MOTIVIC COHOMOLOGY OF FAT POINTS IN MILNOR RANGE

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ABSTRACT. We introduce a new algebraic-cycle model for the motivic cohomology theory of truncated polynomials $k[t]/(t^m)$ in one variable. This approach uses ideas from the deformation theory and non-archimedean analysis, and is distinct from the approaches via cycles with modulus. We compute the groups in the Milnor range when the base field is of characteristic 0, and prove that they give the Milnor K-groups of $k[t]/(t^m)$, whose relative part is the sum of the absolute Kähler differential forms.

1. INTRODUCTION

The objective of this paper is to present a new algebraic-cycle model for the motivic cohomology theory of schemes with singularities over a field k and to compute concretely the simplest case to justify the model.

Bloch's higher Chow groups [3] for smooth k-schemes give the correct motivic cohomology groups as shown by Voevodsky [32], but they fail to do so when the schemes have singularities; a good motivic cohomology group is expected to be part of a conjectural Atiyah-Hirzebruch type spectral sequence that converges to higher algebraic K-groups [24] of Quillen. While the K-groups do detect the difference of a scheme X and X_{red} (see e.g. [31]), the higher Chow groups do not distinguish X from X_{red}. The additive higher Chow groups, initiated by Bloch-Esnault [4], were in a sense born as a way to complement this issue for non-reduced schemes.

The approach through additive higher Chow groups, developed further by e.g. [17], [18], [19], [22], [23], [25], has had several successful aspects; they provide understanding of Witt vectors, de Rham-Witt complexes and crystalline cohomology via algebraic cycles based on the moving lemma of [14] that uses an ingenious method of "weighted translations," and they also spawned a variation, "higher Chow groups with modulus," e.g. [2], that rapidly built connections to various subjects of mathematics such as abelianized fundamental groups [16], Somekawa K-groups and reciprocity functors [11], [26], and motives with modulus [13], to name a few. On the other hand, away from the Milnor range, our attempts to understand the conjectural motivic cohomology for singular schemes through the cycles with modulus bumping into increasingly complex technical and philosophical issues. Some of these hindrances encouraged the authors to return to the starting point, and look for and develop some fundamentally new approaches.

The new approach of this paper may, of course, resolve some of the old issues, while it may create a different set of technical problems; for instance, the Milnor range is no longer represented by 0-cycles, but by higher dimensional cycles, so that harder algebro-geometric challenges await us. Nevertheless, we choose to work with this new model, because as far as we see so far, this seems to be leading us further as well as opening new avenues to handle algebraic cycles via some means and ideas such as deformation theory or non-archimedean analysis, that were thought to be somewhat distant from the subject until now.

The particular case studied in depth in this paper is the truncated polynomial ring $k_m := k[t]/(t^m)$, and we show that the Milnor K-groups of k_m can be expressed in terms of our new cycle groups in the Milnor range. The precursors of these theorems for higher

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Chow groups were the theorems of [21] and [27], and for additive higher Chow groups, the theorems of [4] and [25]. Our theorem in this paper can be regarded as a unification of all those precursors in *ibids.* We repeat however that, unlike those precursors our cycles that represent the Milnor range are now 1-cycles, while the 0-cycles do not appear in our groups (see Remark 2.3.6), so our 1-cycles in the Milnor range form yet the simplest part.

We retain the notations of the cubical version of higher Chow groups (see §2.1), but for smooth k-schemes only. For k_m, we redefine $\mathrm{CH}^q(k_m, n)$ in §2.3, different from the higher Chow groups of [3]. In this new theory, we can easily define the relative group $\mathrm{CH}^q((k_m, (t)), n)$ (see Definition 2.3.7). The main theorem, following immediately from Theorem 3.0.1, is:

Theorem 1.0.1. *Let k be a field of characteristic 0 and let $m, n \geq 1$ be integers. Let $k_m := k[t]/(t^m)$. Then the graph homomorphism $K_n^M(k_m) \to \mathrm{CH}^n(k_m, n)$ to the redefined higher Chow group of k_m is an isomorphism. The isomorphism of the relative parts takes the form $(\Omega^{n-1}_{k/\mathbb{Z}})^{\oplus(m-1)} \simeq \mathrm{CH}^n((k_m, (t)), n)$, where the former is the relative Milnor K-group $K_n^M(k_m, (t))$ of k_m.*

