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p-Adic multi-zeta values

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Abstract

Let X be the projective line minus $0, 1,$ and ∞ over $\mathbb{Q}_p$. The aim of the following is to give a series representations of the p-adic multi-zeta values in the depth two quotient. The approach is to use the lifting $\tilde{F}(z) = z^p$ of the frobenius which is not a good choice near 1, but which gives simple formulas away from 1, and to relate the action of frobenius on the de Rham path from 0 to ∞ and on the one from 0 to 1. Also some relations between the p-adic multi-zeta values of depth two are obtained.
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1. Introduction

The multi-zeta values, invented by Euler, are defined as

$$\zeta(s_k, \dots, s_2, s_1) := \sum_{n_k > \cdots > n_2 > n_1 > 0} \frac{1}{n_k^{s_k} \cdots n_2^{s_2} n_1^{s_1}}$$

for $s_1, \dots, s_{k-1} \geqslant 1$ and $s_k > 1$. Recently, they appeared in the work of Drinfel'd [9] on deformations of Hopf algebras, as the coefficients of the solution of the KZ equation, which was used to prove the existence of an associator over $\mathbb{Q}$; in the work [17] of Zagier who studied the $\mathbb{Q}$-algebra generated by the multi-zeta; and in the works of Broadhurst and Kreimer on knot theory. They are studied extensively by Goncharov (in [12,13]) as periods of the fundamental group of the projective line

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minus three points. And a connection between the geometry of modular varieties is found in [14].

Let $X/\mathbb{Q}$ be a smooth variety with good reduction. The unipotent theory of the fundamental group of X is of motivic nature [5], and when $X/\mathbb{Q}$ is unirational, the unipotent fundamental group can be defined as an object in the category of mixed Tate motives over $\mathbb{Q}$ [7].

Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Let t_{01} be the unit tangent vector at zero pointing towards 1 and t_{10} the unit tangent vector at 1 pointing towards 0. Let ${}_{t_{10}}\mathcal{G}_{t_{01}}(\cdot)$ denote the $\pi_{1,\cdot}(X, t_{01})$ torsor of paths from t_{01} to t_{10}, where the dot denotes the realization under consideration. In the Betti realization the real path defines an element ${}_{t_{10}}e(B)_{t_{01}}$ in ${}_{t_{10}}\mathcal{G}_{t_{01}}(B)$. Similarly the canonical trivialization of unipotent vector bundles with connection by Deligne [5] defines an element ${}_{t_{10}}e(dR)_{t_{01}}$ in ${}_{t_{10}}\mathcal{G}_{t_{01}}(dR)$. By taking the image of ${}_{t_{10}}e(B)_{t_{01}}$ under the de Rham–Betti comparison isomorphism

$$ {}_{t_{10}}\mathcal{G}_{t_{01}}(B) \otimes_{\mathbb{Q}} \mathbb{C} \simeq {}_{t_{10}}\mathcal{G}_{t_{01}}(dR) \otimes_{\mathbb{Q}} \mathbb{C} $$

and using ${}_{t_{10}}e(dR)_{t_{01}}$ to trivialize the torsor ${}_{t_{10}}\mathcal{G}_{t_{01}}(dR) \otimes \mathbb{C}$, we obtain an element $\gamma \in \pi_{1,dR}(X, t_{01}) \otimes \mathbb{C}$.

If A is a (commutative) $\mathbb{Q}$-algebra, an element of $\pi_{1,dR}(X, t_{01}) \otimes_{\mathbb{Q}} A$ corresponds to a group-like element of $A \langle\langle e_0, e_1 \rangle\rangle$, the Hopf algebra of associative formal power series over A in the variables e_0 and e_1. The coproduct

$$ \Delta : A \langle\langle e_0, e_1 \rangle\rangle \to A \langle\langle e_0, e_1 \rangle\rangle \otimes A \langle\langle e_0, e_1 \rangle\rangle $$

is defined by putting

$$ \Delta(e_0) := 1 \otimes e_0 + e_0 \otimes 1 \quad \text{and} \quad \Delta(e_1) := 1 \otimes e_1 + e_1 \otimes 1 $$

and group-like elements g are the ones with constant term 1 and that satisfy $\Delta(g) = g \otimes g$. Because of the last condition such formal power series are determined by the coefficients of e_0, e_1 and of the terms of the form $e_0^{s_k - 1} e_1 \cdots e_0^{s_1 - 1} e_1$, with $s_1, \ldots, s_{k-1} \geqslant 1$ and $s_k > 1$, in them.

The coefficients of e_0 and e_1 in γ are equal to zero. Therefore γ is determined by the coefficients of the terms of the form $e_0^{s_k - 1} e_1 \cdots e_0^{s_1 - 1} e_1$ as above. Let $\{\omega_i(z)\}_{1 \leqslant i \leqslant m}$ be a collection of meromorphic 1-forms on $\mathbb{A}^1_{\mathbb{C}}$, and let $\alpha : [0, 1] \to \mathbb{A}^1_{\mathbb{C}}$ be a path that does not meet the poles of the ω_i, then let

$$ \int_\alpha \omega_m \circ \cdots \circ \omega_2 \circ \omega_1 := \int_{1 \geqslant t_m \geqslant \cdots \geqslant t_2 \geqslant t_1 \geqslant 0} \omega_m(\alpha(t_m)) \wedge \cdots \wedge \omega_2(\alpha(t_2)) \wedge \omega_1(\alpha(t_1)). $$

Let $s_1, \ldots, s_k$ be as above with $s_k > 1$, β the standard inclusion of $[0, 1]$ in $\mathbb{A}^1_{\mathbb{C}}$, $m := \sum_i s_i$, and

$$ \tilde{\omega}_i(z) := \frac{dz}{1 - z} \quad \text{if} \quad i \in \{1, s_1 + 1, \ldots, s_{k-1} + 1\}, $$

$$\tilde{\omega}_i(z) := \frac{dz}{z} \quad \text{otherwise}$$

for $1 \leqslant i \leqslant m$. Then using the Euler–Kontsevich formula

$$\int_\beta \tilde{\omega}_m \circ \cdots \circ \tilde{\omega}_2 \circ \tilde{\omega}_1 = \sum_{n_m > \cdots > n_1 > 0} \frac{1}{n_m^{s_k} \cdots n_1^{s_1}}$$

we see that the coefficient of the term $e_0^{s_k-1}e_1 \cdots e_0^{s-1}e_1$ in γ is equal to

$$(-1)^k \zeta(s_k, \dots, s_1).$$

In the fundamental paper [5] of Deligne, which is the basis of this paper, the crystalline realization of the unipotent fundamental groups are defined. Let p be a prime number. Analogously to the Betti–de Rham comparison, the Crystalline–de Rham comparison gives an element $g \in \mathbb{Q}_p \ll e_0, e_1 \gg$ as follows. The comparison theorem gives an action of frobenius

$$F_* : \mathcal{G}(dR) \otimes \mathbb{Q}_p \to \mathcal{G}(dR) \otimes \mathbb{Q}_p.$$

Let

$$g :=_{t_{01}} e(dR)_{t_{10}} F_*({}_{t_{10}}e(dR)_{t_{01}}),$$

which represents the action of the frobenius on the canonical de Rham path between the tangent vectors t_{01} and t_{10}, and is the analog of γ above. This definition is due to Deligne (unpublished). The action of frobenius on $\pi_{1,dR}(X, t_{01})$ can be described as

$$F_*(e_0) = pe_0 \quad \text{and} \quad F_*(e_1) = g^{-1}pe_1g.$$

Therefore describing the action of the crystalline frobenius on the de Rham fundamental group reduces to describing g, and as above determining g reduces to determining the coefficients of the terms $e_0^{s_k-1}e_1 \cdots e_0^{s_1-1}e_1$ in g. We denote this coefficient by $p^{\sum s_i}\zeta_p(s_k, \dots, s_1)$ and call it a p-adic multi-zeta value.

Our aim in the following is to give a series representation of these values, similar to the one obtained by the Euler–Kontsevich formula above, for depth two, i.e. $k = 2$.

The depth one case was done in [5] using the distribution formula. In the depth two case the method using the distribution formula does not work; first because there are too many unknown terms, and also because the limits of the functions involved as the variable goes to 1 do not exist, even if we restrict to the set of the points whose p-power parts of the ramification indices are bounded, and need to be regularized using the lower depth p-adic multi-zeta values. A direct approach to obtain a formula for these values would be to choose a lifting of frobenius that is a good lifting except on the open disc of radius 1 around ∞, in other words it should be a lifting $\mathcal{F}$ of frobenius and it should satisfy $\mathcal{F}^*(0) = p\,(0)$ and $\mathcal{F}^*(1) = p\,(1)$. However such a choice of a lifting of frobenius makes the computations very complicated even for the depth one p-adic multi-zeta values which are values of the p-adic zeta function.

The approach taken in this chapter is to use the simple lifting $\mathcal{F}(z) = z^p$ of frobenius that is a good choice outside the open unit disk around 1, since it satisfies $\mathcal{F}^*(0) = p(0)$ and $\mathcal{F}^*(\infty) = p(\infty)$. In order to make computations we use the canonical trivialization of the $\pi_{1,dR}(X, t_{01})$ torsor of paths on X starting at t_{01}, noted above, and endow it with its canonical pro-unipotent connection ∇. Then the fact that $\mathcal{F}_*$ has to be a horizontal map from this torsor endowed with ∇ to its pull-back under $\mathcal{F}$ gives a differential equation that involves $g(z) := \mathcal{F}_*({}_z e(dR)_{t_{01}})$ and g. In order to obtain the value of g in small depths we apply frobenius to the relation that the sum of the residues is equal to zero, this gives a relation between $g(\infty)$ and g. Together with the differential equation obtained above we solve this system with respect to $g(z)$ for the depth less than or equal to three using induction on the weight. Since $\tilde{F}$ was a good choice on the rigid analytic space $\mathcal{U} := \mathbb{P}^1_{\mathrm{an}} \backslash D(1, 1^-)$ the coefficients of $g(z)$ are rigid analytic functions on $\mathcal{U}$. These should be thought of as technical, unnatural objects as they depend on the choice of a lifting of frobenius. Then we prove a proposition that is used to compute the values at infinity of the rigid analytic extensions to $\mathcal{U}$ of power series around zero, when these extensions are known to exist, and also give a necessary and sufficient condition for the existence of these extensions. Using this, the relation between $g(\infty)$ and g above and some explicit computations we obtain a formula for g for depth less than or equal to 2. Also some identities are obtained between the depth two p-adic multi-zeta values.

Furusho [11] defines p-adic multiple polylogarithms and also p-adic multiple zeta values as their value at 1. His definition could be thought of as working with the frobenius invariant path rather than working with the action of frobenius as we do. If we let $\gamma \in \mathbb{Q}_p \langle\langle e_0, e_1 \rangle\rangle$ denote the frobenius invariant path between the tangential basepoints at 0 and 1 then the relation between g and γ is described by $g = \gamma \cdot (F_* \gamma)^{-1}$, where the action of F is as above. From this equation it follows that γ and g can be inductively computed in terms of each other. In a future paper, we will compare our results to those of Furusho. Finally, we remark that our Proposition 3 is a very special case of the shuffle product formula of Besser and Furusho [3], conjectured by Deligne and also proved by himself using Voevodsky's theory.

Notation and convention: If we have an element a of the ring of associative formal power series over a ring A, in the variables x_i, $i \in I$, we denote the coefficient of x^J in a, for a multi-index J, by $a[x^J]$. By a variety X over a field K, we mean a geometrically integral K-scheme X, that is separated and of finite type over K.

2. Crystalline and de Rham fundamental groups

2.1. Fundamental group of a tannakian category

Let $\mathcal{M}/K$ be an abelian, K-linear, rigid, ACU, $\otimes$-category with $\mathrm{End}(1) \simeq K$, a *tensor category* in the terminology of Deligne [6, 1.2]). For a K-scheme S, a *fiber* *functor* ω of $\mathcal{M}$ over S is an exact K-linear functor from $\mathcal{M}$ to the tensor category of locally free sheaves of finite rank on S endowed with functorial ACU isomorphisms $\omega(X)\otimes\omega(Y)\simeq\omega(X\otimes Y)$. A *tannakian* category $\mathcal{M}/K$ is a tensor category over K that has a fiber functor over some nonempty K-scheme S. If ω is a fiber functor of $\mathcal{M}$ over S/K and $p_1, p_2 : S\times_K S\to S$ are the two projections, then the functor $\mathrm{Isom}_K(p_2^*\omega, p_1^*\omega)$ which associates to each $\pi : T\to S\times_K S$ the set of isomorphisms between $\pi^*p_2^*\omega$ and $\pi^*p_1^*\omega$ is representable by an affine scheme $\mathrm{Aut}_K(\omega)$ over $S\times_K S$. It is a groupoid acting on S, is called the *fundamental groupoid* of $\mathcal{M}$ at the fiber functor ω, and is denoted by $\mathcal{G}(\mathcal{M},\omega)$, cf. ([6, 1.11]) and ([5, 10.2–10.9, 10.26–10.33]). Pulling back the morphism $\mathcal{G}(\mathcal{M},\omega)\to S\times_K S$ via the diagonal map $\Delta_S : S\to S\times_K S$ gives a group scheme $\pi_1(\mathcal{M},\omega)/S$ over S called the *fundamental group* of $\mathcal{M}$ at ω. If the fiber functor ω is over K, then the natural map from $\mathcal{M}$ to the tannakian category $\mathrm{Rep}\,\pi_1(\mathcal{M},\omega)$ of finite dimensional linear representations of $\pi_1(\mathcal{M},\omega)$ is an equivalence of categories.

2.2. de Rham fundamental group

Let X/K be a smooth variety over a field K of characteristic zero. We denote the category of vector bundles with integrable connection by $\mathrm{Mic}(X/K)$. Note that $H^0_{dR}(X/K)\simeq K$, and hence $\mathrm{End}(1)\simeq K$. The functor $\omega_X : \mathrm{Mic}(X/K)\to\mathrm{Coh}(X/K)$ from this tensor category to the category of coherent sheaves on X that maps (E,∇) to E is a fiber functor. Therefore $\mathrm{Mic}(X/K)$ is a tannakian category over K. Let $\mathrm{Mic}_{\mathrm{uni}}(X/K)$ denote the sub-tannakian category consisting of unipotent objects, that is those objects that have a filtration by subbundles with connection whose graded pieces are the trivial vector bundle with constant connection. Denote the fundamental groupoid $\mathcal{G}(\mathrm{Mic}_{\mathrm{uni}}(X/K),\omega_X)$ corresponding to $\mathrm{Mic}_{\mathrm{uni}}(X/K)$ and ω_X by $\mathcal{G}_{dR}(X/K)$. This is a groupoid acting on X, and is called the *de Rham fundamental groupoid* of X. If $x\in X(K)$ then the functor $\omega(x) : (E,\nabla)\to E(x)$ is a fiber functor over K. The fiber of $\mathcal{G}_{dR}(X/K)$ over (x,x), which is $\mathrm{Aut}_K(\omega(x))$, is called the *de Rham fundamental group* of X at the basepoint x, and denoted by $\pi_{1,dR}(X,x)$.

Remark. Let $X/\mathbb{C}$ be as above. Then by the Riemann–Hilbert correspondence the category of vector bundles with integrable connection that has regular singularities at infinity is equivalent to the category of $\mathbb{C}$-local systems on the underlying topological space of X_{an}. Therefore, the fundamental group of this tannakian category, at a basepoint, is isomorphic to the algebraic envelope of the topological fundamental group. It would thus be natural to define the algebraic de Rham fundamental group of X/K to be the fundamental group of the full tannakian subcategory of $\mathrm{Mic}(X/K)$ consisting of the objects which have regular singularities at infinity. However this definition is not compatible with the base change of the underlying field K, see [5, 10.35] for an example. Since, we are interested in periods, and hence comparison isomorphisms, this is a crucial deficiency. The unipotent version defined above commutes with base change ([5, Section 10]).

2.3. Logarithmic extensions

Let $\bar{X}/K$ be a smooth variety over a field K of characteristic zero, $D \subseteq \bar{X}$ a simple normal crossings divisor, and $X := \bar{X} \backslash D$. The category $\mathrm{Mic}(\bar{X}/K, \log D)$ is the category of vector bundles with integrable connection on $\bar{X}$ which have at most logarithmic singularities along the divisor D. Here we review the theory of logarithmic extensions (see [1, Section 1.4]).

Let $(\bar{E}, \nabla) \in \mathrm{Mic}(\bar{X}/K, \log D)$, $x \in X(L)$ a point over a finite extension L/K, and a system of parameters $t_1, \ldots, t_n$ at x such that D is defined by $t_1 \cdots t_k = 0$ near x. Then the residue

$$\mathrm{res}_{t_i=0}(\bar{E}, \nabla)(x) \in \mathrm{End}\, \bar{E}(x)$$

of $(\bar{E}, \nabla)$ along $t_i = 0$, for $1 \leqslant i \leqslant k$, is the fiber at x of the map $\bar{E} \to \bar{E}$ that sends v to $-(\nabla v, t_i \frac{\partial}{\partial t_i})$. This is independent of the choice of $\{t_i \mid 1 \leqslant i \leqslant n\}$ satisfying the conditions above. If the irreducible component D_i of D is geometrically connected then the similarity class of the residue $\mathrm{res}_{D_i}(\bar{E}, \nabla)(x)$ along D_i is constant for $x \in D_i$.

