exists $y \in \mathbb{F}[\varepsilon]_2^\flat$ such that $y^2 = x$. Then the last equation gives $\langle x \rangle = 2\langle y \rangle = 0$. Therefore $[x] = -[x^{-1}]$.

The proofs of [19, Lemma 1.3, Lemma 1.5] carry over to the $\mathbb{F}[\varepsilon]_2$ case and they imply that $[x] = -[1-x]$, since $\mathbb{F}$ is algebraically closed. □

To finish the proof of the proposition, we only note that Guin's stability theorem [12, §3] gives that the natural map

$$H_3(\mathrm{GL}_3(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \to H_3(\mathrm{GL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z})$$

is an isomorphism. □

3.1.2. Let R be any ring. Then by Whitehead's lemma [15, Proposition 11.1.5], the commutator subgroup $[\mathrm{GL}(R), \mathrm{GL}(R)]$ of $\mathrm{GL}(R)$ is equal to the perfect subgroup $\mathrm{E}(R)$ generated by elementary matrices. Recall that Quillen's plus construction applied to $\mathrm{BGL}(R)$ with respect to $\mathrm{E}(R)$ gives a space $\mathrm{BGL}^+(R)$ such that

$$K_i(R) := \pi_i(\mathrm{BGL}^+(R)), \tag{3.1.4}$$

for $i \geq 1$ [15, §11.2.4].

In this section, let $\mathbb{E} \in \{\mathbb{F}\} \cup \{\mathbb{F}_{p^r} | r \in \mathbb{N}\}$.

Lemma 3.1.4. *With the notation above,* $\mathrm{E}(\mathbb{E}[\varepsilon]_2) = \mathrm{SL}(\mathbb{E}[\varepsilon]_2)$, *and* $H_1(\mathrm{SL}(\mathbb{E}[\varepsilon]_2), \mathbb{Z}) = 0$.

Proof. By the previous paragraph, showing that $\mathrm{E}(R) = \mathrm{SL}(R)$ is equivalent to showing that

$$H_1(\mathrm{SL}(R), \mathbb{Z}) = \mathrm{SL}(R)/[\mathrm{SL}(R), \mathrm{SL}(R)] = 0.$$

By a theorem of Wang [9, Theorem 2.8.12], $\mathrm{E}(\mathbb{E}) = \mathrm{SL}(\mathbb{E})$. Consider the exact sequence

$$1 \to \mathrm{V}(\mathbb{E}) \to \mathrm{SL}(\mathbb{E}[\varepsilon]_2) \to \mathrm{SL}(\mathbb{E}) \to 1,$$

such that $\mathrm{V}(\mathbb{E}) = \cup_n \mathrm{V}_n(\mathbb{E})$, where $\mathrm{V}_n(\mathbb{E})$ is the subgroup of $\mathrm{M}_n(\mathbb{E})$ consisting of matrices of trace 0. This gives a Hochschild-Serre spectral sequence [6, Theorem 6.3]:

$$E^2_{pq} = H_p(\mathrm{SL}(\mathbb{E}), H_q(\mathrm{V}(\mathbb{E}), \mathbb{Z})) \Rightarrow H_{p+q}(\mathrm{SL}(\mathbb{E}[\varepsilon]_2), \mathbb{Z}). \tag{3.1.5}$$

Since $H_1(\mathrm{V}(\mathbb{E}), \mathbb{Z}) = \mathrm{V}(\mathbb{E})$, with the action of $\mathrm{SL}(\mathbb{E})$ on $\mathrm{V}(\mathbb{E})$ given by conjugation, we only need to show that $\mathrm{V}(\mathbb{E})_{\mathrm{SL}(\mathbb{E})} = 0$, where the subscript denotes taking coinvariants.

Note that

$$\mathrm{V}(\mathbb{E})_{\mathrm{SL}(\mathbb{E})} = \mathrm{V}(\mathbb{E})_{\mathrm{GL}(\mathbb{E})},$$

since $\det(T)^{-1}T \in \mathrm{SL}_n(\mathbb{F}_{p^m})$, for $T \in \mathrm{GL}_n(\mathbb{F}_{p^m})$, and $n \equiv 1\ (mod\,(p^m - 1))$. Therefore, it suffices to show that $\mathrm{V}(\mathbb{E})_{\mathrm{GL}(\mathbb{E})} = 0$.