We mention a few further new aspects of our theory. One of them is about the ring k_m that has several presentations $k[t]/(t^m)$, $\mathcal{O}_{\mathbb{A}^1_k,0}/(t^m)$ and $k[[t]]/(t^m)$. Usually those work on algebraic cycle theory may work with the "algebraic" situations, e.g. cycles over either $k[t]$ or its localization $\mathcal{O} := \mathcal{O}_{\mathbb{A}^1_k,0}$. In our approach, we work with cycles over $k[[t]] = \widehat{\mathcal{O}}_{\mathbb{A}^1_k,0}$, which is henselian. This gives more admissible cycles not of algebraic origin from $\mathcal{O}$, but generally they have better rationality properties. For instance, $y = \sqrt{1+t}$ is of degree 2 over $\mathcal{O}$, while it is rational over $\widehat{\mathcal{O}}$ because $\sqrt{1+t} = 1 + \frac{t}{2} - \frac{t^2}{8} + \frac{t^3}{16} - \cdots$. The possibility of using Hensel's lemma could also be a technical benefit. On a pair of integral cycles over $k[[t]]$, we put a "mod t^m relation" when their pull-back to $k[[t]]/(t^m)$ are equal (see Definition 2.3.3) and this allows us to use some intuition from the deformation theory to study cycles. The other fundamental structure that we use in our new model is the non-archimedean t-adic metric on $k((t)) = \mathrm{Frac}(k[[t]])$. This non-archimedean analytic view-point helps us in proving the following new type of mod t^m moving result of Theorem 1.0.2, stated in Theorem 4.3.2. As said, we work with cycles over $k[[t]]$, but nevertheless Theorem 1.0.2 below allows us to transport some technical results already known for cycles over $k[t]$ or $\mathcal{O}_{\mathbb{A}^1_k,0}$, to our cycles over $k[[t]]$, through which it plays an essential role in proving Theorem 1.0.1:

Theorem 1.0.2. *Let k be a field. Let $\mathcal{O} := \mathcal{O}_{\mathbb{A}^1_k,0}$ and let $\widehat{\mathcal{O}} := \widehat{\mathcal{O}}_{\mathbb{A}^1_k,0}$. For the completion homomorphism $\xi^n : z^n_{\mathfrak{m}}(\mathcal{O}, n)^c \to z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$, the composition $\xi^n_m : z^n_{\mathfrak{m}}(\mathcal{O}, n)^c \to z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c \to z^n(k_m, n) := z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c / \sim_{t^m}$ is surjective.*

Here, the superscripts "c" denote the subgroups of the corresponding higher Chow groups consisting of the cycles proper over $\mathcal{O}$ and $\widehat{\mathcal{O}}$, respectively. See Definition 2.2.5. Although there could be more cycles over $k[[t]]$, this theorem shows that modulo t^m in the Milnor range, we can still approximate them by those of algebraic origin. This leads us to reduce the argument of the proof of Theorem 1.0.1 to the graph cycles, giving a great technical simplification.

Since this mod t^m moving lemma of Theorem 1.0.2 is a new type of result for the studies algebraic cycles, to give some motivations to the reader, let us quickly sketch the idea. The essential point behind the proof of Theorem 1.0.2 is, simply put, the notion of "coefficient perturbation": when $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ is an integral cycle, we show that it is possible to choose a "nice" system of equations, for which perturbations of the coefficients may produce "nice" deformations of W. The base parameter space is the space of all choices of the coefficients. Some properties such as non-emptiness of the solutions or proper intersection with the faces are open conditions on this base. However, the integrality that is necessary for moving mod t^m-equivalence is not a constructible property, while even if this is considered with the stronger geometric integrality, we need a flat family. For this we devise a trick that is in a sense an explicit version of the flattening stratification theorem. Using so-obtained locally closed nonempty base, we finally prove that we can deform W mod t^m with all the desired properties preserved, such that it comes from the "algebraic world" over $\mathcal{O}$. In the process, we need to resort to the non-archimedean t-adic metric topology.