Let $\bar{K}$ be an algebraic closure of K and let $\tau : \bar{K}/\mathbb{Z} \to \bar{K}$ be a set theoretic section of the canonical projection $\bar{K} \to \bar{K}/\mathbb{Z}$ with $\tau(0) = 0$. If $(E, \nabla) \in \mathrm{Mic}(X/\bar{K})$ has regular singularities along D then there is a unique object in $(\bar{E}, \nabla)_\tau \in \mathrm{Mic}(\bar{X}/\bar{K}, \log D)$ with restriction (E, ∇) and whose residues along the irreducible components of the divisor have eigenvalues that are in the image of τ. Let D_i be an irreducible component of D, η_i be its generic point, $\hat{\mathcal{O}}_{\bar{X}, \eta_i}$, the completion of the local ring at η_i along its maximal ideal, and s_i a uniformizer at η_i. Then the eigenvalues of $\mathrm{res}_{D_i}(\bar{E}, \nabla)_\tau(x)$ are the elements $\alpha \in \mathrm{Im}\, \tau$ such that $s_i \nabla_{d/ds_i} - \alpha$ has a horizontal section in $\bar{E}|_{\hat{\mathcal{O}}_{\bar{X}, \eta_i}}$, where the connection is considered relative to the field of functions $k(D_i)$ on D_i.

If $(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(X/K)$ then since $\tau(0) = 0 \in K$, by the uniqueness of the extension of $(E, \nabla)_{\bar{K}}$ and descent, the extension can be defined over K. It is called the canonical extension of (E, ∇) and denoted by $(\bar{E}, \nabla)$. It is, up to unique isomorphism, the unique element in $\mathrm{Mic}(\bar{X}/K, \log D)$ with nilpotent residues along D and that restricts to (E, ∇) on X.

We denote the full subcategory of $\mathrm{Mic}(\bar{X}/K, \log D)$ formed by the unipotent objects by the notation $\mathrm{Mic}_{\mathrm{uni}}(\bar{X}/K, \log D)$. By the theory of logarithmic extensions the restriction map

$$\mathrm{Mic}_{\mathrm{uni}}(\bar{X}/K, \log D) \to \mathrm{Mic}_{\mathrm{uni}}(X/K)$$

is an equivalence of tannakian categories.

2.4. Action of the crystalline frobenius

Let K be a complete discrete valuation ring of characteristic zero with ring of integers R and finite residue field k of characteristic p. Let X/K be a smooth variety, and assume that there is a proper, smooth model $\overline{\mathfrak{X}}/R$ and a simple relative normal crossings divisor $\mathfrak{D} \subseteq \overline{\mathfrak{X}}$, whose irreducible components are defined over R, together with a fixed isomorphism $(\overline{\mathfrak{X}} \backslash \mathfrak{D})_K \simeq X$. We let $\mathfrak{X} := \overline{\mathfrak{X}} \backslash \mathfrak{D}$, $D := \mathfrak{D} \otimes_R K$, $\bar{X} := \overline{\mathfrak{X}} \otimes_R K$, $\bar{Y} := \overline{\mathfrak{X}} \otimes_R k$, and $Y := \mathfrak{X} \otimes_R k$. In this section, we recall the definition of $\mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R)$ [2], and the equivalence of the categories $\mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R)$ and $\mathrm{Mic}_{\mathrm{uni}}(X/K)$ ([4, Proposition 2.4.1]).

2.4.1. Unipotent overconvergent isocrystals

For a variety Y/k, let $\mathrm{Isoc}^{\dagger}(Y/R)$ denote the category of overconvergent isocrystals on Y relative to R [2]. This category is defined as follows.

For a formal scheme $\mathcal{Q}/R$, and a locally closed subscheme $Z \subseteq \mathcal{Q} \otimes_R k$, let $]Z[_{\mathcal{Q}} \subseteq \mathcal{Q}_K$ denote the tube of Z in $\mathcal{Q}$. Let $Y \subseteq \bar{Y}$ be a compactification of Y. Assume that there exists a formal scheme $\mathcal{P}/R$, and a closed imbedding $\bar{Y} \hookrightarrow \mathcal{P}$, such that $\mathcal{P}/R$ is smooth in a neighborhood of Y. Let $j :]Y[_{\mathcal{P}} \rightarrow]\bar{Y}[_{\mathcal{P}}$ denote the imbedding of the tube of Y in $\mathcal{P}$ in that of $\bar{Y}$ in $\mathcal{P}$. Then an overconvergent isocrystal on Y is a locally free $j^{\dagger}\mathcal{O}_{]Y[_{\mathcal{P}}}$ module with integrable connection such that the connection converges in a strict neighborhood of $]Y[_{\mathcal{P} \times \mathcal{P}}$ in $]\bar{Y}[_{\mathcal{P} \times \mathcal{P}}$, where $\bar{Y}$ is embedded diagonally in $\mathcal{P} \times_R \mathcal{P}$. The category of overconvergent isocrystals is independent the imbedding $\bar{Y} \hookrightarrow \mathcal{P}$, into a formal scheme $\mathcal{P}/R$.

If the global imbedding as above does not exist then the category is defined by first choosing an open cover $\{U_i\}_{i \in I}$ of $\bar{Y}$ such that an imbedding of U_i in a formal scheme $\mathcal{P}_i/R$, as above, exists. Then an overconvergent isocrystal on Y is a collection of overconvergent isocrystals on $\bar{Y} \cap U_i$ corresponding to the data $U_i \hookrightarrow \mathcal{P}_i$, and isomorphisms between their restrictions on the intersections $\{U_i \cap U_j\}_{i,j \in I}$ satisfying the cocycle condition. The category is independent of the cover and the imbeddings into formal schemes. It is also independent of the compactification $\bar{Y}$ of Y and is denoted by $\mathrm{Isoc}^{\dagger}(Y/R)$. For an object $(E, \nabla) \in \mathrm{Isoc}^{\dagger}(Y/R)$ and data $\{\mathcal{U}_i \rightarrow \mathcal{P}_i\}_{i \in I}$ as above, we denote by $(E, \nabla)_{\mathcal{P}_i}$ the vector bundle with overconvergent connection on $]Y[_{\mathcal{P}_i}$, and call it the *realization* of (E, ∇) on $\mathcal{P}_i$.

The full subcategory $\mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R)$ of $\mathrm{Isoc}^{\dagger}(Y/R)$ that consist of unipotent objects is called the category of unipotent overconvergent isocrystals. Using the notation at the beginning of the section, let $\hat{\overline{\mathfrak{X}}}$ denote the formal scheme over R obtained by completing $\overline{\mathfrak{X}}$ along $\bar{Y}$. Then $]Y[_{\hat{\overline{\mathfrak{X}}}} \subseteq X_{\mathrm{an}}$, and X_{an} is a strict neighborhood of $]Y[_{\hat{\overline{\mathfrak{X}}}}$. This gives a map

$$\alpha_{\overline{\mathfrak{X}}} : \mathrm{Mic}_{\mathrm{uni}}(X/K) \rightarrow \mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R).$$

By [4, Proposition 2.4.1] this is an equivalence of categories.

2.4.2. Description of frobenius

We continue the notation above. Let q be the cardinality of k. Then the q-power frobenius F induces a map $F : (\bar{Y}, Y) \rightarrow (\bar{Y}, Y)$. This gives a map $F^* : \mathrm{Isoc}^{\dagger}(Y/R) \rightarrow \mathrm{Isoc}^{\dagger}(Y/R)$, which is independent of the compactification. If there is a lifting $\overline{\mathfrak{X}}$ as above, this is described as follows. Let $\{U_i\}_{i\in I}$ be an open cover of $\bar{Y}$ and let $\mathcal{P}_i$, for $i\in I$, be the completion of $\overline{\mathfrak{X}}$ along U_i, chosen such that there exists a lifting $\mathcal{F}_i : \mathcal{P}_i \to \mathcal{P}_i$ of the q-power frobenius, for all $i\in I$. Since $\overline{\mathfrak{X}}/R$ is smooth, choosing U_i to be affine is sufficient to ensure that such a lifting exists. If $(E,\nabla)\in \mathrm{Isoc}^{\dagger}(Y/R)$ has the realization $(E,\nabla)_{\mathcal{P}_i}\in \mathrm{Isoc}^{\dagger}(U_i/R)$ as above, then $F^*(E,\nabla)$ has the realization $\mathcal{F}_i^*(E,\nabla)_{\mathcal{P}_i}\in \mathrm{Isoc}^{\dagger}(U_i/R)$. The isomorphisms between the restrictions of $\mathcal{F}_i^*(E,\nabla)_{\mathcal{P}_i}$ and $\mathcal{F}_j^*(E,\nabla)_{\mathcal{P}_j}$ to the intersection $U_i\cap U_j$ are given by pulling back the isomorphisms between $p_1^*(E,\nabla)$ and $p_2^*(E,\nabla)$ on the tube in $\mathcal{P}_i\times\mathcal{P}_j$, where p_1 and p_2 are the projections from $\mathcal{P}_1\times\mathcal{P}_2$ to its factors, by the map induced by

$$\mathcal{F}_i\times\mathcal{F}_j : \mathcal{P}_i\times\mathcal{P}_j \to \mathcal{P}_i\times\mathcal{P}_j.$$

Since $\mathrm{Mic}_{\mathrm{uni}}(X/K)\to \mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R)$ is an equivalence of categories we obtain a tensor functor

$$F^* : \mathrm{Mic}_{\mathrm{uni}}(X/K)\to \mathrm{Mic}_{\mathrm{uni}}(X/K),$$

a priori depending on the model. Let $(E,\nabla)\in \mathrm{Mic}_{\mathrm{uni}}(X/K)$. In order to determine $F^*(E,\nabla)\in \mathrm{Mic}_{\mathrm{uni}}(X/K)$, it is enough to find $(G,\nabla)\in \mathrm{Mic}_{\mathrm{uni}}(X/K)$ such that

$$\alpha_{\overline{\mathfrak{X}}}(G,\nabla)\simeq F^*\alpha_{\overline{\mathfrak{X}}}(E,\nabla).$$

Deligne constructs in [5, Section 11] one such (G,∇) as follows: Let

$$(\bar{E},\nabla)\in \mathrm{Mic}_{\mathrm{uni}}(\bar{X}/K,\log D)$$

be the canonical extension of (E,∇). Let $\{U_i\}_{i\in I}$ be an affine covering of $\bar{Y}$ such that if $\mathcal{P}_i$ is the completion of $\overline{\mathfrak{X}}$ along U_i, then there exist $\mathcal{F}_i : \mathcal{P}_i\to\mathcal{P}_i$ which are liftings of the q-power frobenius and induce maps $\mathcal{P}_i\backslash\mathcal{D}\to\mathcal{P}_i\backslash\mathcal{D}$, where $\mathcal{D}$ is the completion of $\mathfrak{D}$ along $\bar{Y}$. The last condition is equivalent to requiring that $\mathcal{F}_i^*(\mathcal{D}_s)=q\cdot\mathcal{D}_s$ for all components $\mathcal{D}_s$ of $\mathcal{D}$ such that $\mathcal{D}_s\cap U_i\neq\emptyset$. Let $(\overline{\mathfrak{X}}\times\overline{\mathfrak{X}})^{\sim}$ be the blow-up of $\overline{\mathfrak{X}}\times\overline{\mathfrak{X}}$ along the image of the irreducible components of $\mathfrak{D}$ under the diagonal map. Let $\Delta_{\overline{\mathfrak{X}}}^{\sim}$ denote the strict transform of the diagonal. Since the connection $(\bar{E},\nabla)$ has logarithmic singularities along D, and is unipotent the isomorphism $p_1^*E\simeq p_2^*E$ on the first infinitesimal neighborhood of Δ_X in $X\times X$ defined by the connection extends to the tube

$$]\Delta_{\overline{\mathfrak{X}}}^{\sim}\otimes_R k[_{(\overline{\mathfrak{X}}\hat{\times}\overline{\mathfrak{X}})^{\sim}}$$

[5, Section 11], where hat denotes the completion along the closed fiber.

Denote $]U_i[_{\mathcal{P}_i}$ by $\mathcal{U}_i$ then $(\bar{E},\nabla)|_{\mathcal{U}_i}\in \mathrm{Mic}_{\mathrm{uni}}(\mathcal{U}_i,\log D)$. Define

$$F^*(\bar{E},\nabla)|_{\mathcal{U}_i} := \mathcal{F}_{iK}^*(\bar{E},\nabla)|_{\mathcal{U}_i}\in \mathrm{Mic}_{\mathrm{uni}}(\mathcal{U}_i,\log D).$$

For i,j in $I, (\mathcal{F}_i, \mathcal{F}_j)$ induces a map from $\mathcal{U}_i \cap \mathcal{U}_j$ to $]\Delta_{\overline{\mathfrak{X}}}^{\sim} \otimes_R k[_{(\overline{\mathfrak{X}} \hat{\times} \overline{\mathfrak{X}})^{\sim}}$. Therefore pulling back the isomorphisms induced by the connection, via this map gives patching data for the $F^*(\bar{E}, \nabla)|_{\mathcal{U}_i}$, $i \in I$, that are horizontal with respect to the pulled back connections and that satisfy the cocycle condition. Therefore one obtains a global $F^*(\bar{E}, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(\bar{X}_{\mathrm{an}}/K, \log D)$. But since $\bar{X}/K$ is proper, by GAGA it comes from $F^*(\bar{E}, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(\bar{X}/K, \log D)$. Restricting this to X gives the $F^*(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(X/K)$ we were looking for.

3. Tangential Basepoints

3.1. Ordinary basepoints

(i) *de Rham*: Let X/K be a smooth variety over a field of characteristic zero, and $x \in X(K)$. Then the functor

$$\omega(x) : \mathrm{Mic}(X/K) \to K$$

that sends (E, ∇) to $E(x)$ is a fiber functor of $\mathrm{Mic}(X/K)$ over K.

(ii) *Crystalline*: We return to the situation of a variety Y/k as in 2.4.1. A point $y \in Y(k)$ defines a map $\mathrm{Spec}\, k/R \to Y/R$, and hence a fiber functor

$$\omega(y) : \mathrm{Isoc}^{\dagger}(Y/R) \to \mathrm{Isoc}(k/R) \simeq \mathrm{Vec}_K.$$

Let $\bar{Y}$ be a compactification of Y, $U \subseteq \bar{Y}$ an open subvariety, with $y \in U$, $\mathcal{P}/R$ a formal scheme with a closed immersion $U \hookrightarrow \mathcal{P}$ such that $\mathcal{P}/R$ is smooth in a neighborhood of $U \cap Y$, and let $\mathfrak{y} \in \mathcal{P}(R)$ be a point with specialization y, and generic fiber $x \in \mathcal{P}_K$. Corresponding to these data the *realization* of the fiber functor $\omega(y)$ is the functor that sends $(E, \nabla) \in \mathrm{Isoc}^{\dagger}(Y/R)$ to $(E, \nabla)_{\mathcal{P}}(x)$. Note that there are canonical isomorphisms between different realizations of $\omega(y)$.

(iii) *Comparison*: With the notation as in the beginning of Section 2.4, let $\mathfrak{x} \in \mathfrak{X}(R)$ with reduction y, generic point x, and completion along the closed point $\hat{\mathfrak{x}}$. Then the realization of $\omega(y)$ corresponding to $(Y, \hat{\mathfrak{X}})$ and $\hat{\mathfrak{x}}$ maps $\alpha_{\overline{\mathfrak{X}}}(E, \nabla)$ to $(E, \nabla)(x)$, for $(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(X/K)$, and the diagram

$$\begin{array}{ccc} \mathrm{Mic}_{uni}(X/K) & \xrightarrow{\alpha_{\overline{\mathfrak{X}}}} & \mathrm{Isoc}^{\dagger}_{uni}(Y/R) \\ \Big\downarrow{\scriptstyle \omega(x)} & & \Big\downarrow{\scriptstyle \omega(y)} \\ \mathrm{Vec}_K & \xrightarrow{id} & \mathrm{Vec}_{\mathrm{K}} \end{array}$$

commutes.

3.2. Tangential basepoints

From now on we will assume, for simplicity, that X/K, Y/k, and $\mathfrak{X}/R$ are of relative dimension one. Let $\mathcal{T}_0$ be the category whose objects are pairs (V, N), where V is a finite dimensional vector space over K and N is a nilpotent linear operator on V. Morphisms between the objects (V_1, N_1) and (V_2, N_2) are linear maps $T : V_1 \to V_2$ such that $TN_1 = N_2 T$. We define a tensor product on $\mathcal{T}_0$ as $(V_1, N_1) \otimes (V_2, T_2) := (V_1 \otimes V_2, id \otimes N_2 + N_1 \otimes id)$. Then $\mathcal{T}_0$ is naturally a tannakian category over K with this tensor product.

(i) *de Rham*: Let $\bar{X}/K$ be a smooth curve over a field K of characteristic zero and $X \subseteq \bar{X}$ an open subvariety with $D := \bar{X} \backslash X \subseteq \bar{X}(K)$.

Using the equivalence $\mathrm{Mic}_{\mathrm{uni}}(\bar{X}/K, \log D) \simeq \mathrm{Mic}_{\mathrm{uni}}(X/K)$, induced by restriction, in order to give a fiber functor of $\mathrm{Mic}_{\mathrm{uni}}(X/K)$ it is enough to give one of $\mathrm{Mic}_{\mathrm{uni}}(\bar{X}/K, \log D)$.

Let $x \in D$. Then there is a natural tensor functor

$$\psi_x(X) : \mathrm{Mic}_{\mathrm{uni}}(\bar{X}, \log D) \to \mathcal{T}_0$$

defined by sending (E, ∇) to $(E(x), \mathrm{res}_x(E, \nabla))$.

Let

$$\omega : \mathcal{T}_0 \to \mathrm{Vec}_K$$

be the fiber functor that forgets the linear operator. Letting $\omega(x) := \omega \circ \psi_x(X)$ gives a fiber functor

$$\omega(x) : \mathrm{Mic}_{\mathrm{uni}}(\bar{X}/K, \log D) \to \mathrm{Vec}_K$$

that maps (E, ∇) to $E(x)$.