Let $n \geq 2$ and let $E_{ij} \in \mathrm{M}_n(\mathbb{E})$ denote the matrix that has 1 in the i-th row and j-th column, and zero elsewhere. Then E_{ij} is similar to E_{12}, if $i \neq j$, and to E_{11}, if $i = j$. This and the Jordan decomposition theorem imply that, if $N \in \mathrm{V}_n(\mathbb{E})$ is a nilpotent matrix, then $N = \lambda E_{12}$ in $\mathrm{V}_n(\mathbb{E})_{\mathrm{GL}_n(\mathbb{E})}$, for some $\lambda \in \mathbb{F}_p \subseteq \mathbb{E}$. Since in $\mathrm{V}_n(\mathbb{E})_{\mathrm{GL}_n(\mathbb{E})}$, we have $2E_{12} = E_{12} + E_{21} = E_{11} - E_{22} = 0$, and $p \neq 2$, $E_{12} = 0$ and $N = 0$ in $\mathrm{V}_n(\mathbb{E})_{\mathrm{GL}_n(\mathbb{E})}$.

Now note that any $A \in \mathrm{V}_n(\mathbb{E})$ can be written as a sum of two nilpotent matrices and a diagonal matrix D such that $tr(D) = 0$. Then, by the above, $A = D = tr(D)E_{11} = 0$ in $\mathrm{V}_n(\mathbb{E})_{\mathrm{GL}_n(\mathbb{E})}$.

□

3.1.3. By Lemma 3.1.4, the commutator subgroup of $\mathrm{GL}(\mathbb{E}[\varepsilon]_2)$ is the perfect subgroup $\mathrm{SL}(\mathbb{E}[\varepsilon]_2)$. Let $\mathrm{BSL}(\mathbb{E}[\varepsilon]_2)^+$ denote the result of applying Quillen's plus construction [15, §11.2.4] to $\mathrm{BSL}(\mathbb{E}[\varepsilon]_2)$ with respect to $\mathrm{SL}(\mathbb{E}[\varepsilon]_2)$. The natural map

$$\mathrm{BSL}(\mathbb{E}[\varepsilon]_2)^+ \to \mathrm{BGL}(\mathbb{E}[\varepsilon]_2)^+$$

is the universal covering space projection [15, Corollary 11.2.3]. Therefore,

$$K_i(\mathbb{E}[\varepsilon]_2) \simeq \pi_i(\mathrm{BSL}(\mathbb{E}[\varepsilon]_2)^+),$$

for $i \geq 2$.

Note that

Lemma 3.1.5. $K_2(\mathbb{E}[\varepsilon]_2) = 0$.

Proof. We have,

$$K_2(\mathbb{E}[\varepsilon]_2) \simeq K_2(\mathbb{E}) \oplus K_2(\mathbb{E}[\varepsilon]_2, (\varepsilon)).$$

Since $K_2(\mathbb{E}[\varepsilon]_2, (\varepsilon)) \simeq \Omega^1_{\mathbb{E}/\mathbb{Z}} = 0$ [13] and

$$K_2(\mathbb{E}) = K_2^{\mathrm{M}}(\mathbb{E}) = \varinjlim_{\mathbb{F}_{p^n} \subseteq \mathbb{E}} K_2^{\mathrm{M}}(\mathbb{F}_{p^n}) = 0,$$

[16, Theorem 11.1, Corollary 9.13] the assertion follows. □

Therefore $\mathrm{BSL}(\mathbb{E}[\varepsilon]_2)^+$ is 2-connected. Since

$$H_3(\mathrm{SL}(\mathbb{E}[\varepsilon]_2), \mathbb{Z}) \xrightarrow{\sim} H_3(\mathrm{BSL}(\mathbb{E}[\varepsilon]_2)^+, \mathbb{Z}),$$