Some follow-up on-going works will treat the cases of off-Milnor range of the relative Chow group of $(k_m, (t))$, with a cycle-theoretic version of the regulator maps on the additive polylogarithmic complex constructed and studied in [28] and [29]. *cf.* [30]. Its comparison with the regulators in [22] and [23] will also be discussed. Other on-going works deal with the case of Artin local k-algebras with embedding dimensions ≥ 1.

We remark that our cycle complex seems to have a natural generalization which gives a construction of what might be a candidate for the motivic cohomology of any k-scheme with singularities, which offers a way to define the relative version for any pair (X, Z) of a scheme and its closed subscheme. The verification that this is well-defined and is the "correct" definition will require nontrivial work, but we hope that the results of this paper could be taken as an important evidence that our approach or its variation has a potential to reach the goal of constructing the ultimate motivic cohomology theory for all k-schemes.

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Conventions. For a scheme $X \to \mathrm{Spec}\,(R)$ over a discrete valuation ring R, we always denote the special fiber by X_s, and the generic fiber by X_η.

2. Recollections, new definitions and basic results

In this section, we recall some of the basic definitions and results on higher Chow complexes needed in this paper. A new one over the truncated polynomial rings $k[t]/(t^m)$ will be defined in §2.3, which is the main complex we work with.

2.1. **Recollections of higher Chow cycles.** Let k be a field. We recall the cubical version of Bloch's higher Chow complexes (*cf.* [3]). Let $\mathbb{P}^1_k := \mathrm{Proj}(k[u_0, u_1])$, and let $\overline{\square}_k := \mathbb{P}^1_k$, with $y := u_1/u_0$ as the coordinate. Let $\square_k := \overline{\square}_k \setminus \{1\}$. We let $\square^0_k = \overline{\square}^0_k := \mathrm{Spec}\,(k)$, and for $n \geq 1$, we let $\square^n_k$ (resp. $\overline{\square}^n_k$) be the n-fold product of $\square_k$ (resp. $\overline{\square}_k$) with itself over k. A *face* F of $\square^n_k$ (resp. $\overline{\square}^n_k$) is defined to be the closed subscheme given by a finite set of equations of the form $\{y_{i_1} = \epsilon_1, \cdots, y_{i_u} = \epsilon_u\}$, for an increasing sequence of indices $1 \leq i_1 < \cdots < i_u \leq n$, and $\epsilon_j \in \{0, \infty\}$. We allow the case of the empty set of equations, i.e. having $F = \square^n_k$. A *codimension* 1 *face* is given by a single such equation, and we often write F^ϵ_i to be the one given by $\{y_i = \epsilon\}$.

For a smooth k-scheme X, we let $\square^n_X := X \times_k \square^n_k$, $\overline{\square}^n_X := X \times_k \overline{\square}^n_k$, and define the face F_X of $\square^n_X$ to be the pull-back $X \times_k F$, of a face F of $\square^n_k$. We drop the subscript X in $F^\epsilon_{i,X}$ when no confusion should arise. Let $\underline{z}^q(X, n)$ be the free abelian group on the set of codimension q integral closed subschemes $Z \subseteq \square^n_X$ which intersect each face properly on $\square^n_X$. For each codimension 1 face $F^\epsilon_{i,X}$, with $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$, and an irreducible