Lemma 1. *Applying the above consideration to* $\mathbb{P}^1$, $D_{0\infty} := \{0, \infty\}$, *and* $x = 0$, *the map*

$$\psi_0(\mathbb{G}_m) : \mathrm{Mic}_{\mathrm{uni}}(\mathbb{P}^1/K, \log D_{0\infty}) \to \mathcal{T}_0$$

is an equivalence of categories.

Proof. Let $(V, N) \in \mathcal{T}_0$. Then

$$\psi_0\left(V \otimes_K \mathcal{O}_{\mathbb{P}^1}, \quad d - N\frac{dz}{z} \right) = (V, N).$$

This proves the essential surjectivity.

In order to see full-faithfulness it is enough to show, since ψ_0 is a tensor functor, that for (E, ∇) in $\mathrm{Mic}_{\mathrm{uni}}(\mathbb{P}^1/K, \log D_{0\infty})$, taking fibers at 0 induces an isomorphism from $\mathrm{H}^0_{dR}(\mathbb{P}^1_{\log}, (E, \nabla))$ to $\ker_{E(0)}(\mathrm{res}_0(E, \nabla))$, where $\mathbb{P}^1_{\log}$ is $\mathbb{P}^1$ endowed with the log structure induced by $D_{0\infty}$.

First, note that for $(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(\mathbb{P}^1/K, \log D_{0\infty})$ the underlying vector bundle E is trivial. This follows from $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}) = \mathrm{H}^1(\mathbb{P}^1, \mathcal{O}) = 0$ by induction on the nilpotence level [5]. Therefore without loss of generality we will assume that $(E, \nabla) \simeq (\mathcal{O}^{rkE}, d - N\frac{dz}{z})$, where N is a nilpotent matrix. If v is a global (horizontal) section of $(\mathcal{O}^{rkE}, d - N\frac{dz}{z})$ then it is a constant section of $\mathcal{O}^{rkE}$. Therefore, the map

$$\mathrm{H}^0_{dR}(\mathbb{P}^1_{\log}, (E, \nabla)) \to \ker_{E(0)} N$$

is injective. In order to see its surjectivity we note that for any $\alpha \in \ker N$, the constant section of $\mathcal{O}^{rkE}$ with fiber α at 0 is a horizontal section with respect to the connection $d - N\frac{dz}{z}$. $\square$

Let $\bar{T}_x\bar{X}$ denote the smooth compactification of $T_x\bar{X}$. By the lemma above the map $\psi_0(T_x\bar{X}\backslash\{0\})$ is an equivalence of categories. Letting

$$\varphi_x^* := \psi_0(T_x\bar{X}\backslash\{0\})^{-1} \circ \psi_x(X),$$

where $\psi_0(T_x\bar{X}\backslash\{0\})^{-1}$ is an inverse of $\psi_0(T_x\bar{X}\backslash\{0\})$, defined up to canonical isomorphism, we obtain a functor

$$\varphi_x^* : \mathrm{Mic}_{\mathrm{uni}}(\bar{X}/K, \log D) \to \mathrm{Mic}_{\mathrm{uni}}(\bar{T}_x\bar{X}/K, \log D_{0\infty}).$$

Remark. The notation φ_x^* is a bit misleading since, in general, there is not a genuine map from $\bar{T}_x\bar{X}$ to $\bar{X}$ that induces this map.

Note that the definition of φ_x^* does not depend on any choice of a local parameter. Choosing a (nonzero) tangent vector $v \in T_x\bar{X}(K)$ we obtain a fiber functor $\omega(v)$ of $\mathrm{Mic}_{\mathrm{uni}}((\bar{T}_x\bar{X})/K, \log D_{0\infty})$ over K.

Definition 1. Let $v \in (T_x\bar{X}\backslash\{0\})(K)$. Then we continue to denote the fiber functor $\omega(v) \circ \varphi_x^*$ by $\omega(v)$ and call it the fiber functor of $\mathrm{Mic}_{\mathrm{uni}}(X/K)$ at the tangent vector v.

Note that $\omega(v)$ is canonically isomorphic to $\omega(0)(= \omega(x))$ by 4.1 below. In this sense the choice of a tangent vector is not needed. However, it will be needed in order to define the action of frobenius.

(ii) *Crystalline*: Let Z/k be a variety, and M an fs log structure on Z (in the Zariski topology) such that the map $(Z, M) \to \mathrm{Spec}\, k$, where the base is endowed with the trivial log structure, is log smooth. Shiho defines the log convergent site $(Z/R)^{\log}_{\mathrm{conv}}$, and for a locally free isocrystal on the log convergent site he defines the log convergent cohomology $H^{\cdot}_{\mathrm{an}}((Z/R)^{\log}_{\mathrm{conv}}, E)$ [16, Sections 2.1 and 2.2].

Let $\bar{Y}/k$ be a proper, smooth curve, $D \subseteq \bar{Y}(k)$, and $Y := \bar{Y} \backslash D$, with the inclusion $j : Y \to \bar{Y}$. Denote the fs log scheme associated to $\bar{Y}$ with logarithmic structure defined by D, by $\bar{Y}_{\log}$. Then we can define the category $\mathrm{Isoc}((\bar{Y}/R)^{\log}_{\mathrm{conv}})$ of log convergent isocrystals on $\bar{Y}_{\log}$ and the cohomology groups $H^{\cdot}_{\mathrm{an}}((\bar{Y}/R)^{\log}_{\mathrm{conv}}, E)$ for an object $E \in \mathrm{Isoc}((\bar{Y}/R)^{\log}_{\mathrm{conv}})$. There is a restriction functor

$$j^{\dagger} : \mathrm{Isoc}((\bar{Y}/R)^{\log}_{\mathrm{conv}}) \to \mathrm{Isoc}^{\dagger}(Y/R)$$

to the category of overconvergent isocrystals on Y. And if $E \in \mathrm{Isoc}_{\mathrm{uni}}((\bar{Y}/R)^{\log}_{\mathrm{conv}})$ is a unipotent isocrystal then there is an isomorphism of the cohomology groups

$$H^{\cdot}_{\mathrm{an}}((\bar{Y}/R)^{\log}_{\mathrm{conv}}, E) \simeq H^{\cdot}_{\mathrm{rig}}(Y/R, j^{\dagger}E)$$

by (2.4.1) [16, Sections 2.1, 2.2] (in fact there it is proven in the case where $E = \mathcal{O}$. The unipotent version follows from $E = \mathcal{O}$ by induction on the nilpotence level, the cohomology exact sequence, and the five lemma).

Lemma 2. *The canonical functor*

$$j^{\dagger} : \mathrm{Isoc}_{\mathrm{uni}}((\bar{Y}/R)^{\log}_{\mathrm{conv}}) \to \mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R)$$

is an equivalence of categories.

Proof. Since $j^{\dagger}$ is a tensor functor the full-faithfulness follows from $H^0_{\cdot}(.\,, E^* \otimes F) \simeq \mathrm{Hom}_{\cdot}(E, F)$, where dot denotes the log convergent or overconvergent category.

The essential surjectivity is proved by induction on the rank. Let $E \in \mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R)$. Then there is an $F \in \mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R)$, and an exact sequence

$$0 \to F \to E \to \mathcal{O}^{\dagger}_{Y} \to 0.$$

By the induction hypothesis there exists an $\bar{F} \in \mathrm{Isoc}_{\mathrm{uni}}((\bar{Y}/R)^{\log}_{\mathrm{conv}})$, with $j^{\dagger}\bar{F} = F$. The exact sequence above gives an element in

$$\mathrm{Ext}^{\dagger 1}(\mathcal{O}^{\dagger}_{Y}, F) \simeq H^1_{\mathrm{rig}}(Y/R, j^{\dagger}\bar{F}) \simeq H^1_{\mathrm{an}}((\bar{Y}/R)^{\log}_{\mathrm{conv}}, \bar{F}) \simeq \mathrm{Ext}^1_{\log/\mathrm{conv}}(\overline{\mathcal{O}}_Y, \bar{F}).$$

The corresponding element in the last group gives an exact sequence

$$0 \to \bar{F} \to \bar{E} \to \overline{\mathcal{O}}_Y \to 0$$

in $\mathrm{Isoc}_{\mathrm{uni}}((\bar{Y}/R)^{\log}_{\mathrm{conv}})$ with $j^{\dagger}\bar{E} \simeq E$. This proves the essential surjectivity. □

Therefore in order to construct a fiber functor on $\mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R)$ it is enough to construct one on $\mathrm{Isoc}_{\mathrm{uni}}((\bar{Y}/R)^{\log}_{\mathrm{conv}})$. Let $v \in T_y\bar{Y}(k)$. Then we will define a fiber functor

$$\omega(v) : \mathrm{Isoc}_{\mathrm{uni}}((\bar{Y}/R)^{\log}_{\mathrm{conv}}) \to Vec_K$$

analogous to the one in the de Rham case.

Construction of $\omega(v)$: Let $(U, \mathcal{P}, \mathfrak{y}, \mathfrak{v})$ be a data such that $U \subseteq \bar{Y}$ is an open affine subvariety with $U \cap (\bar{Y} \setminus Y) = \{y\}$, $\mathcal{P}/R$ a smooth formal scheme with closed fiber U (this exists by Theorem 6 of [10], $\mathfrak{y}$ a point in $\mathcal{P}(R)$ with reduction y, and $\mathfrak{v}$ a tangent vector in $T_{\mathfrak{y}}\mathcal{P}$ with reduction v. Note that $T_{\mathfrak{y}}\mathcal{P}$ is a lifting of $T_y \bar{Y}$.

Then, we define the realization $\omega(v)_{\mathfrak{v}}$ of $\omega(v)$ corresponding to these data to be the functor that sends $(E, \nabla) \in \mathrm{Isoc}_{\mathrm{uni}}((\bar{Y}/R)^{\log}_{\mathrm{conv}})$ to $E_{\mathcal{P}}(\mathfrak{y}_K)$, where $E_{\mathcal{P}}$ denotes the underlying bundle of $(E, \nabla)_{\mathcal{P}}$.

Let $(U, \mathcal{Q}, \mathfrak{z}, \mathfrak{w})$ be a choice of a different data as above. Let $(\mathcal{P} \times \mathcal{Q})^{\sim}$ denote the blow-up of $\mathcal{P} \times \mathcal{Q}$ at $(\mathfrak{y}, \mathfrak{z})$. Then since (E, ∇) is a logarithmic isocrystal, there is a canonical horizontal isomorphism between $p_1^*(E, \nabla)_{\mathcal{P}}$ and $p_2^*(E, \nabla)_{\mathcal{Q}}$ in the tube $]\Delta_U^{\sim}[_{(\mathcal{P} \times \mathcal{Q})^{\sim}}$, where $\Delta_U^{\sim}$ is the strict transform of the diagonal Δ_U in the blow-up, and where p_i, for $i = 1, 2$, are the projections from $(\mathcal{P} \times \mathcal{Q})^{\sim}$ to $\mathcal{P}$ and $\mathcal{Q}$. The exceptional divisor of the blow-up is canonically isomorphic to $\mathbb{P}(T_{\mathfrak{y}}\mathcal{P}(R) \oplus T_{\mathfrak{z}}\mathcal{Q}(R))$, with special fiber $\mathbb{P}(T_y \bar{Y}(k) \oplus T_y \bar{Y}(k))$, and $[v, v] = \mathbb{P}(T_y \bar{Y}(k) \oplus T_y \bar{Y}(k)) \cap \Delta_U^{\sim}$. Therefore, since the tangent vectors $\mathfrak{v}$ and $\mathfrak{w}$ have reduction v, $[\mathfrak{v}, \mathfrak{w}]$ is in the tube of $[v, v]$. In order to give an isomorphism between the realizations $\omega(v)_{\mathfrak{v}}$ and $\omega(v)_{\mathfrak{w}}$ of $\omega(v)$ corresponding to the two data above, we need to give isomorphisms between $E_{\mathcal{P}}(\mathfrak{y}_K)$ and $E_{\mathcal{Q}}(\mathfrak{z}_K)$. We define this isomorphism to be the evaluation of the isomorphism between $p_1^*(E, \nabla)_{\mathcal{P}}$ and $p_2^*(E, \nabla)_{\mathcal{Q}}$ at the point $[\mathfrak{y}_K, \mathfrak{z}_K]$. By the integrability of the connection these isomorphisms satisfy the cocycle condition.

Definition 2. With the notation as above, for $v \in T_y \bar{Y}(k)$ the fiber functor

$$\omega(v) : \mathrm{Isoc}^{\dagger}_{\mathrm{uni}}(Y/R) \to \mathrm{Vec}_K$$

is called the fiber functor at the tangent vector v.

Lemma 3. *Let* $(U, \mathcal{P}, \mathfrak{y}, \mathfrak{v})$ *and* $(U, \mathcal{P}, \mathfrak{y}, \mathfrak{w})$ *be two data as above. And let* $(E, \nabla) \in \mathrm{Isoc}_{\mathrm{uni}}((\bar{Y}/R)^{\log}_{\mathrm{conv}})$. *Then the automorphism of* $(E, \nabla)_{\mathcal{P}}(\mathfrak{y}_K)$ *induced by the identifications* $\omega(v)_{\mathfrak{v}} \to \omega(v) \to \omega(v)_{\mathfrak{w}}$ *is given by* $\exp(\log(\mathfrak{w}/\mathfrak{v})\, \mathrm{res}_{\mathfrak{y}_K}(E, \nabla)_{\mathcal{P}})$.

First, note that since $\mathfrak{v}$ and $\mathfrak{w}$ have the same reduction, $|\mathfrak{w}/\mathfrak{v} - 1| < 1$. We would like to compute the action of parallel transport along the connection explicitly between two tangent vectors. This is a local question. Let $D(0, \varepsilon^-)$ denote the open disc of radius ε in $\mathbb{A}^1_K$. Let $\varphi : D(0, \varepsilon^-) \to]U[_{\mathcal{P}}$ be a map with $\varphi(0) = \mathfrak{y}_K$, that is an isomorphism onto its image. By pulling back (E, ∇) via φ, it suffices to answer the analogous question on $D(0, \varepsilon^-)$.

In order to do this, let $(E,\nabla)\in \mathrm{Mic}_{\mathrm{uni}}(D(0,\varepsilon^-),\log 0)$ and v, and $w\in T_0D(0,\varepsilon^-)$ with $|w/v-1|$ (note that this property, satisfied for the vectors above, remains valid after pulling back via φ). By the argument in the proof of Lemma 1 we may assume without loss of generality, since $H^i(D(0,\varepsilon^-),\mathcal{O})=0$ for $1\leqslant i$, that $(E,\nabla)=(\mathcal{O}^{rkE},d-N\frac{dz}{z})$, where the nilpotent N is the residue of (E,∇) at 0.

Let (z_1,z_2) be the coordinates for $D(0,\varepsilon^-)\times D(0,\varepsilon^-)$. Letting $h:=z_2/z_1-1$, the strict transform $\Delta^\sim$ of the diagonal in $(D(0,\varepsilon^-)\times D(0,\varepsilon^-))^\sim$ is defined by $h=0$. Then the isomorphism between E_z and $E_{z+z\cdot h}$, in the formal neighborhood of $\Delta^\sim$, defined by the connection is given by $\exp(\log(1+h)N)$, which makes sense since N is nilpotent. This formula extends uniquely to give an isomorphism between E_z and $E_{z+z\cdot h}$ for $|h|<1$, since in this disc $\log(1+h)$ is an analytic function. Therefore, the isomorphism between E_v and E_w is given by $\exp(\log(w/v)N)$. $\square$

Frobenius: This section will be needed in order to define the action of frobenius on the de Rham fundamental group at a tangential basepoint.

Let X_1/R and X_2/R be smooth (formal) schemes, and let D_i, for $i=1,2$, be effective relative divisors in X_i. For a map $f: X_1\to X_2$ such that $D_1\subseteq f^*(D_2)$, we obtain a map

$$\mathrm{Sym}^{\cdot}\, I_{D_2}/I_{D_2}^2\simeq \mathrm{Gr}^{\cdot}_{I_{D_2}}\mathcal{O}_{X_2}\xrightarrow{\mathrm{Gr}(f)}\mathrm{Gr}^{\cdot}_{I_{D_1}}\mathcal{O}_{X_1}\simeq \mathrm{Sym}^{\cdot}\, I_{D_1}/I_{D_1}^2,$$

corresponding to the graded algebras associated to the filtrations $\{I^n_{D_i}\}$, $i=1,2$. This gives a map

$$\mathrm{Spec}\,(\mathrm{Gr}^{\cdot}f):\ C(N_{D_1/X_1})\to C(N_{D_2/X_2})$$

between the normal cones of D_i in X_i which is called the principal part of f along D_1 and D_2, and denoted by $P(f)$ when the divisors under consideration are fixed. When the D_i are smooth we denote $C(N_{D_i/X_i})$ by N_{D_i/X_i}, which is the notation for the normal bundle of D_i in X_i. If the D_i are smooth and $f^*(D_2)=n\cdot D_1$ for some $n\in\mathbb{Z}^+$, this gives a map $N_{D_1/X_1}\to N_{D_2/X_2}$ between the normal bundles homogeneous of degree n.

Lemma 4. *With the notation and assumptions as above, the diagram*

$$\begin{array}{ccc}\mathrm{Isoc}_{uni}((\overline{Y}/R)^{\log}_{conv}) & \xrightarrow{F^*} & \mathrm{Isoc}_{uni}((\overline{Y}/R)^{\log}_{conv})\\ {\scriptstyle\omega(v)}\downarrow & & \downarrow{\scriptstyle\omega(v)}\\ \mathrm{Vec}_K & \xrightarrow{id} & \mathrm{Vec}_K\end{array}$$

is commutative.