[15, Theorem 11.2.2] Hurewicz theorem applied to $\mathrm{BSL}(\mathbb{E}[\varepsilon]_2)^+$ gives an isomorphism

$$K_3(\mathbb{E}[\varepsilon]_2) \xrightarrow{\sim} H_3(\mathrm{SL}(\mathbb{E}[\varepsilon]_2), \mathbb{Z}). \tag{3.1.6}$$

The same argument also gives,

$$H_2(\mathrm{SL}(\mathbb{E}[\varepsilon]_2), \mathbb{Z}) = K_2(\mathbb{E}[\varepsilon]_2) = 0. \tag{3.1.7}$$

Proposition 3.1.6. *The natural map*

$$H_3(\mathrm{SL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \to \ker(\delta)$$

is a surjection.

Proof. Since $\mathrm{SL}(\mathbb{F}[\varepsilon]_2)$ is the commutator subgroup of $\mathrm{GL}(\mathbb{F}[\varepsilon]_2)$, [19, Lemma 5.3] shows that there is a homotopy equivalence

$$\mathrm{BSL}(\mathbb{F}[\varepsilon]_2)^+ \times \mathrm{B}\mathbb{F}[\varepsilon]_2^\times \to \mathrm{BGL}(\mathbb{F}[\varepsilon]_2)^+. \tag{3.1.8}$$

Noting that $H_i(\mathrm{BGL}(\mathbb{F}[\varepsilon]_2)^+, \mathbb{Z}) \xrightarrow{\sim} H_i(\mathrm{GL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z})$ and

$$H_i(\mathrm{BSL}(\mathbb{F}[\varepsilon]_2)^+, \mathbb{Z}) \xrightarrow{\sim} H_i(\mathrm{SL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z}),$$

for $i \geq 0$, the above computations and the Künneth theorem give an isomorphism

$$H_3(\mathrm{SL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \oplus H_3(\mathbb{F}[\varepsilon]_2^\times, \mathbb{Z}) \xrightarrow{\sim} H_3(\mathrm{GL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z}).$$

Since the map (3.1.8) is induced by a map which sends $\mathbb{F}[\varepsilon]_2^\times$ into $\mathrm{GL}_1(\mathbb{F}[\varepsilon]_2) \subseteq T_2(\mathbb{F}[\varepsilon]_2) \subseteq \mathrm{GL}(\mathbb{F}[\varepsilon]_2)$, we see by Proposition 3.1.1 and Proposition 3.1.2 that the natural map

$$H_3(\mathrm{SL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \to \ker(\delta)$$

is a surjection. □

3.2. In this section, using $\mathfrak{Li}_2$ and li_2 below, we will compute the cohomology of the Bloch complex of weight two.

3.2.1. We constructed a map $\mathrm{Li}_{2,2} : B_2(k[\varepsilon]_2) \to k$, where k is a field of characteristic 0, in [21, Definition 2.2.4]. Since $p \geq 5$, the same construction, with the same proof, gives a map

$$li_2 : B_2(\mathbb{F}[\varepsilon]_2) \to \mathbb{F}.$$

Namely,

$$li_2(s + \alpha\varepsilon) := -\frac{\alpha^3}{2s^2(1-s)^2},$$

for $s + \alpha\varepsilon \in \mathbb{F}[\varepsilon]_2^\flat$ (cf. [21, §2.1]).

3.2.2. Combining $\mathfrak{Li}_2$ and li_2, we have a map

$$\mathfrak{Li}_2 \oplus li_2 : B_2(\mathbb{F}[\varepsilon]_2) \to \mathbb{F}(1) \oplus \mathbb{F}(3).$$

Lemma 3.2.1. *The maps* $\mathfrak{Li}_2 : \ker(\delta)^\circ \to \mathbb{F}(1)$ *and* $li_2 : \ker(\delta)^\circ \to \mathbb{F}(3)$ *are surjective.*

Proof. Note that since $\pounds_1(x)$ is of degree $p-1$ in x, there is an $s_0 \in \mathbb{F}_p$ such that $\pounds_1(s_0) \neq 0$. Since $\pounds_1(0) = \pounds_1(1) = 0$, $s_0 \in \mathbb{F}^\flat$ and this gives that $\mathfrak{Li}_2(\langle s_0 \rangle) \neq 0$ and $li_2(\langle s_0 \rangle) \neq 0$.