$Z \in \underline{z}^q(X,n)$, we let $\partial_i^\epsilon(Z)$ be the cycle associated to the scheme-theoretic intersection $Z \cap F_{i,X}^\epsilon$. By definition, $\partial_i^\epsilon(Z) \in \underline{z}^q(X,n-1)$. This forms a cubical abelian group $(\underline{n} \mapsto \underline{z}^q(X,n))$, where $\underline{n} = \{0,1,\cdots,n\}$, in the sense of [20, §1.1]. Let $\partial := \sum_{i=1}^n (-1)^i(\partial_i^\infty - \partial_i^0)$ on $\underline{z}^q(X,n)$. One checks immediately from the formalism of cubical abelian groups that $\partial \circ \partial = 0$ and hence one obtains the associated nondegenerate complex $z^q(X,\bullet) := \underline{z}^q(X,\bullet)/\underline{z}^q(X,\bullet)_{\rm degn}$, where $\underline{z}^q(X,n)_{\rm degn}$ is the subgroup of degenerate cycles, i.e. sums of those obtained by pulling back via one of the standard projections $\square_X^n \to \square_X^{n-1}$, for $0 \le i \le n$, which omits one of the coordinates on $\square_X^n$. This complex $(z^q(X,\bullet),\partial)$ is called the (cubical) *higher Chow complex* of X and its homology groups ${\rm CH}^q(X,n) := {\rm H}_n(z^q(X,\bullet))$ are the higher Chow groups of X. It is a theorem of Voevodsky [32] that the higher Chow groups form a universal bigraded ordinary cohomology theory ${\rm H}_{\mathcal{M}}^{2q-n}(X,\mathbb{Z}(q)) := {\rm CH}^q(X,n)$ on the category of smooth k-varieties X.

2.2. **Some subgroups.** If we are given an integral closed subscheme $W \subseteq X$, we have a subcomplex $z_W^q(X,\bullet) \subseteq z^q(X,\bullet)$ defined as follows: first, let $\underline{z}_W^q(X,n) \subseteq \underline{z}^q(X,n)$ be the subgroup generated by integral closed subschemes $Z \subseteq \square_X^n$ which intersect each $W \times F$ properly on $\square_X^n$, as well as each $F_X = X \times F$, for every face F of $\square_k^n$. More precisely, we require the codimension of $Z \cap (W \times F)$ in $W \times F$ to be $\ge q$. Modding out by degenerate cycles, we then have a subcomplex $z_W^q(X,\bullet) \subseteq z^q(X,\bullet)$. In this paper, we are interested only in the cases when (X,W) is $({\rm Spec}\,(\widehat{\mathcal{O}}),\widehat{\mathfrak{m}})$ or possibly $({\rm Spec}\,(\mathcal{O}),\mathfrak{m})$ where:

Definition 2.2.1. Let $\mathcal{O} := \mathcal{O}_{\mathbb{A}_k^1,0}$ with the maximal ideal $\mathfrak{m}$, and let $\widehat{\mathcal{O}} := \widehat{\mathcal{O}}_{\mathbb{A}_k^1,0}$, which is the completion of $\mathcal{O}$ at $\mathfrak{m}$, with $\widehat{\mathfrak{m}}$ its unique maximal ideal. Here $\widehat{\mathcal{O}} \simeq k[[t]]$. For $m \ge 1$, let $k_m := \widehat{\mathcal{O}}/(t^m) = k[[t]]/(t^m)$. We use these notations throughout this paper.

Remark 2.2.2. We have $z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}},n) = 0$. To see this, suppose that $z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}},n) \neq 0$ and let $\mathfrak{p} \in z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}},n)$ be a closed point on $\square_{\widehat{\mathcal{O}}}^n$. Here, $[k(\mathfrak{p}):k] < \infty$ so that the image of the composition $\mathfrak{p} \hookrightarrow \square_{\widehat{\mathcal{O}}}^n \to {\rm Spec}\,(\widehat{\mathcal{O}})$ must be the unique closed point $\widehat{\mathfrak{m}}$ of ${\rm Spec}\,(\widehat{\mathcal{O}})$, i.e. $\mathfrak{p}$ lies in the special fiber of the morphism $\square_{\widehat{\mathcal{O}}}^n \to {\rm Spec}\,(\widehat{\mathcal{O}})$, contradicting the assumption that $\mathfrak{p} \in z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}},n)$. Hence $z_{\widehat{\mathfrak{m}}}^{n+1}(\widehat{\mathcal{O}},n) = 0$.