Proof. In order to see the commutativity of the diagram above let $(U,\mathcal{P},\mathfrak{y},\mathfrak{v},\mathcal{F})$ be a data such that $(U,\mathcal{P},\mathfrak{y},\mathfrak{v})$ is as in the last section and $\mathcal{F}:\mathcal{P}\to\mathcal{P}$ is a lifting of frobenius with $\mathcal{F}^*(\mathfrak{y})=q\cdot(\mathfrak{y})$. We let $P(\mathcal{F})$ denote the principal part of $\mathcal{F}$ along $\mathfrak{y}$. If z is a local coordinate on $\mathcal{P}$ at $\mathfrak{y}$ relative to R then the assumption that $\mathcal{F}^*(\mathfrak{y})=$

$q \cdot (\mathfrak{y})$ and that $\mathcal{F}$ lifts the q-power frobenius on $\bar{Y}$ implies that $\mathcal{F}^*(z) = a_q z^q + \sum_{q<n} a_n z^n$ as functions on the completion of $\mathcal{P}$ along $\mathfrak{y}$, where $a_q - 1$, and a_n, for $q<n$, are in $\mathfrak{m}_R$. In particular $P(\mathcal{F})^*(dz) = a_q(dz)^q$. Therefore, since by assumption the residue field k has cardinality q and $\mathfrak{v}$ is a lifting of $v, P(\mathcal{F})(\mathfrak{v})$ has reduction v. The identity $\mathcal{F}_K^*(E,\nabla)_{\mathcal{P}}(\mathfrak{y}_K) = (E,\nabla)_{\mathcal{P}}(\mathfrak{y}_K)$ gives an isomorphism between $\omega(v)_{\mathfrak{v}} \circ \mathcal{F}^*$ and $\omega(v)_{P(\mathcal{F})(\mathfrak{v})}$.

We need to give isomorphisms for the different choices. Let $(U, \mathcal{Q}, \mathfrak{z}, \mathfrak{w}, \mathcal{F}')$ be another choice as above. Then we obtain a map

$$(\mathcal{F} \times \mathcal{F}')^{\sim} : (\mathcal{P} \times \mathcal{Q})^{\sim} \rightarrow (\mathcal{P} \times \mathcal{Q})^{\sim}.$$

On the exceptional divisor the map is given by

$$[P(\mathcal{F}), P(\mathcal{F}')] : \mathbb{P}(T_{\mathfrak{y}}\mathcal{P}(R) \oplus T_{\mathfrak{z}}\mathcal{Q}(R)) \rightarrow \mathbb{P}(T_{\mathfrak{y}}\mathcal{P}(R) \oplus T_{\mathfrak{z}}\mathcal{Q}(R)).$$

Pulling back the isomorphism between $\omega(v)_{P(\mathcal{F})(\mathfrak{v})}$ and $\omega(v)_{P(\mathcal{F}')(\mathfrak{w})}$ via this map we obtain the isomorphism between $\omega(v)_{\mathfrak{v}} \circ \mathcal{F}^*$ and $\omega(v)_{\mathfrak{w}} \circ \mathcal{F}'^*$. This shows that we have a commutative diagram

$$\begin{array}{ccc} \omega(v)_{\mathfrak{v}} \circ \mathcal{F}^* & \longrightarrow & \omega(v)_{\mathfrak{w}} \circ \mathcal{F}'^* \\ \downarrow & & \downarrow \\ \omega(v)_{P(\mathcal{F})(\mathfrak{v})} & \longrightarrow & \omega(v)_{P(\mathcal{F}')(\mathfrak{w})}. \end{array}$$

Since, the connection is integrable the cocycle condition is satisfied for three different data. This shows the commutativity of the diagram. □

Comparison: Let $\overline{\mathfrak{X}}/R$ be a smooth proper curve, $\mathfrak{D} \subseteq \overline{\mathfrak{X}}(R)$, $\mathfrak{X} := \overline{\mathfrak{X}} \backslash \mathfrak{D}$, $\bar{Y} := \overline{\mathfrak{X}} \otimes_R k$, $Y := \mathfrak{X} \otimes_R k$, $\bar{X} := \overline{\mathfrak{X}} \otimes_R K$, $X := \mathfrak{X} \otimes_R K$, $\mathfrak{y} \in \mathfrak{D}$, $\mathfrak{v} \in (T_{\mathfrak{y}}\overline{\mathfrak{X}} \backslash \{0\})(R)$, with $x := \mathfrak{y} \otimes K$, $\mathfrak{v}_K := \mathfrak{v} \otimes K$, $y := \mathfrak{y} \otimes k$, and $v := \mathfrak{v} \otimes k$.

Lemma 5. *With the notation as above, the diagram*

$$\begin{array}{ccc} \mathrm{Mic}_{uni}(X/K) & \longrightarrow & \mathrm{Isoc}^{\dagger}_{uni}(Y/R) \\ \downarrow{\scriptstyle \omega(\mathfrak{v}_K)} & & \downarrow{\scriptstyle \omega(v)} \\ \mathrm{Vec}_K & \xrightarrow{id} & \mathrm{Vec}_K \end{array}$$

commutes.

Proof. We obtain the commutativity by identifying $\omega(\mathfrak{v}_K)$ with the realization of $\omega(v)$ corresponding to the data obtained by completing $\overline{\mathfrak{X}}$ along $\bar{Y}$. □

Combining this with Lemma 4 gives a frobenius action

$$F_* : \pi_{1,dR}(X, \mathfrak{v}_K) \to \pi_{1,dR}(X, \mathfrak{v}_K)$$

on the de Rham fundamental group with a tangential basepoint. Note that

$$\pi_{1,dR}(X, \mathfrak{v}_K) \simeq \pi_{1,dR}(X, x)$$

canonically. However, the frobenius obtained on $\pi_{1,dR}(X, x)$ depends on the choice of a tangent vector.

Remark. Let $\overline{\mathfrak{X}} := \mathbb{P}^1/R$ and $\mathfrak{X} := \mathbb{P}^1 \backslash \{0, \infty\}$. We claim that the fiber functor

$$\omega(0) : \mathrm{Mic}_{\mathrm{uni}}(X/K) \to \mathrm{Vec}_K$$

does not commute with the action of the frobenius on $\mathrm{Mic}_{\mathrm{uni}}(X/K)$. This can be seen as follows. Let U be an affine open set in $\bar{Y}$ containing 0, and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two liftings of frobenius to the completion $\mathcal{U}$ of $\overline{\mathfrak{X}}$ along U, with $\mathcal{F}_1^*(0) = \mathcal{F}_2^*(0) = q \cdot (0)$. Then for $(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(X)$ the isomorphism

$$\bar{E}(0) = \mathcal{F}_1^* \bar{E}(0) \simeq \mathcal{F}_2^* \bar{E}(0) = \bar{E}(0)$$

is in general nontrivial, in fact it is $\exp(\mathrm{res}_0(\bar{E}, \nabla) \lim_{t \to 0} \frac{\mathcal{F}_2(t)}{\mathcal{F}_1(t)})$.

4. p-adic multi-zeta values

4.1. de Rham section of the fundamental groupoid

Using the notation of Section 3, assume that $\bar{X} = \mathbb{P}^1$. Then for any $(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(X/K)$, the underlying vector bundle $\bar{E}$ of the canonical extension $(\bar{E}, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(\bar{X}, \log D)$ is trivial [5]. This gives that the canonical map

$$\Gamma(\bar{X}, \bar{E}) \otimes \mathcal{O}_{\bar{X}} \simeq \bar{E}$$

is an isomorphism and the functor

$$\omega(dR) : \mathrm{Mic}_{\mathrm{uni}}(X/K) \to \mathrm{Vec}_K$$

that maps (E, ∇) to $\Gamma(\bar{X}, \bar{E})$ is compatible with tensor products and hence is a fiber functor. For any K-rational (tangential) basepoint x the natural transformation from $\omega(dR)$ to $\omega(x)$ which associates to each $(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(X/K)$ the linear transformation $\Gamma(\bar{X}, \bar{E}) \to \bar{E}(x)$ induced by restriction is an isomorphism. Therefore, the fiber functors $\omega(dR)$ and $\omega(x)$ are canonically isomorphic. And hence the fiber functors $\omega(x)$ and $\omega(y)$ are canonically isomorphic for any two (tangential) basepoints x and y. We denote this path from x to y by ${}_ye(dR)_x$ and call it the canonical de Rham path from x to y.

4.2. Pro-unipotent algebraic groups

In this section, we collect what we will need from the theory of algebraic groups. More details and proofs can be found in [15].

If K is a field of characteristic zero the map that sends G/K to $\mathrm{Lie}\, G/K$ is an equivalence of categories

$$\mathrm{Lie} : \left(\left(\begin{matrix}\text{unipotent algebraic groups}\\ \text{over } K\end{matrix}\right)\right) \to \left(\left(\begin{matrix}\text{nilpotent Lie algebras}\\ \text{over } K\end{matrix}\right)\right).$$

Furthermore for a unipotent G/K the natural functor that associates to a finite dimensional linear representation of G/K the corresponding nilpotent representation of $\mathrm{Lie}\, G$ is an equivalence of categories

$$\mathrm{Lie} : ((\text{reprs. of } G/K)) \to ((\text{nilpotent reprs. of } \mathrm{Lie}\, G/K)).$$

Let G/K be unipotent and let $\mathcal{U}(\mathrm{Lie}\, G)$ be the universal enveloping algebra of $\mathrm{Lie}\, G$. Then it can be realized as the ring of differential operators on G at 1. The central descending series on $\mathrm{Lie}\, G$ induces a filtration on $\mathcal{U}(\mathrm{Lie}\, G)$, which coincides with the filtration with respect to the powers of the augmentation ideal of $\mathcal{U}(\mathrm{Lie}\, G)$. Let $\hat{\mathcal{U}}(\mathrm{Lie}\, G)$ be its completion with respect to this filtration. There is a natural pairing

$$\mathcal{U}(\mathrm{Lie}\, G) \otimes \Gamma(G, \mathcal{O}) \to K$$

that maps $D \otimes f$ to $D(f)(1)$. This pairing induces a duality between $\hat{\mathcal{U}}(\mathrm{Lie}\, G)$ and $\Gamma(G, \mathcal{O})$.

The ring multiplication on $\Gamma(G, \mathcal{O})$ dualizes to give a coproduct

$$\Delta : \hat{\mathcal{U}}(\mathrm{Lie}\, G) \to \hat{\mathcal{U}}(\mathrm{Lie}\, G) \hat{\otimes} \hat{\mathcal{U}}(\mathrm{Lie}\, G)$$

that makes $\hat{\mathcal{U}}(\mathrm{Lie}\, G)$ a Hopf algebra over K. If $\delta : G \to G \times G$ denotes the diagonal imbedding then the product on $\Gamma(G, \mathcal{O})$ is given by

$$\delta^* : \Gamma(G, \mathcal{O}) \otimes \Gamma(G, \mathcal{O}) \to \Gamma(G, \mathcal{O}).$$

The dual of this map is the coproduct Δ on $\hat{\mathcal{U}}(\mathrm{Lie}\, G)$. Since $d\delta$ maps $e \in T_1 G \simeq \mathrm{Lie}\, G$ to $e \oplus e \in T_1 G \oplus T_1 G \simeq \mathrm{Lie}(G \times G)$, we see that

$$\Delta(e) = 1 \otimes e + e \otimes 1 \quad \text{for} \quad e \in \mathrm{Lie}\, G \subseteq \hat{\mathcal{U}}(\mathrm{Lie}\, G).$$

Giving a rational point $G(K)$ is the same as giving a linear map $h : \Gamma(G, \mathcal{O}) \to K$, with $h(1) = 1$, such that the following diagram:

$$\begin{array}{ccc} \Gamma(G,\mathcal{O}) \otimes \Gamma(G,\mathcal{O}) & \xrightarrow{\delta^*} & \Gamma(G,\mathcal{O}) \\ {\scriptstyle h\otimes h}\downarrow & & {\scriptstyle h}\downarrow \\ K & \xrightarrow{id} & K \end{array}$$

commutes. This shows, by the duality above, that the rational points $G(K)$ correspond to elements a of $\hat{\mathcal{U}}(\text{Lie}\, G)$, with $\epsilon(a) = 1$ and $\Delta(a) = a \otimes a$, where $\epsilon : \hat{\mathcal{U}}(\text{Lie}\, G) \to K$ is the augmentation map. The statements above naturally generalize to the pro-unipotent case.

4.3. p-Adic multi-zeta values

In the following, when we write $\mathbb{P}^1/\mathbb{Q}$ we will always assume that it is endowed with a specific choice of a coordinate function, i.e. a rational function $z \in k(\mathbb{P}^1/\mathbb{Q})$ such that $\mathbb{Q}(z) = k(\mathbb{P}^1/\mathbb{Q})$, where $k(\mathbb{P}^1/\mathbb{Q})$ denotes the field of rational functions on $\mathbb{P}^1$ defined over $\mathbb{Q}$. For $i, j \in \{0, 1, \infty\}$ let t_{ij} denote the unit tangent vector at the point i that points in the direction from i to j. For example, $t_{01} = \frac{d}{dz}$ at 0, $t_{10} = -\frac{d}{dz}$ at 1, $t_{\infty 0} = z^2 \frac{d}{dz}$ at ∞, etc.

Let $X_0 = (\mathbb{P}^1 \backslash D)/\mathbb{Q}$, where $D \subseteq \{0, 1, \infty\}$. If p is a prime number, we will always be working with the standard models $\mathfrak{X}_0 := (\mathbb{P}^1_{\mathbb{Z}_p} \backslash \mathfrak{D})/\mathbb{Z}_p$ of these varieties over $\mathbb{Z}_p$, where $\mathfrak{D}$ is the closure of D in $\mathbb{P}^1_{\mathbb{Z}_p}$. For each $i \in D$, there is an endomorphism res_i of the fiber functor $\omega(dR)$ that associates to each $(E, \nabla) \in \text{Mic}_{\text{uni}}(\bar{X}_0/\mathbb{Q}, \log D)$ the endomorphism $\text{res}_i(E, \nabla)$ of $E(i) \simeq \Gamma(\bar{X}_0, E)$. If D_j is an irreducible component of a simple normal crossings divisor D on a smooth variety Z over a field K, and (E, ∇), and (F, ∇) two vector bundles with connection that have logarithmic singularities along D_j then one sees that $\text{res}_{D_j}((E, \nabla) \otimes (F, \nabla)) = id \otimes \text{res}_{D_j}(F, \nabla) + \text{res}_{D_j}(E, \nabla) \otimes id$. Therefore, the maps res_i defined above are in fact derivations of the fiber functor $\omega(dR)$ hence give elements of the Lie algebra $\text{Lie}\, \pi_{1,dR}(X_0, \omega(dR))$ of the pro-unipotent algebraic group $\pi_{1,dR}(X_0, \omega(dR))$.

Let $\mathcal{T}_D$ denote the tannakian category of finite-dimensional vector spaces over K endowed with nilpotent linear operators N_i, for each $i \in D$, such that $\sum_{i \in D} N_i = 0$.

The functor that sends $(E, \nabla) \in \text{Mic}_{\text{uni}}(\bar{X}_0, \log D)$ to the vector space $\Gamma(\bar{X}_0, E)$ endowed with the linear operators $\text{res}_i(E, \nabla)$ is an equivalence of categories.

That this functor is well-defined and defines an equivalence is seen as follows.

By Section 4.1, we know that (E, ∇) is canonically isomorphic to $(\Gamma(\bar{X}_0, \bar{E}) \otimes \mathcal{O}, \nabla')$ for some connection ∇' with logarithmic singularities along D, and nilpotent residues res_i, at $i \in D$. Then ∇' is determined by $\nabla'(1) \in \text{End}(\Gamma(\bar{X}_0, E)) \otimes \Gamma(\bar{X}_0, \Omega^1_{\bar{X}_0}(\log D))$ or equivalently by the $< \nabla'(1), r_i > = \text{res}_i \in \text{End}(\Gamma(\bar{X}_0, E))$, for $i \in D$, where r_i denotes the residue map from

$\Gamma(\bar{X}_0, \Omega^1_{\bar{X}_0}(\log D))$ to $\mathbb{Q}$ at i. Since $\sum_{i \in D} r_i = 0$, we have $\sum_{i \in D} \text{res}_i = 0$. This shows that the functor is well-defined and using the argument in the proof of Lemma 1 one sees that it is fully faithful.

In order to show the essential surjectivity, assume without loss of generality that $\infty \in D$. Let N_i, for $i \in D$ be a set of nilpotent operators on a vector space V with $\sum_{i \in D} N_i = 0$. Then

$$\left(V \otimes \mathcal{O}_{\bar{X}_0}, d - \sum_{i \in D \setminus \{\infty\}} \right) N_i d \log(z - i)$$

is an object of $\text{Mic}_{\text{uni}}(\bar{X}_0, \log D)$ whose image under the functor defined above is V endowed with the N_i.

Let $\text{Lie}_{\text{nil}} < \text{res}_i >_{i \in D}$ denote the free pro-nilpotent Lie algebra generated by the symbols res_i, for $i \in D$. Then by tannaka duality the above equivalence of categories implies that the natural map

$$\text{Lie}_{\text{nil}} < \text{res}_i >_{i \in D} \Big/ \sum_{i \in D} \text{res}_i \to \text{Lie}\, \pi_{1,dR}(X_0, \omega(dR))$$

is an isomorphism.

(i) *The case when* $X_0 = \mathbb{G}_m$: Let $X_0 = \mathbb{G}_m$, then

$$\pi_{1,dR}(X_0, \omega(dR)) \simeq \pi_{1.dR}(X_0, 1)$$

by the canonical de Rham path between $\omega(dR)$ and $\omega(1)$. Denote by e the element of $\text{Lie}\pi_{1,dR}(X_0, 1)$ that corresponds to $\text{res}_0 \in \text{Lie}\, \pi_{1,dR}(X_0, \omega(dR))$ under this isomorphism. Note that $\text{Lie}_{\text{nil}} < e > \simeq \text{Lie}\, \pi_{1,dR}(X_0, 1)$. Let p be a prime number. In order to compute the action of frobenius on $\pi_{1,dR}(X_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p, 1)$, let $\mathfrak{X}_0 := \mathbb{G}_m/\mathbb{Z}_p$ and let $\mathcal{F} : \mathfrak{X}_0 \to \mathfrak{X}_0$ be the lifting of frobenius $\mathcal{F}(z) = z^p$. This has the property that $\mathcal{F}^*(\mathfrak{D}) = p\, \mathfrak{D}$, and maps the basepoint 1 to 1. Therefore, $F_*(e) = p\, e$, since pulling back by $\mathcal{F}$ multiplies the residue at 0 of a logarithmic connection by p.