For $\alpha \in \mathbb{F}^\times$ and $\beta \in \mathbb{F}$, we have $\alpha \otimes \beta = \alpha^{1/p} \otimes (p\beta) = 0$ in $\mathbb{F}^\times \otimes \mathbb{F}$. Hence

$$\Lambda^2 \mathbb{F}[\varepsilon]_2^\times = \Lambda^2 \mathbb{F}^\times \oplus \Lambda^2 (\mathbb{F}[\varepsilon]_2^\times)^\circ \xrightarrow{\sim} \Lambda^2 \mathbb{F}^\times \oplus \Lambda^2(\varepsilon\mathbb{F})$$

and

$$(\Lambda^2 \mathbb{F}[\varepsilon]_2^\times)^\circ \xrightarrow{\sim} \Lambda^2(\varepsilon\mathbb{F}). \tag{3.2.1}$$

For $\alpha \in B_2(\mathbb{F}[\varepsilon]_2)$, let $\underline{\alpha} \in B_2(\mathbb{F}) \subseteq B_2(\mathbb{F}[\varepsilon]_2)$ be the reduction of α modulo (ε), and $\alpha^\circ := \alpha - \underline{\alpha}$. Let $\lambda_0 \in \mathbb{N}_{\geq 2}$. Then $\beta(\lambda_0, s_0) := \lambda_0 \star \langle s_0 \rangle^\circ - \lambda_0^2 \langle s_0 \rangle^\circ \in \ker(\delta)^\circ$ and

$$\mathfrak{Li}_2(\beta(\lambda_0, s_0)) = (\lambda_0 - \lambda_0^2)\mathfrak{Li}_2(\langle s_0 \rangle) \neq 0$$

and

$$li_2(\beta(\lambda_0, s_0)) = (\lambda_0^3 - \lambda_0^2) li_2(\langle s_0 \rangle) \neq 0.$$

Therefore $\mathfrak{Li}_2$ and li_2 are nonzero on $\ker(\delta)^0$. Surjectivity follows, since $\mathbb{F}$ is algebraically closed. □

Note that $K_n(\mathbb{F}[\varepsilon]_2)^\circ = K_n(\mathbb{F}[\varepsilon]_2, (\varepsilon))$.

Lemma 3.2.2. *We have an isomorphism*

$$K_3(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} \mathbb{F}(1) \oplus \mathbb{F}(3), \tag{3.2.2}$$

as $\mathbb{Z}[\mathbb{F}^\times]$*-modules.*

Proof. First note that for $\mathbb{E} \in \{\mathbb{F}\} \cup \{\mathbb{F}_{p^r} | r \in \mathbb{N}\}$, we have by (3.1.6),

$$K_3(\mathbb{E}[\varepsilon]_2) \xrightarrow{\sim} H_3(\mathrm{SL}(\mathbb{E}[\varepsilon]_2), \mathbb{Z}).$$

Similarly, $K_3(\mathbb{E}) \xrightarrow{\sim} H_3(\mathrm{SL}(\mathbb{E}), \mathbb{Z})$. Therefore,

$$\begin{aligned} K_3(\mathbb{F}[\varepsilon]_2)^\circ &= \ker(H_3(\mathrm{SL}(\mathbb{F}[\varepsilon]_2), \mathbb{Z}) \to H_3(\mathrm{SL}(\mathbb{F}), \mathbb{Z})) \\ &= \varinjlim_n \ker(H_3(\mathrm{SL}(\mathbb{F}_{p^n}[\varepsilon]_2), \mathbb{Z}) \to H_3(\mathrm{SL}(\mathbb{F}_{p^n}), \mathbb{Z})) \\ &= \varinjlim_n K_3(\mathbb{F}_{p^n}[\varepsilon]_2)^\circ. \end{aligned}$$

By [1, Theorem 1.1], there are canonical isomorphisms

$$K_3(\mathbb{F}_{p^n}[\varepsilon]_2)^\circ \xrightarrow{\sim} K_1(\mathbb{F}_{p^n}[\varepsilon]_4, (\varepsilon))/(1+\alpha\varepsilon^2|\alpha \in \mathbb{F}_{p^n}).$$