Remark 2.2.3. We have $z_{\mathfrak{m}}^{n+1}(\mathcal{O},n) = 0$ as well. The proof is identical to that of Remark 2.2.2 by simply replacing $(\widehat{\mathcal{O}},\widehat{\mathfrak{m}})$ by $(\mathcal{O},\mathfrak{m})$. We use Remarks 2.2.2 and 2.2.3 later in Corollary 4.3.3.

Corollary 2.2.4. *For $n \ge 1$, let $Z \in z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}},n)$ be an integral cycle. Then for any proper face $F \subset \square_{\widehat{\mathcal{O}}}^n$, we have $Z \cap F = \emptyset$. In particular, we have $\partial_i^\epsilon(Z) = 0$ for any $1 \le i \le n$ and $i \in \{0,\infty\}$, thus $\partial(Z) = 0$. A similar result holds for $Z \in z_{\mathfrak{m}}^n(\mathcal{O},n)$.*

Proof. If $Z \cap F \neq \emptyset$ for a codimension 1 face $F \subset \square_{\widehat{\mathcal{O}}}^n$, given by $\{y_i = \epsilon\}$, then $\partial_i^\epsilon(W) \neq 0$ in $z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}},n-1)$. But this contradicts Remark 2.2.2 that says $z_{\widehat{\mathfrak{m}}}^n(\widehat{\mathcal{O}},n-1) = 0$. So Z does not intersect any codimension 1 face at all. However any proper face is contained in a codimension 1 face, so the corollary follows. □

Definition 2.2.5. Let $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}},n)^c \subset z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}},n)$ be the subgroup generated by the integral cycles $Z \in z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}},n)$ that are proper over ${\rm Spec}\,(\widehat{\mathcal{O}})$. Similarly, define $z_{\mathfrak{m}}^q(\mathcal{O},n)^c \subset z_{\mathfrak{m}}^q(\mathcal{O},n)$.

There are some technical advantages in working with cycles in $z_{\widehat{\mathfrak{m}}}^q(\widehat{\mathcal{O}},n)^c$. Firstly we have the following, inspired by [18, Lemma 5.1.4]:

Lemma 2.2.6. *Let X be a k-scheme. Let $W \subset \square_X^n$ be a nonempty closed subscheme and let $\overline{W} \subset \overline{\square}_X^n$ be its Zariski closure. Then $W \to X$ is proper if and only if $W = \overline{W}$.*

Proof. ($\Rightarrow$) The structure morphism $W \to X$ factors into the composite $W \hookrightarrow \overline{\square}^n_X \to X$. Since the composite is assumed to be proper and the second morphism is separated by [10, Theorem II-4.9, p.103], the inclusion $W \hookrightarrow \overline{\square}^n_X$ is proper by [10, Corollary II-4.8(e), p.102]. In particular W is closed in $\overline{\square}^n_X$ and $W = \overline{W}$.

($\Leftarrow$) If $W = \overline{W}$, then $W \hookrightarrow \overline{\square}^n_X$ is closed, in particular it is a proper morphism by [10, Corollary II-4.9(a), p.102]. Hence composed with the proper projective morphism $\overline{\square}^n_X \to X$, the composite $W \to X$ is proper by [10, Corollary II-4.8(b), p.102]. $\square$

Lemma 2.2.7. *Let $W \in z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral cycle. Then we have:*

1. *W is closed in $\overline{\square}^n_{\widehat{\mathcal{O}}}$, so that its Zariski closure $\overline{W}$ is W itself.*
2. *The structure morphism $W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is surjective.*

A similar assertion holds for $W \in z^q_{\mathfrak{m}}(\mathcal{O}, n)^c$.

Proof. (1) is a corollary to Lemma 2.2.6.