(ii) *The case when* $X_0 = \mathbb{P}^1 \setminus \{0, 1, \infty\}$: Let e_0, e_1, and e_∞ be the elements of $\text{Lie}\, \pi_{1,dR}(X_0, t_{01})$ that are the images of the elements res_0, res_1, res_∞ in $\text{Lie}\, \pi_{1,dR}(X_0, \omega(dR))$ under the isomorphism

$$\text{Lie}\, \pi_{1,dR}(X_0, \omega(dR)) \simeq \text{Lie}\, \pi_{1,dR}(X_0, t_{01})$$

given by the canonical de Rham path between $\omega(dR)$ and $\omega(t_{01})$. Let $\mathcal{U}_{dR}$ be the universal enveloping algebra of $\text{Lie}\, \pi_{1,dR}(X, t_{01})$ and $\hat{\mathcal{U}}_{dR}$ be its completion with respect to its augmentation ideal. By the above, it is a cocommutative Hopf algebra, and its topological dual is the Hopf algebra of functions on $\pi_{1,dR}(X, t_{01})$. Since $\text{Lie}\, \pi_{1,dR}(X, t_{01}) \simeq \text{Lie}_{\text{nil}} < e_0, e_1 >$, $\hat{\mathcal{U}}_{dR}$ is isomorphic to the ring of associative formal power series on e_0 and e_1 with the coproduct Δ given by $\Delta(e_0) = 1 \otimes e_0 + e_0 \otimes 1$, and $\Delta(e_1) = 1 \otimes e_1 + e_1 \otimes 1$. By the duality above the K-rational points, $\pi_{1,dR}(X, t_{01})(K)$

correspond to associative formal power series a in e_0 and e_1 with coefficients in K, that start with 1 and satisfy $\Delta(a) = a \otimes a$. Such formal power series are called group-like.

Lemma 6. *If $a \in \hat{\mathcal{U}}_{dR}$ is group-like, then it is determined by the coefficients of e_0 and of the terms of the form $e_0^{s_k-1}e_1 \cdots e_0^{s_1-1}e_1$, where $s_1, \ldots, s_k \geqslant 1$, in a.*

Proof. Let $m := e_{i_1} \cdots e_{i_n}$, where $e_{i_j} \in \{e_0, e_1\}$, be a general monomial. Then comparing the coefficients of $m \otimes e_0$ on both sides of the equality $\Delta(a) = a \otimes a$, we obtain

$$a[m]a[e_0] = \sum_{m' \in Sh(m,\, e_0)} a[m'],$$

where $Sh(m, e_0)$ denotes the set of monomials obtained by putting e_0 either between two of the terms in m, to the leftmost place, or to the rightmost place (Sh is for shuffle). Therefore $a[me_0]$ is determined by terms $a[e_0]$ and $a[m'']$ such that m'' has fewer e_0 terms on its right than m. Therefore by induction, if d is any monomial $a[d]$ is determined by $a[e_0]$ and by terms of the form $a[e]$, where e has an e_1 on its rightmost place, and hence is of the form $e_0^{s_k-1}e_1 \cdots e_0^{s_1-1}e_1$. $\square$

Let p be a prime number. From now on we put $X := X_0 \otimes_{\mathbb{Q}} \mathbb{Q}_p$, and use the notation of Section 2.4, i.e. $\overline{\mathfrak{X}} = \mathbb{P}^1/\mathbb{Z}_p$, D is the divisor $(0) + (1) + (\infty)$ etc. By the last paragraph of Section 3.2, there is an action of frobenius

$$F_* : \pi_{1,dR}(X, t_{01}) \to \pi_{1,dR}(X, t_{01}).$$

Lemma 7. *With the notations as above $F_*(e_0) = pe_0$.*

Proof. By the definitions above

$$\omega(t_{01}) : \mathrm{Mic}_{\mathrm{uni}}(X/K) \to \mathrm{Vec}_K$$

is the functor that sends $(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(X/K)$ to the fiber $\bar{E}(0)$ at zero of its canonical extension $(\bar{E}, \nabla)$ and e_0 is the element of Lie $\pi_{1,dR}(X, t_{01})$ that associates to each object $(E, \nabla) \in \mathrm{Mic}_{\mathrm{uni}}(X/K)$ the residue $\mathrm{res}_0(\bar{E}, \nabla) \in \mathrm{End}(\bar{E}(0))$. Since $F^*(\bar{E}, \nabla)$ was defined by choosing local liftings of frobenius, with the lifting $\mathcal{F}$ of frobenius around 0 such that $\mathcal{F}^*(0) = p\,(0)$, we have $\mathrm{res}_0(F^*(\bar{E}, \nabla)) = p \cdot \mathrm{res}_0(\bar{E}, \nabla)$. Therefore $F_* e_0 = p \cdot e_0$. $\square$

Let $\sigma : X \to X$ be the automorphism that maps z to $\sigma(z) = 1 - z$, then we see that the induced map

$$\sigma_* : \pi_{1,dR}(X, t_{01}) \to \pi_{1,dR}(X, t_{10})$$

maps e_0 to ${}_{t_{10}}e(dR)_{t_{01}}e_1{}_{t_{01}}e(dR)_{t_{10}}$. Since the action of σ commmutes with the action of frobenius, we obtain

$$F_*({}_{t_{10}}e(dR)_{t_{01}}e_1{}_{t_{01}}e(dR)_{t_{10}}) = p{}_{t_{10}}e(dR)_{t_{01}}e_1{}_{t_{01}}e(dR)_{t_{10}}.$$

Therefore if we let

$$g := {}_{t_{01}}e(dR)_{t_{10}}F_*({}_{t_{10}}e(dR)_{t_{01}}) \in \pi_{1,dR}(X, t_{01}),$$

the action of frobenius is described by

$$F_*(e_0) = pe_0 \quad \text{and} \quad F_*(e_1) = g^{-1}pe_1g.$$

Therefore describing the action of frobenius on the de Rham fundamental group of X reduces to describing g. The following definition is due to Deligne (unpublished).

Definition 3. The coefficient of the term $e_0^{s_k-1}e_1 \cdots e_0^{s_1-1}e_1$ in g is denoted by $p^{\sum s_i}\zeta_p(s_k, \ldots, s_1)$ and called a p-adic multi-zeta value.

Note that this coefficient is also denoted by $g[e_0^{s_k-1}e_1 \cdots e_0^{s_1-1}e_1]$ below. We will see below that $g[e_0] = 0$. Therefore, by Lemma 6, g is determined by the p-adic multi-zeta values.

5. Computations

5.1. Lifting of frobenius

In the following let $R := \mathbb{Z}_p$, in order to avoid multiple indices. Let $U := \bar{Y} \backslash \{1\}$, and let $\mathcal{U}/R$ be the formal scheme over R that is the completion of $\overline{\mathfrak{X}}$ along U. Let

$$\mathcal{F} : \mathcal{U} \to \mathcal{U}$$

be $\mathcal{F}(z) = z^p$. Since $\mathcal{F}^*(0) = p(0)$, and $\mathcal{F}^*(\infty) = p(\infty)$, this is a good lifting of frobenius to $\mathcal{U}$. Note that $\mathcal{U}_K = \mathbb{P}^1 \backslash D(1, 1^-)$. Moreover, note that $P(\mathcal{F})(t_{01}) = t_{01}$.

The functor $\alpha_X : \mathrm{Mic}_{\mathrm{uni}}(X/K) \to \mathrm{Coh}(\bar{X})$ that associates to each (E, ∇) the underlying vector bundle $\bar{E}$ of the canonical extension is a fiber functor. Therefore, we obtain a groupoid $\mathcal{G} := \mathcal{G}(\mathrm{Mic}_{\mathrm{uni}}(X/K), \alpha_X)$ acting on $\bar{X}$. It is endowed with a *connection logarithmic along D*. By this we mean that the identity section of the map $\mathcal{G} \to \bar{X} \times \bar{X}$ over the diagonal $\Delta_{\bar{X}}$, extends to a section on the first infinitesimal neighborhood of the strict transform $\Delta_{\tilde{\bar{X}}}$ of the blow-up $(\bar{X} \times \bar{X})^{\sim}$ of $\bar{X} \times \bar{X}$ along $\Delta(D)$. As usual this connection satisfies the cocycle condition and converges in the tube of $\Delta_{\tilde{\bar{X}}}$ (we are always using the standard model). The following will be essential for the computations.

Claim. *$\mathcal{F}$ induces a horizontal map*

$$\mathcal{F}_*\colon\ \mathcal{G}|_{\mathcal{U}_K} \times \mathcal{U}_K \to \mathcal{F}^*(\mathcal{G}|_{\mathcal{U}_K\times\mathcal{U}_K})$$

between the groupoid restricted to the affinoid $\mathcal{U}_K \times \mathcal{U}_K$ and its pull-back via $\mathcal{F}^*$.

Proof. This is a direct consequence of the construction of the frobenius. Let s denote the restriction of the connection of $\mathcal{G}$ to $\mathcal{U}_K$. This is a section of $\mathcal{G}$ on the first infinitesimal neighborhood $\Delta^{(1)}_{\mathcal{U}_K}$ of the diagonal. In order to show the horizontality of $\mathcal{F}_*$ we need to show that $\mathcal{F}_*$ maps the section s of $\mathcal{G}$ to the section $\mathcal{F}^*s$ of $\mathcal{F}^*\mathcal{G}$. The section s of $\mathcal{G}$ is the one that associates to each $(E,\nabla)\in \mathrm{Mic}_{\mathrm{uni}}(\bar{X}, \log D)$ and each scheme valued point $(x,y)\in\mathcal{U}_K\times\mathcal{U}_K$ lying in $\Delta^{(1)}_{\mathcal{U}_K}$, the isomorphism between $E(x)$ and $E(y)$ defined by the connection ∇ of E, where $E(z)$ is an abuse of notation for $z^*(E)$. By definition $\mathcal{F}_*(s)$ is the section $F_*(s)|_{\mathcal{U}_K}$ of $F^*(\mathcal{G})|_{\mathcal{U}_K\times\mathcal{U}_K} \simeq \mathcal{F}^*\mathcal{G}|_{\mathcal{U}_K\times\mathcal{U}_K}$. Then $\mathcal{F}_*(s)$ is the section of $F^*\mathcal{G}|_{\mathcal{U}_K\times\mathcal{U}_K}$ that associates to each (E,∇) as above the isomorphism between $F^*(E)(x)|_{\mathcal{U}_K}$ and $F^*E(y)|_{\mathcal{U}_K}$ defined by the connection $F^*\nabla|_{\mathcal{U}_K}$. But by the construction, with $\mathcal{F}$ as the lifting of frobenius on $\mathcal{U}$, we have $F^*E|_{\mathcal{U}_K}\simeq\mathcal{F}^*E|_{\mathcal{U}_K}$ and $F^*\nabla|_{\mathcal{U}_K}\simeq\mathcal{F}^*\nabla|_{\mathcal{U}_K}$. Therefore $\mathcal{F}_*(s)$ is the section that associates to each (E,∇) as above the isomorphism between $\mathcal{F}^*(E)(x)$ and $\mathcal{F}^*E(y)$ defined by $\mathcal{F}^*\nabla$. But this is nothing other than the section $\mathcal{F}^*s$ of $\mathcal{F}^*\mathcal{G}|_{\mathcal{U}_K\times\mathcal{U}_K}$. □

Taking the fiber product

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{G} \\ \downarrow & & \downarrow \\ \overline{X} & \longrightarrow & \overline{X}\times\overline{X}, \end{array}$$

where the lower horizontal map is $x\to(0,x)$, we obtain the fundamental $\pi_{1,dR}(X,\omega(0))$-torsor of paths that start at the point 0 (note that even though $0\notin X$, this is well-defined in the de Rham realization).

With this notation, the horizontality of the frobenius means that the diagram

$$\begin{array}{ccc} \mathcal{T}|_{\mathcal{U}_K} & \xrightarrow{\mathcal{F}_*} & \mathcal{F}^*(\mathcal{T}|_{\mathcal{U}_K}) \\ \big\downarrow \nabla & & \big\downarrow \mathcal{F}^*\nabla \\ \mathrm{Lie}\pi_{1,dR}(X,t_{01})\otimes\Omega^1_{\mathcal{U}_K/K} & \xrightarrow{\mathrm{Lie}F_*\otimes\mathrm{id}} & \mathrm{Lie}\pi_{1,dR}(X,t_{01})\otimes\Omega^1_{\mathcal{U}_K/K} \end{array}$$

is commutative. The canonical de Rham path ${}_ze(dR)_{t_{01}}$ gives a section of $\mathcal{T}$. We let

$$g(z) \coloneqq {}_{t_{01}}e(dR)_{\mathcal{F}(z)}\mathcal{F}_*({}_ze(dR)_{t_{01}})\in\pi_{1,dR}(X,t_{01})$$

for $z \in \mathcal{U}_K$, and view it as a group-like element of $\hat{\mathcal{U}}_{dR}$. As z varies this defines a rigid analytic section $g(z)$ of $\mathcal{U}_{dR} \hat{\otimes} \mathcal{O}_{\text{an}}(\mathcal{U}_K)$, i.e. the coordinates of the e^J are rigid analytic functions on $\mathcal{U}_K$, for each monomial e^J in e_0 and e_1.

Proposition 1. *Taking the image of* ${}_ze(dR)_{t_{01}}$ *in the two different ways in the commutative diagram above gives the differential equation*

$$dg(z) = (e_0 g(z) - g(z)e_0)\frac{p\,dz}{z} + e_1 g(z)\frac{pz^{p-1}\,dz}{z^p - 1} - g(z)g^{-1}e_1 g\frac{p\,dz}{z-1}. \tag{1}$$

Proof. First, we will give a more explicit description of $(\mathcal{T}, \nabla)$.

Claim. *The* $\pi_{1,dR}(X, t_{01})$*-torsor with connection* $(\mathcal{T}, \nabla)$ *is naturally isomorphic to the trivial torsor* $\pi_{1,dR}(X, t_{01}) \times \bar{X}$ *endowed with the connection* $\nabla' := d - e_0\frac{dz}{z} - e_1\frac{dz}{z-1}$. *The isomorphism is the one that sends the unit section* 1 *of* $\pi_{1,dR}(X, t_{01}) \times \bar{X}$ *to the section* ${}_ze(dR)_{t_{01}}$ *of* $\mathcal{T}$.

Proof. We only need to check that the isomorphism of torsors described in the statement is horizontal with respect to the connections. This can be seen as follows. Let $(\bar{E}, \nabla) \in \text{Mic}_{\text{uni}}(\bar{X}/K, \log D)$. Let

$$\rho : \pi_{1,dR}(X, t_{01}) \to GL(\bar{E}(0))$$

be the corresponding representation, and denote the representation $\hat{\mathcal{U}}_{dR} \to \text{End}(\bar{E}(0))$ by the same symbol. Viewing $(\bar{E}, \nabla)$ as a $GL(\bar{E}(0))$-torsor with connection it is isomorphic to $(\mathcal{T}, \nabla) \times_\rho GL(\bar{E}(0))$, where the last notation denotes the $GL(\bar{E}(0))$-torsor with connection obtained by changing the structure group of $(\mathcal{T}, \nabla)$ via the map ρ. On the other hand $(\bar{E}, \nabla)$ is isomorphic to

$$\left(\Gamma(\bar{X}, \bar{E}) \otimes_K \mathcal{O}_{\bar{X}}, d - \rho(e_0)\frac{dz}{z} - \rho(e_1)\frac{dz}{z-1}\right)$$

by the unique map that induces the identity map on the global sections. That the residues are of the last connection at 0 and 1 are equal to $\rho(e_0)$ and $\rho(e_1)$ is immediate from the definitions of e_0 and e_1. This gives a horizontal isomorphism from

$$\left(GL(\bar{E}(0)) \times \bar{X}, d - \rho(e_0)\frac{dz}{z} - \rho(e_1)\frac{dz}{z-1}\right)$$

to $(\mathcal{T}, \nabla) \times_\rho GL(\bar{E}(0))$ that is induced by the natural map described in the statement of the claim. In other words the isomorphism between $\pi_{1,dR}(X, t_{01}) \times \bar{X}$ and $\mathcal{T}$ is horizontal if it is pushed-forward by an algebraic representation ρ :

$\pi_{1,dR}(X, t_{01}) \to GL(V)$, where V is a finite dimensional K vector space. Since $\pi_{1,dR}(X, t_{01})$ is pro-unipotent this implies that the original isomorphism is horizontal. $\square$

In the remaining part of the proof of the proposition we will use this trivialization of $(\mathcal{T}, \nabla)$. Note that with this notation if $\gamma(z)$ is a section of the torsor $\pi_{1,dR}(X, t_{01}) \times \bar{X}$ then

$$\nabla'(\gamma(z)) = \gamma(z)^{-1} d\gamma(z) - \gamma(z)^{-1} e_0 \gamma(z) \frac{dz}{z} - \gamma(z)^{-1} e_1 \gamma(z) \frac{dz}{z-1}$$

as elements in Lie $\pi_{1,dR}(X, t_{01}) \otimes \Omega^1_{\bar{X}, \log D/K}$. Where as usual we view $\pi_{1,dR}(X, t_{01})(K)$ and Lie $\pi_{1,dR}(X, t_{01})$ as subsets of $\hat{\mathcal{U}}_{dR}$ as in Section 4.3.