Since the right hand side is isomorphic, as a $\mathbb{Z}[\mathbb{F}_{p^n}^\times]$-module, to $\varepsilon\mathbb{F}_{p^n} \oplus \varepsilon^3\mathbb{F}_{p^n}$, this gives

$$K_3(\mathbb{F}[\varepsilon]_2)^\circ \simeq \varinjlim_n (\varepsilon\mathbb{F}_{p^n} \oplus \varepsilon^3\mathbb{F}_{p^n}) \simeq \mathbb{F}(1) \oplus \mathbb{F}(3).$$

□

We denote $V(\mathbb{F})$ by V to ease the notation.

Corollary 3.2.3. *There is a natural isomorphism*

$$K_3(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} H_2(\mathrm{SL}(\mathbb{F}), V) \oplus H_0(\mathrm{SL}(\mathbb{F}), \Lambda^3 V).$$

Proof. Let us look at the $\mathbb{F}^\times$-action on the terms of the Hochschild-Serre spectral sequence (3.1.5) that contribute to $K_3(\mathbb{F}[\varepsilon]_2)^\circ$. Only those parts of the terms on which the $\mathbb{F}^\times$-action has weight 1 or 3 will have a contribution to $K_3(\mathbb{F}[\varepsilon]_2)^\circ$, by Proposition 3.2.2. Since the action of $\mathbb{F}^\times$ on $\mathbb{F}[\varepsilon]_2$ is by dilatation, the induced action of $\mathbb{F}^\times$ on V is simply by multiplication. We will only consider the restriction of this action to $\mathbb{F}_p^\times \subseteq \mathbb{F}^\times$. This will suffice to distinguish between the distinct weight pieces since $p \geq 5$. One reason for only looking at the $\mathbb{F}_p^\times$-action is that the various tensor power constructions below are over $\mathbb{Z}$, instead of $\mathbb{F}$.

Since $H_1(V, \mathbb{Z}) = V$, the $\mathbb{F}_p^\times$-action on E^2_{21} has weight 1.

Since $H_2(V, \mathbb{Z}) = \Lambda^2 V$ [6, Theorem 6.4, §V], the action of $\mathbb{F}_p^\times$ on $H_2(V, \mathbb{Z})$ has weight 2, hence the same is true for the action on E^2_{12}. Hence the E^2_{12} term in the spectral sequence does not contribute to $K_3(\mathbb{F}[\varepsilon]_2)^\circ$.

The long exact sequence for homology associated to the exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0$, gives an exact sequence $0 \to H_3(V, \mathbb{Z}) \to H_3(V, \mathbb{Z}/p) \to H_2(V, \mathbb{Z}) \to 0$. Since $H_3(V, \mathbb{Z}/p) = \Lambda^3 V \oplus (V \otimes V)$ [6, Theorem 6.6, §V] and $H_2(V, \mathbb{Z}) = \Lambda^2 V$, the only part of $H_3(V, \mathbb{Z})$ that has weight 1 or 3 under the $\mathbb{F}_p^\times$-action is $\Lambda^3 V$, and this weight is 3.

Combining all of these, we see that the only terms that have a contribution are $H_2(\mathrm{SL}(\mathbb{F}), V) = E^2_{21}$ and $H_0(\mathrm{SL}(\mathbb{F}), \Lambda^3 V) \subseteq E^2_{03}$. Note that the latter is a direct summand. By using the same arguments, we see that all the differentials $d^2_{\cdot\cdot}$ restricted to the groups above are 0. This shows that there is a filtration on $K_3(\mathbb{F}[\varepsilon]_2)^\circ$, whose graded pieces are the homology groups above. Since the $\mathbb{F}_p^\times$ action on these homology groups have weights 1 and 3, $K_3(\mathbb{F}[\varepsilon]_2)^\circ$ is a direct sum of these homology groups. □

Theorem 3.2.4. *The composition of the maps in (3.1.6) and Proposition 3.1.6 induce an isomorphism*

$$K_3(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} \ker(\delta)^\circ = \mathrm{H}^1(\gamma_2(2)^\circ). \tag{3.2.3}$$

The following two maps, which are induced by (3.2.3) and Corollary 3.2.3:

$$H_2(\mathrm{SL}(\mathbb{F}), V) \to K_3(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} \ker(\delta)^\circ \xrightarrow{\mathfrak{Li}_2} \mathbb{F}(1) \tag{3.2.4}$$

and

$$H_0(\mathrm{SL}(\mathbb{F}), \Lambda^3 V) \to K_3(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} \ker(\delta)^\circ \xrightarrow{li_2} \mathbb{F}(3) \tag{3.2.5}$$

are also isomorphisms.