If $W \to \mathrm{Spec}(\widehat{\mathcal{O}})$ is not dominant, then $W = W_s$, which violates the assumption that $W \in z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$. Hence $W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is dominant. Now, being proper and dominant, it must be surjective, proving (2). $\square$

Lemma 2.2.8. *The boundary operator ∂ of the complex $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)$ induces a boundary operator on each $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$, thus $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^c$ is a complex of abelian groups.*

Proof. For each integral $Z \in z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$, the composite of proper morphisms $Z \cap F^\epsilon_{i,\widehat{\mathcal{O}}} \hookrightarrow Z \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is proper, where $Z \cap F^\epsilon_{i,\widehat{\mathcal{O}}}$ is the scheme-theoretic intersection. Hence ∂^ϵ_i maps $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ into $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n-1)^c$. That $\partial \circ \partial = 0$ is obvious. $\square$

Unfortunately, although it will not be so apparent in this manuscript, this subgroup $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ of cycles proper over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ may be too restrictive, except for the Milnor range. For instance, some significant cycles in off-Milnor range, which we need later in a follow-up paper, are not proper over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$. Due to this reason, we introduce the following a bit bigger group defined inductively:

Definition 2.2.9. Let $q \leq n$ be integers. Define the subgroup $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \subset z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$ inductively as follows:

1. If $n = q$, we let $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} := z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$.
2. Suppose $n > q$. Inductively, suppose $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, i)^{pc}$ is defined for each $q \leq i \leq n-1$. Then let $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc}$ be the subgroup generated by the integral cycles $Z \in z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$ such that $\partial^\epsilon_i(Z) \in z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n-1)^{pc}$ for each $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$.

(N.B. Here "pc" stands for "partially compact".) Apparently by construction, the face operator ∂^ϵ_i maps $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc}$ into $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n-1)^{pc}$, and we have $\partial \circ \partial = 0$ so that $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^{pc}$ is a complex of abelian groups. By definition, we have

$$z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^c \subseteq z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^{pc} \subseteq z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet).$$

We can similarly define $z^q_{\mathfrak{m}}(\mathcal{O}, n)^{pc}$. Define $\mathrm{CH}^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} := \mathrm{H}_n(z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^{pc})$ and similarly define $\mathrm{CH}^q_{\mathfrak{m}}(\mathcal{O}, n)^{pc}$.

2.3. Cycles modulo t^m.

Definition 2.3.1. Let $m \geq 1$ be an integer. Let X be an integral $\mathrm{Spec}(\widehat{\mathcal{O}})$-scheme and let $Z_1, Z_2 \subset X$ be two *integral* closed subschemes of X. We allow the case when Z_1 or Z_2 is the empty scheme. We say that Z_1 *and* Z_2 *are equivalent mod* t^m, if we have the equality $Z_1 \times_{\mathrm{Spec}\,(\widehat{\mathcal{O}})} \mathrm{Spec}\,(\widehat{\mathcal{O}}/(t^m)) = Z_2 \times_{\mathrm{Spec}\,(\widehat{\mathcal{O}})} \mathrm{Spec}\,(\widehat{\mathcal{O}}/(t^m))$ as closed subschemes of $X \times_{\mathrm{Spec}\,(\widehat{\mathcal{O}})} \mathrm{Spec}\,(\widehat{\mathcal{O}}/(t^m))$.

We can extend this notion to algebraic cycles on X by extending $\mathbb{Z}$-linearly.

Remark 2.3.2. It might be tempting to define the mod t^m equivalence on each pair of closed subschemes Z_1 and Z_2 as long as we have $Z_1 \times_{\mathrm{Spec}\,(\widehat{\mathcal{O}})} \mathrm{Spec}\,(\widehat{\mathcal{O}}/(t^m)) = Z_2 \times_{\mathrm{Spec}\,(\widehat{\mathcal{O}})} \mathrm{Spec}\,(\widehat{\mathcal{O}}/(t^m))$. But this finer relation may result in some technically very undesirable effects in dealing with algebraic cycles. One of such problems is that this "tempting" definition often identifies an irreducible closed subscheme with possibly reducible ones, and this makes an analysis of the behaviors of algebraic cycles very intractable. We thus insist to put this mod t^m equivalence only on pairs of integral closed subschemes with the above equality.