Then

$$\nabla({}_z e(dR)_{t_{01}}) = \nabla'(1) = -e_0 \frac{dz}{z} - e_1 \frac{dz}{z-1}.$$

Hence by the formula at the end of 4.3.(ii), where Lie F_* is denoted by F_*,

$$\text{Lie}\, F_* \otimes id({}_z e(dR)_{t_{01}}) = -pe_0 \frac{dz}{z} - g^{-1} pe_1 g \frac{dz}{z-1}.$$

On the other hand the pull-back connection $\mathcal{F}^*\nabla$ on $\mathcal{F}^*(\mathcal{T}|_{\mathcal{U}_K})$ is given by

$$\mathcal{F}^*(\nabla') = \mathcal{F}^*\left(d - e_0 \frac{dz}{z} - e_1 \frac{dz}{z-1}\right) = d - e_0 \frac{pdz}{z} - e_1 \frac{pz^{p-1}dz}{z^p - 1}$$

in the trivialization above. Finally $\mathcal{F}_*({}_z e(dR)_{t_{01}})$ is mapped to $g(z)$ in this trivialization, and we get

$$\mathcal{F}^*\nabla(\mathcal{F}_*({}_z e(dR)_{t_{01}}) = \mathcal{F}^*\nabla(g(z)) = g(z)^{-1}\, dg(z) - g(z)^{-1}\left(e_0 \frac{pdz}{z} + e_1 \frac{pz^{p-1}}{z^p - 1}\right) g(z)$$

Since $\mathcal{F}^*\nabla(\mathcal{F}_*({}_z e(dR)_{t_{01}}) = (\text{Lie}\, F_* \otimes id)(\nabla({}_z e(dR)_{t_{01}})$, we obtain

$$g(z)^{-1} dg(z) - g(z)^{-1}\left(e_0 \frac{p\, dz}{z} + e_1 \frac{pz^{p-1}dz}{z^p - 1}\right) g(z) = -p\, e_0 \frac{dz}{z} - g^{-1} p\, e_1 g \frac{dz}{z-1}. \qquad \square$$

Since

$$P(\mathcal{F})(t_{01}) = t_{01}, \quad P(\mathcal{F})(t_{\infty 0}) = (-1)^{p-1} t_{\infty 0},$$

and $\log(-1)^{1-p} = 0$, we obtain the relations

$$g(0) = 1 \quad \text{and} \quad g(\infty) = {}_{t_{01}} e(dR)_{t_{\infty 0}} F_*({}_{t_{\infty 0}} e(dR)_{t_{01}}),$$

by Lemma 3.

5.2. Relation between g and $g(\infty)$

The following relation will be fundamental for the computations. Let $\tau : X \to X$ be $\tau(z) = \frac{z-1}{z}$. Then

$$\tau_*(e_0) = {}_{t_{\infty 0}}e(dR)_{t_{01}}\, e_\infty\, {}_{t_{01}}e(dR)_{t_{\infty 0}}.$$

Since τ_* commutes with the action of frobenius we obtain, from Section 9, that

$$F_*(e_\infty) = pg(\infty)^{-1} e_\infty g(\infty),$$

using the notation of Section 5.1.

Applying F_* to

$$e_0 + e_1 + e_\infty = 0,$$

we obtain

$$e_0 + g^{-1} e_1 g = g(\infty)^{-1}(e_0 + e_1)g(\infty). \tag{2}$$

5.3. Uniqueness

If we know $g(\infty)$, the equation above together with the facts that g is group-like, and $g[e_1] = 0$ (see 5.6 below) determines g uniquely. If h is another such element of $\hat{\mathcal{U}}_{dR}$ then $\mathrm{Ad}(gh^{-1})(e_1) = e_1$. Writing $\exp(a) = gh^{-1}$ this gives $\exp(\mathrm{ad}(a))(e_1) = e_1$. An induction on the weight of the leading term of a, starting with $a[e_0] = 0$ and $a[e_1] = 0$, gives $a = 0$.

5.4. Analytic functions on $\mathcal{U}$

We will need the following proposition.

Proposition 2. *Let $f(z) = \sum_{0<n} a_n z^n$ be a power series with $a_n \in K$. Then $f(z)$ is the power series expansion at 0 of a rigid analytic function on $\mathcal{U}$ if and only if the map $\alpha : \mathbb{N}\backslash\{0\} \to K$, defined by $\alpha(n) = a_n$ is continuous, where $\mathbb{N}\backslash\{0\} \subseteq \mathbb{Z}$ is endowed with the p-adic metric. And in this case the sequence of rational functions $\frac{1}{1-z^{p^N}} \sum_{0<n\leqslant p^N} a_n z^n$ converges uniformly to this unique extension to $\mathcal{U}$, and the value at infinity of this extension is given by $-\lim_{N\to\infty} a_{p^N}$.*

Proof. First note that functions of the form

$$\frac{1}{(1-az)^k} - 1 = \sum_{0<n} \binom{n+k-1}{k-1} a^n z^n$$

for $k>0$, satisfy both the analyticity property and the congruence condition in the statement of the lemma if and only if $|a-1|_p<1$ or $a=0$.

Assume that $f(z)$ extends to a rigid analytic function on $\mathcal{U}$. Then it is a uniform limit with respect to the supremum norm on $\mathcal{U}$ of rational functions which do not have poles on $\mathcal{U}$. Without loss of generality we may assume that these rational functions are 0 at the point 0 and hence, by the method of partial fractions, are linear combinations of the functions considered above with $|a-1|<1$. Since we have seen above that each of these rational functions satisfy the congruence property, to deduce it for the coefficients of $f(z)$ we only need to note the following inequality

$$\sup_{0\leqslant n}|b_n|\leqslant \sup_{|z|<1}|g(z)|,$$

for a power series $g(z)=\sum_{0\leqslant n} b_n z^n$, which converges on $D(0,1^-)$.

Conversely, assume that we have the congruence condition for the coefficients of $f(z)$. Note that if

$$f_N(z)=\frac{1}{1-z^{p^N}}\sum_{0<n\leqslant p^N} a_n z^n,$$

we have

$$f_{N+1}(z)-f_N(z)=\frac{1}{1-z^{p^{N+1}}}\sum_{0<n\leqslant p^{N+1}}(a_n-a_{n(\mathrm{mod}\,p^N)})z^n,$$

where by $n(\mathrm{mod}\,p^N)$ we denote the unique integer m satisfying $0<m\leqslant p^N$ and $n\equiv m(\mathrm{mod}\,p^N)$. From this we see that

$$|f_{N+1}(z)-f_N(z)|\leqslant \sup_{0<n\leqslant p^{N+1}}|a_n-a_{n(\mathrm{mod}\,p^N)}|,$$

if $1\leqslant|z-1|$. This shows that $\{f_N(z)\}$ is a uniform Cauchy sequence of rational functions without poles on $\mathcal{U}$, and hence converges to an analytic function on $\mathcal{U}$. Note that the congruence condition implies that $f(z)$ converges on $D(0,1^-)$, and as above we see that

$$|f(z)-f_N(z)|\leqslant \sup_{0<n}|a_n-a_{n(\mathrm{mod}\,p^N)}|,$$

if $|z|<1$. This shows that $f(z)$ is the power series expansion at 0 of the uniform limit of the sequence $\{f_N(z)\}$, and finishes the proof of the proposition if we note that $f_N(\infty)=-a_{p^N}$. □

5.5. Parallel transport

Let $\mathcal{V}$ be the completion of $\overline{\mathfrak{X}}$ along Y. If $x, y \in \mathcal{V}$ have the same specialization then there is a canonical isomorphism,

$$ {}_y\gamma_x : \omega(x) \simeq \omega(y) $$

from $\omega(x)$ to $\omega(y)$. We let

$$ {}_y\mathrm{par}_x := {}_{t_{01}}e(dR)_{y\,y}\gamma_{x\,x}e(dR)_{t_{01}} \in \pi_{1,dR}(X, t_{01}), $$

and call it the parallel transport along the connection from x to y.

Lemma 8. *We have*

$$ {}_y\mathrm{par}_x[e_0] = \log\frac{y}{x} \quad \textit{and} \quad {}_y\mathrm{par}_x[e_1] = \log\frac{y-1}{x-1}. $$

Proof. The second formula follows from the first one by using the functoriality of the canonical connection with respect to the map $\sigma : X \to X$ that maps z to $\sigma(z) := 1-z$. In order to prove the first formula, let $f(z) :=_z \mathrm{par}_z[e_0]$, where z ranges over an open disc of radius one around x. Then we have

$$ \frac{d}{dz}f(z) = \frac{1}{z} $$

and $f(x) = 0$, which gives $f(z) = \log\frac{z}{x}$. □

5.6. Computation of $g[e_0]$ and $g[e_1]$

Eq. (1) together with the initial condition $g(0) = 1$, gives

$$ g(z)[e_0] = 0 \quad \text{and} \quad g(z)[e_1] = p \sum_{\substack{0<n \\ p \nmid n}} \frac{z^n}{n} $$

on $D(0, 1^-)$, which has the rigid analytic extension $-\log\frac{(1-z)^p}{1-z^p} =: p\ell_1^{(p)}(z)$ to $\mathcal{U}$. To obtain the value of $g[e_0]$ and $g[e_1]$ we need a lift $\mathcal{F}'$ of frobenius to the completion $\mathcal{U}_1$ of $\overline{\mathfrak{X}}$ along $\bar{Y}\backslash\{0\}$, that satisfies $\mathcal{F}'^*((1)) = p(1)$ and $P(\mathcal{F}')(t_{10}) = t_{10}$. We choose $\mathcal{F}'(z) = 1-(1-z)^p$. And we obtain

$$ \begin{aligned} g[e_0] &= \lim_{z\to 1}\left({}_{\mathcal{F}'(z)}\mathrm{par}_{\mathcal{F}(z)}[e_0] + \mathcal{F}_*({}_z e(dR)_{t_{01}})[e_0]\right) \\ &= \lim_{z\to 1} \log\frac{1-(1-z)^p}{z^p} = 0 \end{aligned} $$

and

$$g[e_1] = \lim_{z\to 1}({}_{\mathcal{F}'(z)}\mathrm{par}_{\mathcal{F}(z)}[e_1] + \mathcal{F}_*({}_ze(dR)_{t_{01}})[e_1])$$
$$= \lim_{z\to 1}\left(\log\frac{-(1-z)^p}{z^p-1} - \log\frac{(1-z)^p}{1-z^p}\right) = 0.$$

5.7. Computation of $g(z)[e_0^{s-1}e_1]$

By a direct inductive computation using (1) and 5.6 we see that

$$g(z)[e_0^{s-1}e_1] = p^s \sum_{\substack{0<n \\ p\nmid n}} \frac{z^n}{n^s}$$

on $D(0,1^-)$ which has the rigid extension

$$p^s\,\ell_s^{(p)}(z) := p^s \lim_{N\to\infty} \frac{1}{1-z^{p^N}} \sum_{\substack{0<n<p^N \\ p\nmid n}} \frac{z^n}{n^s}$$

to $\mathcal{U}$ (cf. Proposition 2).

5.8. Computation of $g(\infty)[e_0^a\, e_1 e_0^b]$

By 5.4 and 5.7 we see that $g(\infty)[e_0^{s-1}e_1] = p^s\ell_s^{(p)}(\infty) = 0$. Using this and noting that $g(\infty)$ is a group-like element with $g(\infty)[e_0] = 0$ (by 5.6) we see that $g(\infty)[e_0^a e_1 e_0^b] = 0$ as in the proof of Lemma 6.

5.9. Computation of $g[e_0^a e_1 e_0^b]$

Using the facts that g is group-like and $g[e_0] = 0$ we see that

$$g[e_0^a\, e_1 e_0^b] = (-1)^b \binom{a+b}{a} g[e_0^{a+b}e_1].$$

5.10. p-adic analogue of Euler's formula

Comparing the coefficients of $e_1 e_0^{s-1} e_1$ on both sides of the equation

$$(e_0+e_1)g(\infty) = g(\infty)(e_0 + g^{-1}e_1 g)$$

and noting that $g(\infty)[e_0^{s-1}e_1] = g(\infty)[e_1e_0^{s-1}] = 0$ (by 5.8), we obtain that

$$g^{-1}[e_1e_0^{s-1}] + g[e_0^{s-1}e_1] = 0$$

or

$$g[e_0^{s-1}e_1] = g[e_1e_0^{s-1}] = (-1)^{s-1}g[e_0^{s-1}e_1].$$

Therefore

$$p^s\zeta_p(s) = g[e_0^{s-1}e_1] = 0,$$

if s is even.

5.11. A formula for $g[e_0^{s-1}e_1]$

Comparing the coefficients of $e_0e_1e_0^{s-1}e_1$ on both sides of

$$(e_0 + e_1)g(\infty) = g(\infty)(e_0 + g^{-1}e_1g)$$

we obtain

$$g(\infty)[e_1e_0^{s-1}e_1] = g^{-1}[e_0e_1e_0^{s-1}] = (-1)^s sg[e_0^s e_1].$$

Using Eq. (2) we obtain

$$dg(z)[e_1e_0^{s-1}e_1] = g(z)[e_0^{s-1}e_1]\frac{pz^{p-1}\,dz}{z^p - 1} + (g(z)[e_1e_0^{s-1}] + g^{-1}[e_1e_0^{s-1}] + g[e_0^{s-1}e_1])\frac{p\,dz}{1-z}.$$

Using 5.7, 5.10, and the fact that $g(z)$ is group-like this reduces to

$$dg(z)[e_1e_0^{s-1}e_1] = p^{s+1}(-1)^{s-1}\ell_s^{(p)}(z)\,\frac{dz}{1-z} - p^{s+1}\ell_s^{(p)}(z)\,\frac{z^{p-1}\,dz}{1-z^p}.$$

Hence, we obtain

$$g(z)[e_1e_0^{s-1}e_1] = p^{s+1}\left((-1)^{s-1}\sum_{\substack{0<n_1<n_2\\ p\nmid n_1}}\frac{z^{n_2}}{n_1^s n_2} - \sum_{\substack{0<n_1<n_2\\ p\nmid n_1\\ n_1\equiv n_2 (\mathrm{mod}\ p)}}\frac{z^{n_2}}{n_1^s n_2}\right)$$

on $D(0,1^-)$. Since we know that $g(z)[e_1e_0^{s-1}e_1]$ is rigid analytic on $\mathcal{U}$ (see 5.1), Proposition 2 gives

$$g(\infty)[e_1e_0^{s-1}e_1] = p^{s+1}(-1)^s\lim_{N\to\infty}\frac{1}{p^N}\sum_{\substack{0<n<p^N\\ p\nmid n}}\frac{1}{n^s}.$$

And this gives the expression

$$g[e_0^{s-1}e_1] = \frac{p^s}{s-1} \lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<n<p^N \\ p\nmid n}} \frac{1}{n^{s-1}}$$

for $g[e_0^{s-1}e_1]$.

Note that if $0<r<p$ and $P(x)\in\mathbb{Z}[x]$ then

$$\lim_{N\to\infty} \frac{1}{p^N} \sum_{0<n<p^{N-1}} \frac{P(n)}{(r+pn)^{s-1}} \in \mathbb{Z}_p.$$

Therefore

$$\lim_{k\to\infty} \lim_{N\to\infty} \frac{1}{p^N} \sum_{0<n<p^{N-1}} \frac{(r+pn)^{(p-1)p^k}}{(r+pn)^{s-1}} = \lim_{k\to\infty} \lim_{N\to\infty} \frac{1}{p^N} \sum_{0<n<p^{N-1}} \frac{(1+p\,Q(n))^{p^k}}{(r+pn)^{s-1}}$$
$$= \lim_{N\to\infty} \frac{1}{p^N} \sum_{0<n<p^{N-1}} \frac{1}{(r+pn)^{s-1}}.$$

If B_n denotes the n-th Bernoulli number, that is

$$\frac{ze^z}{e^z-1} = \sum_{0\leqslant n} B_n \frac{z^n}{n!},$$

then

$$g[e_0^{s-1}e_1] = p^s \lim_{k\to\infty} \left(\frac{1}{s-1-(p-1)p^k} \lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<n<p^N \\ p\nmid|n}} \frac{1}{n^{s-1-(p-1)p^k}} \right)$$
$$= p^s \lim_{k\to\infty} \left(\frac{1-p^{(p-1)p^k-s}}{s-1-(p-1)p^k} \lim_{N\to\infty} \frac{1}{p^N} \sum_{0<n<p^N} \frac{1}{n^{s-1-(p-1)p^k}} \right)$$
$$= p^s \lim_{k\to\infty} \left(\frac{1-p^{(p-1)p^k-s}}{s-1-(p-1)p^k} B_{(p-1)p^k-s+1} \right)$$
$$= p^s \lim_{k\to\infty} \left(\left(1-\frac{1}{p^{s-p^k(p-1)}}\right) \zeta(s-(p-1)p^k) = p^s \zeta_p(s) \right),$$

where ζ_p is as in [5], a meromorphic function on the space of characters that interpolates the values at the negative integers of the ordinary zeta function without its Euler factors at p. This also shows that the notation we are using for the p-adic multi-zeta is compatible with the notation used for the p-adic zeta function. Note that this formula was proven in [5] using the distribution formula.