Proof. Combining (3.2.3) above with the isomorphism (3.2.2) and the surjective maps in Lemma 3.2.1 gives a surjective map from $\mathbb{F}(1)\oplus\mathbb{F}(3)$ onto itself, which therefore is an isomorphism and hence so is (3.2.3). The other two statements follow immediately from this and Corollary 3.2.3. □

Lemma 3.2.5. *We have* $\mathrm{H}^2(\gamma_2(2)^\circ) = 0$.

Proof. By the definition of Milnor K-theory, $\mathrm{H}^2(\gamma_2(2)) = K_2^{\mathrm{M}}(\mathbb{F}[\varepsilon]_2)$ [15, 11.1.16]. By [12, §4.2], $K_2^{\mathrm{M}}(\mathbb{F}[\varepsilon]_2) = K_2(\mathbb{F}[\varepsilon]_2)$. The assertion follows from Lemma 3.1.5. □

Proposition 3.2.6. *The maps* $\mathfrak{Li}_2$, li_2 *and* δ *induce an isomorphism:*

$$B_2(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} \mathbb{F}(1)\oplus\mathbb{F}(3)\oplus\Lambda^2\mathbb{F}(1).$$

Proof. This follows from Theorem 3.2.4 and Lemma 3.2.5 and the fact that $\mathbb{F}\otimes\mathbb{F}^\times = 0$. □

4. Application to Deformations of Aomoto Dilogarithms

4.1. Our main references for Aomoto polylogarithms are [3], [10, §1.16], and [23]. First we define an infinitesimal version of Aomoto dilogarithms, as in [22, §3.2].

We call a closed subscheme $L \subseteq \mathbb{P}^2_{\mathbb{F}[\varepsilon]_2} = \mathrm{Proj}(\mathbb{F}[\varepsilon]_2[z_0, z_1, z_2])$, a *line*, if

$$L = \mathrm{Proj}(\mathbb{F}[\varepsilon]_2[z_0, z_1, z_2]/(a_0z_0 + a_1z_1 + a_2z_2)),$$

for some $a_0, a_1, a_2 \in \mathbb{F}[\varepsilon]_2$, at least one of which is invertible in $\mathbb{F}[\varepsilon]_2$. We denote by

$$\underline{L} := \mathrm{Proj}(\mathbb{F}[z_0, z_1, z_2]/(\underline{a}_0z_0 + \underline{a}_1z_1 + \underline{a}_2z_2)) \subseteq \mathbb{P}^2_{\mathbb{F}},$$

the reduction of L.

A *simplex* in $\mathbb{P}^2_{\mathbb{F}[\varepsilon]_2}$ is an ordered triple $H := (H_0, H_1, H_2)$ of lines $H_i \subseteq \mathbb{P}^2_{\mathbb{F}[\varepsilon]_2}$, for $0 \le i \le 2$. We denote the reduction of H to a simplex in $\mathbb{P}^2_{\mathbb{F}}$ by $\underline{H} := (\underline{H}_0, \underline{H}_1, \underline{H}_2)$. H is said to be *non-degenerate*, if $\cap_{0\le i\le 2}\underline{H}_i = \emptyset$. A *face* of H is an intersection $\cap_{i\in I}H_i$, for some $I \subset \{0, 1, 2\}$. A pair of simplices (L, M) is said to be *admissible* if $\underline{L}$ and $\underline{M}$ do not have a common face.

Let $A_2(\mathbb{F}[\varepsilon]_2)$ be the abelian group generated by pairs of *admissible* simplices (L, M) modulo the following relations:

(i) $(L, M) = 0$, if L or M is degenerate

(ii) For $\sigma \in Sym(2)$, the group of permutations of $\{0, 1, 2\}$, let

$$\sigma(L_0, L_1, L_2) := (L_{\sigma(0)}, L_{\sigma(1)}, L_{\sigma(2)}).$$

Then for every $\sigma \in Sym(2)$,

$$(\sigma(L), M) = (L, \sigma(M)) = \mathrm{sgn}(\sigma)(L, M).$$

(iii) If L_0, L_1, L_2, L_3 are lines in $\mathbb{P}^2_{\mathbb{F}[\varepsilon]_2}$ such that for all $0 \le i \le 3$,

$$((L_0, \cdots, \hat{L}_i, \cdots, L_3), M)$$

is admissible then we have the additivity relation for the first component:

$$\sum_{0\le i\le 3} (-1)^i((L_0, \cdots, \hat{L}_i, \cdots, L_3), M) = 0.$$

We have the analogous additivity relation for the second component.

(iv) For every $\alpha \in \mathrm{PGL}_3(\mathbb{F}[\varepsilon]_2)$,

$$(\alpha L, \alpha M) = (L, M).$$

4.2. There is a natural map [3, §2.2]:

$$m : S^2\mathbb{F}[\varepsilon]_2^\times \to A_2(\mathbb{F}[\varepsilon]_2).$$

If $\alpha, \beta \in \mathbb{F}[\varepsilon]_2^\flat$, and $\alpha \odot \beta$ denotes the image of $\alpha \otimes \beta$ in $S^2\mathbb{F}[\varepsilon]_2^\times$ then $m(\alpha \odot \beta) := P(\alpha, \beta)$, where $P(\alpha, \beta) \in A_2(\mathbb{F}[\varepsilon]_2)$ is the *prism* defined as follows. Let z_0, z_1, z_2 denote the coordinates in $\mathbb{P}^2_{\mathbb{F}[\varepsilon]_2}$, $L := (z_0 = 0, z_1 = 0, z_2 = 0)$, and

$$\Delta_1 := (z_2 - z_0 = 0, z_1 - \alpha z_0 = 0, z_2 - z_0 - \frac{\beta - 1}{\alpha - 1}(z_1 - z_0) = 0)$$

$$\Delta_2 := (z_2 - z_0 - \frac{\beta - 1}{\alpha - 1}(z_1 - z_0) = 0, z_2 - \beta z_0 = 0, z_1 - z_0 = 0).$$

Then let $P(\alpha, \beta) = (L, \Delta_1) + (L, \Delta_2)$, $P_2(\mathbb{F}[\varepsilon]_2) = m(S^2\mathbb{F}[\varepsilon]_2^\times)$ and

$$B_2'(\mathbb{F}[\varepsilon]_2) := A_2(\mathbb{F}[\varepsilon]_2)/P_2(\mathbb{F}[\varepsilon]_2).$$

4.3. For $x \in \mathbb{F}[\varepsilon]_2^\flat$, let (L, M_x) be the configuration in $\mathbb{P}^2_{\mathbb{F}[\varepsilon]_2}$, where L is the simplex above and M_x is $(z_1 - z_0 = 0, z_1 + z_2 = z_0, z_2 - xz_0 = 0)$, cf. [3, Fig. 1.4]. This then defines a map $l_2 : \mathbb{Z}[\mathbb{F}[\varepsilon]_2^\flat] \to A_2(\mathbb{F}[\varepsilon]_2)$, by letting $l_2(x) := (L, M_x)$.