5.12. Computation of $g(\infty)[e_0^a e_1 e_0^b e_1 e_0^c]$

Comparing the coefficients of $e_0^{a+1}e_1e_0^be_1$ on both sides of the equation

$$g(\infty)^{-1}(e_0+e_1)g(\infty)=e_0+g^{-1}e_1g$$

we obtain

$$g(\infty)[e_0^ae_1e_0^be_1]=g^{-1}[e_0^{a+1}e_1e_0^b]=(-1)^{b+1}\binom{a+b+1}{b}g[e_0^{a+b+1}e_1].$$

Using the above and the facts that $g(\infty)$ is group-like, and $g(\infty)[e_0]=0$ we have

$$\begin{aligned}g(\infty)[e_0^ae_1e_0^be_1e_0^c]&=(-1)^c\sum_{0\leqslant k\leqslant c}\binom{a+c-k}{a}\binom{b+k}{b}g(\infty)[e_0^{a+c-k}e_1e_0^{b+k}e_1]\\&=(-1)^{b+c+1}\frac{(a+b+c+1)!}{a!b!c!}\sum_{0\leqslant k\leqslant c}\frac{(-1)^k}{a+c+1-k}\binom{c}{k}g[e_0^{a+b+c+1}e_1].\end{aligned}$$

In particular,

$$\begin{aligned}g(\infty)[e_1e_0^be_1e_0^c]&=(-1)^{b+c+1}\binom{b+c+1}{c+1}\sum_{0\leqslant k\leqslant c}(-1)^k\binom{c+1}{k}g[e_0^{b+c+1}e_1]\\&=(-1)^{b+1}\binom{b+c+1}{b}g[e_0^{b+c+1}e_1]=g^{-1}[e_0^be_1e_0^{c+1}].\end{aligned}$$

5.13. A relation between p-adic multi-zeta values of depth 2

Comparing the coefficients of $e_1e_0^{t-1}e_1e_0^{s-1}e_1$ in

$$g(\infty)(e_0+g^{-1}e_1g)=(e_0+e_1)g(\infty)$$

we obtain

$$\begin{aligned}g[e_0^{t-1}e_1e_0^{s-1}e_1]+g^{-1}[e_1e_0^{t-1}e_1e_0^{s-1}]&=g(\infty)[e_0^{t-1}e_1e_0^{s-1}e_1]-g(\infty)[e_1e_0^{t-1}e_1e_0^{s-1}]\\&\quad-g^{-1}[e_1e_0^{t-1}]g[e_0^{s-1}e_1]\\&=g^{-1}[e_0^te_1e_0^{s-1}]-g^{-1}[e_0^{t-1}e_1e_0^s]\\&\quad+g[e_0^{t-1}e_1]g[e_0^{s-1}e_1].\end{aligned}$$

Corollary 1. *If we assume* (*for simplicity*) *that the weight* $s+t$ *is odd*, *the formula above gives*

$$\zeta_p(t,s)+(-1)^s\sum_{0\leqslant k<s}\binom{t-1+k}{k}\zeta_p(s-k,t+k)=(-1)^s\binom{s+t}{t}\zeta_p(s+t).$$

5.14. A formula for $g[e_0^{t-1}e_1e_0^{s-1}e_1]$ *in odd weights*

(i) Comparing the coefficients of $e_0^{t-1}e_1e_0^{s-1}e_1^2$, for $t\geqslant 2$, in

$$(e_0+e_1)g(\infty)=g(\infty)(e_0+g^{-1}e_1g)$$

we obtain

$$g(\infty)[e_0^{t-2}e_1e_0^{s-1}e_1^2]=g(\infty)[e_0^{t-1}e_1e_0^{s-1}e_1]+g^{-1}[e_0^{t-1}e_1e_0^{s-1}e_1].$$

(ii) Similarly, comparing the coefficients of $e_0^qe_1e_0^{s-1}e_1e_0^{r-1}e_1e_0$ we obtain, for $q\geqslant 1$,

$$g(\infty)[e_0^{q-1}e_1e_0^{s-1}e_1e_0^{r-1}e_1e_0]=g(\infty)[e_0^qe_1e_0^{s-1}e_1e_0^{r-1}e_1]+g^{-1}[e_0^qe_1e_0^{s-1}]g[e_0^{r-1}e_1e_0].$$

Assuming for simplicity that the weight is odd, since then there are no non-zero products of zeta values of this weight, this gives, for $q\geqslant 1$

$$g(\infty)[e_0^qe_1e_0^{s-1}e_1e_0^{r-1}e_1]=\frac{-1}{q+1}\Big(sg(\infty)[e_0^{q-1}e_1e_0^se_1e_0^{r-1}e_1]+rg(\infty)[e_0^{q-1}e_1e_0^{s-1}e_1e_0^re_1]\Big).$$

Therefore, for $q\geqslant 1$,

$$\begin{aligned}g(\infty)[e_0^qe_1e_0^{s-1}e_1^2]&=\frac{(-1)^q}{(q+1)!}\sum_{0\leqslant l\leqslant q}\binom{q}{l}(q-l)!\frac{(s-1+l)!}{(s-1)!}g(\infty)[e_1e_0^{s-1+l}e_1e_0^{q-l}e_1]\\&=\frac{(-1)^q}{q+1}\sum_{0\leqslant l\leqslant q}\binom{s-1+l}{l}g(\infty)[e_1e_0^{s-1+l}e_1e_0^{q-l}e_1].\end{aligned}$$

Combining (i) and (ii) we obtain that, if the weight is odd,

$$\begin{aligned}g[e_0^{t-1}e_1e_0^{s-1}e_1]=&\frac{(-1)^{t-1}}{t-1}\sum_{0\leqslant l\leqslant t-2}\binom{s-1+l}{l}g(\infty)[e_1e_0^{s-1+l}e_1e_0^{t-2-l}e_1]\\&+g(\infty)[e_0^{t-1}e_1e_0^{s-1}e_1]\end{aligned}$$

for $t\geqslant 2$. In 5.17 below we will try to describe the right-hand side of the last equation.

5.15. Regularized iterated sums

The following type of sums will appear naturally in the following. In order to find formulas for the coefficients of $g(z)$ we will generally begin with a rigid analytic function of the form

$$f(z) = \lim_{N\to\infty} \frac{1}{1-z^{p^N}} \sum_{0<n\leqslant p^N} a_n z^n$$

on $\mathcal{U}$, (as in Proposition 2). In fact, in the following we will know that the functions extend to analytic functions on $\mathbb{P}^1 \backslash D(1, |p|^{1/(p-1)}-)$. Then we will find the antiderivative of these functions after multiplying them with

$$\frac{dz}{z}, \quad \frac{z^{p-1}dz}{1-z^p} \text{ or } \frac{dz}{1-z}$$

under the initial condition that they be zero at 0. Since the residue of this product at infinity is in general non-zero the anti-derivative is only a locally analytic function, which depends on the choice of a branch of the p-adic logarithm. However, if we add

$$C\frac{dz}{1-z}$$

to this product, where C is minus the residue of the product, the sum will be an analytic function on $\mathcal{U}$. And if we know that our original function f extends to an open disc around $\mathcal{U}$, this will be true in the applications, then the anti-derivative of the sum will be an analytic function on $\mathcal{U}$. The addition of the term above will come naturally from the fundamental differential equation (1). The residues are given by

$$\operatorname{res}_\infty f(z)\frac{dz}{z} = -f(\infty) = \lim_{N\to\infty} a_{p^N},$$

$$\operatorname{res}_\infty f(z)\frac{z^{p-1}dz}{1-z^p} = f(\infty) = -\lim_{N\to\infty} a_{p^N},$$

$$\operatorname{res}_\infty f(z)\frac{dz}{1-z} = f(\infty) = -\lim_{N\to\infty} a_{p^N}.$$

The first non-trivial example of this situation arises when we need to regularize the series

$$\lim_{N\to\infty} \frac{1}{1-z^{p^N}} \sum_{\substack{0<n_1<n_2\leqslant p^N \\ p\nmid n_1}} \frac{z^{n_2}}{n_1^s n_2^t}$$

which is obtained from

$$\lim_{N\to\infty} \frac{1}{1-z^{p^N}} \sum_{\substack{0<n\leqslant p^N \\ p\nmid n}} \frac{z^n}{n^s}$$

by multiplication with

$$\frac{dz}{1-z} \quad \text{and} \quad \frac{dz}{z}$$

and integration. We denote the series obtained by successive regularization with the above method by

$$\lim_{N\to\infty} \frac{1}{1-z^{p^N}} \sideset{}{'}\sum_{\substack{0<n_1<n_2\leqslant p^N \\ p\nmid n_1}} \frac{z^{n_2}}{n_1^s n_2^t}$$

and its value at infinity by

$$-\lim_{N\to\infty} \sideset{}{'}\sum_{\substack{0<n<p^N \\ p\nmid n}} \frac{1}{n^s}\frac{1}{p^{tN}}.$$

By the description above we see that we can write

$$\sum_{\substack{0<n<p^N \\ p\nmid n}} \frac{1}{n^s} = \gamma_1(s)p^N + \gamma_2(s)p^{2N} + \cdots + \gamma_t(s)p^{tN} + O(p^{(t+1)N})$$

for some $\gamma_i(s)$ and the regularized sum above is nothing other than $-\gamma_t(s)$. Then 5.16 below combined with 5.11 gives that

$$\begin{aligned} p^{s+t}\gamma_t(s) &= (-1)^{s+t+1} g(\infty)[e_1 e_0^{s-1} e_1 e_0^{t-1}] = (-1)^{s+t+1} g^{-1}[e_0^{s-1} e_1 e_0^t] \\ &= (-1)^s \binom{s+t-1}{t} g[e_0^{s+t-1} e_1] = (-1)^s p^{s+t} \binom{s+t-1}{t} \zeta_p(s+t). \end{aligned}$$

In general, if an iterated series

$$\sum \frac{z^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

is obtained from a series, successively after multiplication by one of the functions above and integration from a rigid analytic function on $\mathcal{U}$, we denote by

$$\lim_{N\to\infty} \frac{1}{1-z^{p^N}} \sideset{}{'}\sum_{n_k\leqslant p^N} \frac{z^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

the function regularized by the method above, and by

$$\sideset{}{'}\sum \frac{z^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

the restriction of the regularized function to $D(0,1^-)$.

In particular,

$$\sum_{\substack{0<n_1<n_2\\ p\nmid n_1}}{}' \frac{z^{n_2}}{n_1^s n_2^t} = \sum_{\substack{0<n_1<n_2\\ p\nmid n_1}} \frac{z^{n_2}}{n_1^s n_2^t} - \sum_{1\leqslant i<t} \sum_{0<n} \gamma_i(s) \frac{z^n}{n^{t-i}}.$$

5.16. Computation of $g(z)[e_0^{t-1}e_1e_0^{s-1}e_1]$

Claim. For $s, t \geqslant 1$ we have

$$g(z)[e_1e_0^{s-1}e_1e_0^{t-1}] = p^{s+t}(-1)^t \sum_{0\leqslant k<t} \left(\binom{s-1+k}{k} \sum_{\substack{0<n_1<n_2\\ p\nmid n_1\\ n_1\equiv n_2 (\mathrm{mod}\, p)}} \frac{z^{n_2}}{n_1^{s+k} n_2^{t-k}} \right) + p^{s+t}(-1)^{s+t} \sum_{\substack{0<n_1<n_2\\ p\nmid n_1}}{}' \frac{z^{n_2}}{n_1^s n_2^t}.$$

Proof. The formula for $t = 1$ is

$$g(z)[e_1e_0^{s-1}e_1] = p^{s+1} \left((-1)^{s-1} \sum_{\substack{0<n_1<n_2\\ p\nmid n_1}} \frac{z^{n_2}}{n_1^s n_2} - \sum_{\substack{0<n_1<n_2\\ p\nmid n_1\\ n_1\equiv n_2 (\mathrm{mod}\, p)}} \frac{z^{n_2}}{n_1^s n_2} \right)$$

which was proven in 5.11. By Eq. (1), $g(z)[e_1e_0^{s-1}e_1e_0^{t-1}]$ is a linear combination of two terms: the term obtained by multiplying $g(z)[e_1e_0^{s-1}e_1e_0^{t-2}]$ with $\frac{dz}{z}$ integrating and regularizing as in 5.11, and the term obtained by multiplying $g(z)[e_0^{s-1}e_1e_0^{t-1}]$ with $\frac{z^{p-1}dz}{z^p-1}$ and integrating. The contribution from the first term is computed using the formula in the claim with t replaced with $t-1$. The contribution coming from the second term is computed using

$$g(z)[e_0^{s-1}e_1e_0^{t-1}] = (-1)^{t-1}\binom{s-1+t-1}{t-1} g(z)[e_0^{s+t-2}e_1],$$

since $g(z)$ is group-like, and

$$g(z)[e_0^{s+t-2}e_1] = p^{s+t-1} \sum_{\substack{0<n\\ p\nmid n}} \frac{z^n}{n^{s+t-1}},$$

proved in 5.7. Induction on t gives the formula in the claim. □

By the analogous induction on t starting with the same formula we obtain

$$g(z)[e_0^{t-1}e_1e_0^{s-1}e_1] = p^{s+t}(-1)^{s-1}\sum_{0\leqslant k<t}\left(\binom{s-1+k}{k}\sideset{}{'}\sum_{\substack{0<n_1<n_2\\ p\nmid n_1}}\frac{z^{n_2}}{n_1^{s+k}n_2^{t-k}}\right) - p^{s+t}\sum_{\substack{0<n_1<n_2\\ p\nmid n_1\\ n_1\equiv n_2 (\mathrm{mod}\, p)}}\frac{z^{n_2}}{n_1^s n_2^t}.$$

By Proposition 2 the first expression gives

$$g(\infty)[e_1e_0^{s-1}e_1e_0^{t-1}] = p^{s+t}(-1)^{s+t+1}\lim_{N\to\infty}\sideset{}{'}\sum_{\substack{0<n<p^N\\ p\nmid n}}\frac{1}{n^s}\frac{1}{p^{tN}}$$

and this together the fact that $g(\infty)$ is group-like, and $g(\infty)[e_0] = 0$ gives

$$g(\infty)[e_0^{t-1}e_1e_0^{s-1}e_1] = p^{s+t}(-1)^s\binom{s+t-2}{s-1}\lim_{N\to\infty}\sideset{}{'}\sum_{\substack{0<n<p^N\\ p\nmid n}}\frac{1}{n^s}\frac{1}{p^{tN}}.$$

5.17. Computation of $g(z)[e_1e_0^{b-1}e_1e_0^{a-1}e_1]$

Using 5.16, the differential equation (1), and relation (2) we obtain

$$\begin{aligned} g(z)[e_1e_0^{b-1}e_1e_0^{a-1}e_1] = p^{a+b+1}\Bigg(& (-1)^a\sum_{0\leqslant k<b}\left(\binom{a-1+k}{k}\sideset{}{'}\sum_{\substack{0<n_1<n_2<n_3\\ p\nmid n_1\\ n_2\equiv n_3 (\mathrm{mod}\, p)}}\frac{z^{n_3}}{n_1^{a+k}n_2^{b-k}n_3}\right) \\ & + (-1)^a\sum_{0\leqslant k<a}\left(\binom{b-1+k}{k}\sum_{\substack{0<n_1<n_2<n_3\\ p\nmid n_1\\ n_1\equiv n_2 (\mathrm{mod}\, p)}}\frac{z^{n_3}}{n_1^{b+k}n_2^{a-k}n_3}\right) \\ & + \sum_{\substack{0<n_1<n_2<n_3\\ p\nmid n_1\\ n_1\equiv n_2\equiv n_3 (\mathrm{mod}\, p)}}\frac{z^{n_3}}{n_1^a n_2^b n_3} + (-1)^{a+b}\sideset{}{'}\sum_{\substack{0<n_1<n_2<n_3\\ p\nmid n_1}}\frac{z^{n_3}}{n_1^b n_2^a n_3} \end{aligned}$$

$$+\sum_{0\leqslant k<b-1}\left((-1)^{a+b-2-k}\binom{a+k}{a-1}g[e_0^{a+k}e_1]\sum_{\substack{0<n_1<n_2\\ p\nmid n_1}}\frac{z^{n_2}}{n_1^{b-1-k}n_2}\right)\Bigg).$$

Therefore

$$g(\infty)[e_1e_0^{b-1}e_1e_0^{a-1}e_1]=-p^{a+b+1}\Bigg(\lim_{N\to\infty}\frac{1}{p^N}\left((-1)^{a+b}\sideset{}{'}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1}}\frac{1}{n_1^bn_2^a}\right.$$
$$+(-1)^a\sum_{0\leqslant k<a}\left(\binom{b-1+k}{k}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ n_1\equiv n_2 (\mathrm{mod}\, p)}}\frac{1}{n_1^{b+k}n_2^{a-k}}\right)$$
$$\left.+(-1)^a\sum_{0\leqslant k<b}\left(\binom{a-1+k}{k}\sideset{}{'}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ p\mid n_2}}\frac{1}{n_1^{a+k}n_2^{b-k}}\right)\right)$$
$$+\sum_{0\leqslant k<b-1}\left((-1)^{a+b-2-k}(b-k-1)\binom{a+k}{a-1}\right.$$
$$\left.\times\zeta_p(a+k+1)\zeta_p(b-k)\right)\Bigg).$$

For odd weights, using 5.18 below, this reduces to

$$g(\infty)[e_1e_0^{b-1}e_1e_0^{a-1}e_1]=-p^{a+b+1}\Bigg(\lim_{N\to\infty}\frac{1}{p^N}\left((-1)^{a+b}\sideset{}{'}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1}}\frac{1}{n_1^bn_2^a}\right.$$
$$\left.+(-1)^a\sum_{0\leqslant k<b}\left(\binom{a-1+k}{k}\sideset{}{'}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ p\mid n_2}}\frac{1}{n_1^{a+k}n_2^{b-k}}\right)\right)$$
$$+(-1)^{a+1}\frac{a+b}{2}\sum_{0\leqslant k<a}\binom{b-1+k}{k}\zeta_p(a+b+1)\Bigg).$$

5.18. *Computation of* $\lim_{N\to\infty}\frac{1}{p^N}\sum^{p^N}_{\substack{p\nmid n_1\\ n_1\equiv n_2 (\mathrm{mod}\ p)}}\frac{1}{n_1^a n_2^b}$ *for even* $a+b$

We have

$$\sum_{\substack{0<n<p^N\\ p|n-q}}\frac{1}{n^a}\sum_{\substack{0<n<p^N\\ p|n-q}}\frac{1}{n^b}=\sum_{\substack{0<n_1<n_2<p^N\\ p|n_1-q\\ p|n_2-q}}\frac{1}{n_1^a n_2^b}+\sum_{\substack{0<n_1<n_2<p^N\\ p|n_1-q\\ p|n_2-q}}\frac{1}{n_1^b n_2^a}+\sum_{\substack{0<n<p^N\\ p|n-q\\ p|n-q}}\frac{1}{n^{a+b}},$$

for $0<q<p$. Multiplying this with $1/p^N$ taking the limit as $N\to\infty$, and summing over $0<q<p$ we obtain

$$\lim_{N\to\infty}\frac{1}{p^N}\left(\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ n_1\equiv n_2(\mathrm{mod}\ p)}}\frac{1}{n_1^a n_2^b}+\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ n_1\equiv n_2(\mathrm{mod}\ p)}}\frac{1}{n_1^b n_2^a}\right)=-\lim_{N\to\infty}\frac{1}{p^N}\sum_{\substack{0<n<p^N\\ p\nmid n}}\frac{1}{n^{a+b}}$$

since the limits

$$\frac{1}{p^N}\sum_{\substack{0<n<p^N\\ p|n-q}}\frac{1}{n^c}$$

exist.

On the other hand, since

$$\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ n_1\equiv n_2(\mathrm{mod}\ p)}}\frac{1}{n_1^b n_2^a}=\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ n_1\equiv n_2(\mathrm{mod}\ p)}}\frac{1}{(p^N-n_1)^a(p^N-n_2)^b}$$

and

$$\frac{1}{n-p^N}=\sum_{0\leqslant k}\frac{p^{kN}}{n^{k+1}}$$

we obtain

$$\lim_{N\to\infty}\frac{1}{p^N}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ n_1\equiv n_2(\mathrm{mod}\ p)}}\frac{1}{n_1^b n_2^a}=\lim_{N\to\infty}\frac{1}{p^N}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ n_1\equiv n_2(\mathrm{mod}\ p)}}\frac{1}{n_1^a n_2^b},$$

for even $a+b$.

Combining this with the above we obtain

$$\lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<n_1<n_2<p^N \\ p\nmid n_1 \\ n_1\equiv n_2 (\mathrm{mod}\ p)}} \frac{1}{n_1^a n_2^b} = -\frac{a+b}{2}\zeta_p(a+b+1).$$

By the same argument we also obtain

$$\lim_{N\to\infty} \frac{1}{p^N} \sum_{0\leqslant k\leqslant s-2} \sum_{\substack{0<n_1<n_2<p^N \\ p\nmid n_1 \\ p\nmid n_2}} \frac{1}{n_1^{k+1} n_2^{s-1-k}} = \frac{-s(s-1)}{2}\zeta_p(s+1),$$

for $s\geqslant 2$.

Example. An expression for $g[e_0^{s-1}e_1^2]$, for even s. From 5.14 we obtain, for even s,

$$g[e_0^{s-1}e_1^2] = \frac{(-1)^{s-1}}{s-1} \sum_{0\leqslant k\leqslant s-2} g(\infty)[e_1 e_0^k e_1 e_0^{s-2-k} e_1] + g(\infty)[e_0^{s-1}e_1^2].$$

After using 5.17 and simplifying we obtain

$$g[e_0^{s-1}e_1^2] = \frac{p^{s+1}}{1-s} \lim_{N\to\infty} \frac{1}{p^N} \left(- \sum_{0\leqslant k\leqslant s-2} \sideset{}{'}\sum_{\substack{0<n_1<n_2<p^N \\ p\nmid n_1}} \frac{1}{n_1^{k+1} n_2^{s-1-k}}. \right.$$
$$\left. + \sideset{}{'}\sum_{\substack{0<n_1<n_2<p^N \\ p\nmid n_1 \\ p\mid n_2}} \frac{1}{n_1 n_2^{s-1}} + \sum_{\substack{0<n_1<n_2<p^N \\ p\nmid n_1 \\ n_1\equiv n_2 (\mathrm{mod}\ p)}} \frac{1}{n_1 n_2^{s-1}} \right) - g[e_0^s e_1].$$

Using 5.19 and 5.20 this gives

$$\begin{aligned} g[e_o^{s-1}e_1^2] &= p^{s+1}\zeta_p(s+1)\left(\frac{1}{1-s}\left(-\sum_{0\leqslant k\leqslant s-2}\left(\frac{1}{2}\binom{s}{k}(-1)^k\right) + \frac{s(s-1)}{2} + \frac{1}{2} - \frac{s}{2}\right) - 1\right) \\ &= -\frac{s}{2}p^{s+1}\zeta_p(s+1). \end{aligned}$$

Proposition 3. *We have*

$$\zeta_p(s,1) = -\frac{s}{2}\zeta_p(s+1), \quad \zeta_p(1,s) = \left(\frac{s}{2}+1\right)\cdot\zeta_p(s+1),$$

for even s, and the shuffle formula

$$\zeta_p(t,2) + \zeta_p(2,t) = \zeta_p(t+2),$$

for odd t.

Proof. The first equality was shown in the argument before the proposition. The second one follows from the first one by using Corollary 1 to obtain

$$\zeta_p(s,1) - \zeta_p(1,s) = -(s+1)\zeta_p(s+1).$$

The shuffle formula follows from the second formula by using Corollary 1 to obtain

$$\zeta_p(t,2) + \zeta_p(2,t) + t\cdot\zeta_p(1,t+1) = \frac{(t+2)(t+1)}{2}\cdot\zeta_p(t+2). \qquad \square$$

5.19. Partial sums

The following will be used to explicitly compute the effect of regularization in the regularized iterated sums. Let $\Gamma_p(z)$ denote Morita's Gamma function, which is the unique continuous extension to $\mathbb{Z}_p$ of the function from $\mathbb{N}\backslash\{0,1\}$ to $\mathbb{Z}_p^*$ that sends n to $(-1)^n \prod_{\substack{1\leqslant j<n \\ p\nmid j}} j$. Let log denote the Iwasawa extension of the logarithm then $\log\Gamma_p(z)$ extends to an analytic function on $D(0,1^-)$ and

$$\log\Gamma_p(z) = \gamma_p z - \sum_{1\leqslant s}\frac{\zeta_p(s+1)}{s+1}z^{s+1}$$

holds [8], where

$$\gamma_p = \lim_{N\to\infty}\frac{1}{p^N}\sum_{\substack{0<n<p^N \\ p\nmid n}}\log n$$

is the p-adic Euler constant.

Comment. Note that in the p-adic case $-\zeta_p(s+1) = (-1)^{s+1}\zeta_p(s+1)$.

(i) *Harmonic*: We would like to see that the following equality holds:

$$\sum_{\substack{0<j<n \\ p\nmid j}} \frac{1}{j} = -\sum_{1\leqslant s} \zeta_p(s+1)n^s = \log {\Gamma_p}'(n) - \gamma_p$$

for $n \in p\mathbb{Z}$.

Note that

$$\begin{aligned}
\sum_{1\leqslant s} \zeta_p(s+1)n^s &= \sum_{1\leqslant s} \frac{1}{s}\left(\lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p\nmid j}} \frac{1}{j^s}\right) n^s \\
&= \lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p\nmid j}} \sum_{1\leqslant s} \frac{1}{s}\left(\frac{n}{j}\right)^s \\
&= \lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p\nmid j}} -\log\left(1-\left(\frac{n}{j}\right)\right) \\
&= -\lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p\nmid j}} \log(j-n) - \log j \\
&= -\lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<j<n \\ p\nmid j}} \log(-j) - \log(p^N - j) \\
&= \lim_{N\to\infty} \frac{1}{p^N} \sum_{\substack{0<j<n \\ p\nmid j}} \log\left(1-\frac{p^N}{j}\right) \\
&= -\sum_{\substack{0<j<n \\ p\nmid j}} \frac{1}{j},
\end{aligned}$$

for $n \in p\,\mathbb{Z}$. The change of the order of limits is valid since if we let

$$a_N(s) = \frac{1}{s}\frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p\nmid j}} \frac{1}{j^s}$$

then $|a_N(s)|(1-\varepsilon)^s \to 0$, for $0<\varepsilon<1$, uniformly in N.

(ii) *Generalization*: Similar computations as above give, for $n \in p\,\mathbb{Z}$,

$$(-1)^k(k-1)! \sum_{\substack{0<j<n \\ p\nmid j}} \frac{1}{j^k} = \sum_{1\leqslant s} (s+1)_{k-1}\zeta_p(s+k)\, n^s = -\log \Gamma_p^{(k)}(n) + \log \Gamma_p^{(k)}(0),$$

c.f. the regularization in Section 5.15. Here for $k \geqslant 1$, $(x)_k := x \cdot (x+1) \cdots (x+(k-1))$. To see the above expression note that

$$\sum_{1 \leqslant s} (s+1)_{k-1} \zeta_p(s+k) n^s = \sum_{1 \leqslant s} (s+1)_{k-2} \left(\lim_{N \to \infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p \nmid j}} \frac{1}{j^{s+k-1}} \right) n^s$$

$$= \lim_{N \to \infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p \nmid j}} \frac{1}{j^{k-1}} \sum_{1 \leqslant s} (s+1)_{k-2} \left(\frac{n}{j} \right)^s$$

$$= (k-2)! \lim_{N \to \infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p \nmid j}} \frac{1}{j^{k-1}} \sum_{1 \leqslant s} \binom{s+k-2}{k-2} \left(\frac{n}{j} \right)^s$$

$$= (k-2)! \lim_{N \to \infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p \nmid j}} \frac{1}{j^{k-1}} \left(\left(1 - \left(\frac{n}{j} \right) \right)^{-(k-1)} - 1 \right)$$

$$= (k-2)! \lim_{N \to \infty} \frac{1}{p^N} \sum_{\substack{0<j<p^N \\ p \nmid j}} \frac{1}{(j-n)^{k-1}} - \frac{1}{j^{k-1}}$$

$$= (-1)^{k-1} (k-2)! \lim_{N \to \infty} \frac{1}{p^N} \sum_{\substack{0<j<n \\ p \nmid j}} \left(\frac{1}{j^{k-1}} - \frac{1}{(j-p^N)^{k-1}} \right)$$

$$= (-1)^k (k-1)! \sum_{\substack{0<j<n \\ p \nmid j}} \frac{1}{j^k}.$$

5.20. Computation of $\lim_{N \to \infty} \frac{1}{p^N} \sum'_{\substack{0<n_1<n_2<p^N \\ p \nmid n_1 \\ p | n_2}} \frac{1}{n_1^a n_2^b}$

Let $f(n) := \sum_{\substack{0<j<n \\ p \nmid j}} \frac{1}{j^a}$. Then

$$\lim_{N \to \infty} \frac{1}{p^N} \sum'_{\substack{0<n_1<n_2<p^N \\ p \nmid n_1 \\ p | n_2}} \frac{1}{n_1^a n_2^b} = \lim_{N \to \infty} \frac{1}{p^N} \sum'_{\substack{0<n<p^N \\ p | n}} \frac{f(n)}{n^b}$$

Recall that

$$- \lim_{N \to \infty} \frac{1}{p^N} \sum'_{\substack{0<n_1<n_2<p^N \\ p \nmid n_1 \\ p | n_2}} \frac{1}{n_1^a n_2^b}$$

is defined to be the value at infinity of the function obtained by successively multiplying

$$\lim_{N\to\infty} \frac{1}{1-z^{p^N}} \sum_{\substack{0<n_1<n_2\leqslant p^N \\ p\nmid n_1}} \frac{z^{n_2}}{n_1^a n_2} = \lim_{N\to\infty} \frac{1}{1-z^{p^N}} \sum_{0<n\leqslant p^N} \frac{f(n)z^n}{n}$$

with $\frac{dz}{z}$, integrating and regularizing $(b-1)$-times and finally multiplying with $\frac{z^{p-1}\,dz}{1-z^p}$, integrating and regularizing.

The formula in 5.19 gives

$$f(n) = (-1)^a \sum_{1\leqslant s} \binom{s+a-1}{a-1} \zeta_p(s+a) n^s.$$

When we successively regularize as in 5.15 the residues that are used in the regularizations are the Taylor coefficients of $f(n)$ in the expansion above. That is, if we let $a_s := (-1)^a \binom{s+a-1}{a-1}\zeta_p(s+a)$ be the Taylor coefficients above then the first residue is $\lim_{N\to\infty} f(p^N)/p^N = a_1$. In the second step a_1 needs to be subtracted, and the residue in the second regularization is $\lim_{N\to\infty}(\frac{f(p^N)}{p^{2N}} - \frac{a_1}{p^N}) = a_2$. We do this $(b-1)$-times until we reach the second to last step where the residue is

$$\lim_{N\to\infty} \frac{1}{p^{(b-1)N}} \left(f(p^N) - \sum_{1\leqslant i\leqslant b-2} a_i p^{iN} \right) = a_{b-1}.$$

In the final step, where we need to regularize after multiplication with $\frac{z^{p-1}dz}{1-z^p}$, the residue will be $-\lim_{N\to\infty} \frac{1}{p^{bN}}(f(p^N) - \sum_{1\leqslant i\leqslant b-1} a_i p^{iN}) = -a_b$. Note that the residue of $g(z)\frac{z^{p-1}dz}{1-z^p}$ at infinity is $g(\infty)$. Therefore

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{p^N} {\sum_{\substack{0<n<p^N \\ p\mid n}}}' \frac{f(n)}{n^b} &= \lim_{N\to\infty} \frac{1}{p^N} \left(\sum_{\substack{0<n\leqslant p^N \\ p\mid n}} a_b + \sum_{\substack{0<n<p^N \\ p\mid n}} \sum_{b+1\leqslant s} a_s n^{s-b} \right) \\
&= (-1)^a \frac{1}{p} \binom{a+b-1}{a-1} \zeta_p(a+b) \\
&\quad + (-1)^{a+1} \frac{1}{2} \binom{a+b}{a-1} \zeta_p(a+b+1) \\
&\quad + (-1)^a \sum_{2\leqslant s} \binom{s+a+b-1}{a-1} \zeta_p(s+a+b) B_s p^{s-1}
\end{aligned}$$

which is equal to $(-1)^{a+1}\frac{1}{2}\binom{a+b}{a-1}\zeta_p(a+b+1)$ if $a+b$ is even, since the zeta values of even weight and the Bernoulli numbers B_s, with $3\leqslant s$ and odd, are equal to zero.

5.21. Value of $\zeta_p(t,s)$

Assuming, for simplicity, that $s+t$ is odd and $2\leqslant t$, and combining Section 5.13,

$$g[e_0^{t-1}e_1e_0^{s-1}e_1]=\frac{(-1)^{t-1}}{t-1}\sum_{0\leqslant l\leqslant t-2}\binom{s-1+l}{l}g(\infty)[e_1e_0^{s-1+l}e_1e_0^{t-2-l}e_1] + g(\infty)[e_0^{t-1}e_1e_0^{s-1}e_1],$$

Section 5.12,

$$\begin{aligned}g(\infty)[e_0^{t-1}e_1e_0^{s-1}e_1] &= (-1)^{t+1}\binom{s+t-2}{s-1}g(\infty)[e_1e_0^{s-1}e_1e_0^{t-1}]\\ &= p^{s+t}(-1)^{s+t+1}\binom{s+t-2}{s-1}\binom{s+t-1}{s-1}\zeta_p(s+t)\end{aligned}$$

Sections 5.17 and 5.20,

$$\begin{aligned}g(\infty)[e_1e_0^{s-1+l}e_1e_0^{t-2-l}e_1] = p^{s+t}\Bigg(&\lim_{N\to\infty}\frac{1}{p^N}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1\\ p\nmid n_2}}\frac{(-1)^{s+t}}{n_1^{s+l}n_2^{t-l-1}}\\ &+\zeta_p(s+t)\Bigg(\frac{(-1)^{t+l+1}}{2}\binom{s+t-1}{s+l-1}\\ &+\sum_{0\leqslant k<s+l}\frac{(-1)^k}{2}\binom{s+t-1}{t-2-l+k}\binom{t-l-2+k}{k}\\ &+(-1)^{t+l+1}\frac{s+t-1}{2}\sum_{0\leqslant k<t-l-1}\binom{s+l-1+k}{k}\Bigg)\Bigg)\end{aligned}$$

and putting

$$\begin{aligned}C(t,s) := \frac{(-1)^{t-1}}{t-1}\sum_{0\leqslant l\leqslant t-2}\binom{s-1+l}{l}\Bigg(&(-1)^{t+l+1}\binom{s+t-1}{s+l-1}\Big/2\\ &+\sum_{0\leqslant k<s+l}\frac{(-1)^k\,(s+t-1)!}{2\,(s+l-k+1)!(t-l-2)!k!}\end{aligned}$$

$$+(-1)^{t+l+1}\frac{s+t-1}{2}\sum_{0\leqslant k<t-l-1}\binom{s+l-1+k}{k}\Bigg)\Bigg)$$
$$+\frac{(-1)^{s+t+1}t}{s+t-1}\binom{s+t-1}{s-1}^2$$

we obtain the following.

Theorem 1. *The depth two p-adic multi-zeta values are given by*

$$\zeta_p(t,s)=\frac{(-1)^s}{1-t}\sum_{0\leqslant k\leqslant t-2}\left(\binom{s-1+k}{k}\lim_{N\to\infty}\frac{1}{p^N}\sum_{\substack{0<n_1<n_2<p^N\\ p\nmid n_1n_2}}\frac{1}{n_1^{s+k}n_2^{t-k-1}}\right)$$
$$+C(t,s)\zeta_p(s+t),$$

for odd weights.

Remark. In fact the proof gives a formula in the even weight case too. However it becomes too messy to write down. The only reason for restricting to odd weights was to simplify the formula (using $\zeta_p(2k)=0$ and (5.18)).

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