Proposition 4.3.1. *The map l_2 above induces an isomorphism*

$$B_2(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} B_2'(\mathbb{F}[\varepsilon]_2)^\circ.$$

Proof. For $a \in \mathbb{F}[\varepsilon]_2^\flat$ let $H(a) \in A_2(\mathbb{F}[\varepsilon]_2)$ be the *half-square with side* a, cf. [3, Fig. 3.1]. Namely, $H(a)$ is $(L, S(a))$, where L is as above and $M(a)$ is defined as:

$$(z_1 - az_0 = 0, z_2 - z_1 = 0, z_2 - z_0 = 0).$$

Since $\mathbb{F}[\varepsilon]_2^\times$ is 2-divisible, by bisecting the sides of $M(a)$, we see that

$$H(a) = 2P(a^{1/2}, a^{1/2}) \in P_2(\mathbb{F}[\varepsilon]_2).$$

Therefore half-squares lie in the group generated by the prisms. Also in the notation of [3, §3.6], $\delta(1) \in A_2(\mathbb{F})$. Using these two facts, the proof of Main Theorem 2 [3, §3.8] carries over in our case to prove the statement. □

Remark 4.3.2. The inverse to the map in Proposition 4.3.1 is the map which is induced by $\eta : A_2(\mathbb{F}[\varepsilon]_2) \to B_2(\mathbb{F}[\varepsilon]_2)$, defined in [3, §3.3]:

$$\eta(L, M) = \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma)(L_0 \cap M_{\sigma(1)}, L_1 \cap M_{\sigma(1)}, L_2 \cap M_{\sigma(1)}, M_{\sigma(0)} \cap M_{\sigma(1)}).$$

4.4. There is a homomorphism $\nu_{1,1} : A_2(\mathbb{F}[\varepsilon]_2) \to \mathbb{F}[\varepsilon]_2^\times \otimes \mathbb{F}[\varepsilon]_2^\times$ induced by a comultiplication map which makes the following diagram commutative up to sign [3, Proposition, §2.14], [23, Example 5.1]:

$$\begin{array}{ccc} A_2(\mathbb{F}[\varepsilon]_2) & \xrightarrow{\nu_{1,1}} & \mathbb{F}[\varepsilon]_2^\times \otimes \mathbb{F}[\varepsilon]_2^\times \\ \eta\downarrow & & \downarrow \\ B_2(\mathbb{F}[\varepsilon]_2) & \xrightarrow{\delta} & \Lambda^2\mathbb{F}[\varepsilon]_2^\times. \end{array}$$

Moreover,

$$\nu_{1,1}(m(\alpha \odot \beta)) = \alpha \otimes \beta + \beta \otimes \alpha \tag{4.4.1}$$

[3, Proposition, §2.12].

Theorem 4.4.1. *The maps $\mathfrak{Li}_2 \circ \eta$, $li_2 \circ \eta$ and $\nu_{1,1}$ induce an isomorphism*

$$A_2(\mathbb{F}[\varepsilon]_2)^\circ \xrightarrow{\sim} \mathbb{F}(1) \oplus \mathbb{F}(3) \oplus (\mathbb{F}(1) \otimes \mathbb{F}(1)).$$

Proof. Since $\mathbb{F}$ is algebraically closed and $p > 2$, $\mathbb{F}[\varepsilon]_2^\times$ is 2-divisible. This, together with (4.4.1), implies that in the sequence of homomorphisms

$$S^2\mathbb{F}[\varepsilon]_2^\times \xrightarrow{m} P_2(\mathbb{F}[\varepsilon]_2) \xrightarrow{\nu_{1,1}} \ker((\mathbb{F}[\varepsilon]_2^\times \otimes \mathbb{F}[\varepsilon]_2^\times) \to \Lambda^2\mathbb{F}[\varepsilon]_2^\times),$$

both m and $\nu_{1,1}$ are isomorphisms. Since $(\mathbb{F}[\varepsilon]_2^\times \otimes \mathbb{F}[\varepsilon]_2^\times)^\circ = \mathbb{F}(1) \otimes \mathbb{F}(1)$, this implies that $\nu_{1,1}$ induces an isomorphism

$$P_2(\mathbb{F}[\varepsilon]_2)^\circ \to \ker(\mathbb{F}(1) \otimes \mathbb{F}(1) \to \Lambda^2\mathbb{F}(1)). \tag{4.4.2}$$

Since Proposition 4.3.1 states that $A_2(\mathbb{F}[\varepsilon]_2)^\circ / P_2(\mathbb{F}[\varepsilon]_2)^\circ \simeq B_2(\mathbb{F}[\varepsilon]_2)^\circ$, (4.4.2), the above commutative diagram and Proposition 3.2.6 give the statement. □

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