Definition 2.3.3. For two integral schemes $Z_1, Z_2 \in z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$, we say that Z_1 and Z_2 are *naively mod t^m-equivalent*, if their Zariski closures $\overline{Z}_1$, $\overline{Z}_2$ in $\overline{\square}^n_{\widehat{\mathcal{O}}}$ are mod t^m-equivalent in the sense of Definition 2.3.1. Extend this notion $\mathbb{Z}$-linearly to cycles. We say that Z_1 and Z_2 are *mod t^m-equivalent as higher Chow cycles* and write $Z_1 \sim_{t^m} Z_2$, if the pair (Z_1, Z_2) and all pairs of faces $(Z_1 \cap F, Z_2 \cap F)$ for each face $F \subset \square^n_{\widehat{\mathcal{O}}}$ are all naively mod t^m-equivalent.

For simplicity, when Z_1, Z_2 are mod t^m-equivalent as higher Chow cycles, we will simply say they are mod t^m-equivalent.

This definition of mod t^m-equivalence immediately satisfies:

Lemma 2.3.4. *The boundary operator ∂ of the complex $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)$ induces the boundary operator, also denoted by ∂, on the mod t^m groups $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)/\sim_{t^m}$, turning them into a complex. Similarly, we obtain the mod t^m complexes $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^{pc}/\sim_{t^m}$ and $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^{c}/\sim_{t^m}$.*

To avoid a technical difficulty (see Remark 5.5.1), we will use $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^{pc}/\sim_{t^m}$:

Definition 2.3.5. Let $m \geq 1$, $q, n \geq 0$ be integers. Define

$$z^q(k_m, n) := z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc}/\sim_{t^m}, \tag{2.3.1}$$

where $\sim_{t^m}$ is the mod t^m-equivalence in Definition 2.3.3. By Lemma 2.3.4, this $z^q(k_m, \bullet)$ is a complex of abelian groups. We denote its homology by $\mathrm{CH}^q(k_m, n)$.

Remark 2.3.6. The group $z^{n+1}(k_m, n)$ is 0 because $z^{n+1}_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n) = 0$ by Remark 2.2.2. Hence the group $z^n(k_m, n)$ is the simplest nontrivial group in our cycle theory.

Note that we have k-algebra homomorphisms $k \to \widehat{\mathcal{O}} \to k$, where the first is the natural k-algebra map and the second is reduction modulo (t). The composition is the identity of k. The first map induces the structure morphism $p : \mathrm{Spec}\,(\widehat{\mathcal{O}}) \to \mathrm{Spec}\,(k)$, which induces the pull-back map $p^* : z^q(k, \bullet) \to z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^{pc}$ and the second map induces the closed immersion $s : \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$, which induces the intersection-restriction to the special fiber $s^* : z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, \bullet)^{pc} \to z^q(k, \bullet)$. By definition, we have $s^* \circ p^* = \mathrm{Id}$, so that we can regard $z^q(k, \bullet)$ as a subcomplex of $z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc}$ via p^*. Going modulo t^m, which does not do anything on $z^q(k, \bullet)$, we obtain $z^q(k, \bullet) \overset{p^*}{\to} z^q(k_m, \bullet) \overset{s^*}{\to} z^q(k, \bullet)$, where we still have $s^* \circ p^* = \mathrm{Id}$. This gives a splitting

$$z^q(k_m, \bullet) = z^q(k, \bullet) \oplus \ker s^*, \tag{2.3.2}$$

of the complex $z^q(k_m, \bullet)$.

Definition 2.3.7. We define the relative mod t^m cycle complex to be $z^q((k_m, (t)), \bullet) := \ker s^*$, and its homology is denoted by $\mathrm{CH}^q((k_m, (t)), n) := \mathrm{H}_n(z^q((k_m, (t)), \bullet))$.

2.4. **The non-archimedean norm.** We recall some basic facts on the non-archimedean t-adic metric topology associated to the local field $k((t))$, needed in §4. Recall that the field $k((t))$ of Laurent series has a natural discrete valuation $v : k((t)) \to \mathbb{Z}$ given by the order of vanishing function $v = \mathrm{ord}_t$ with $v(0) := \infty$. Its ring of integers $\mathcal{O}_{k((t))} = \widehat{\mathcal{O}} = k[[t]]$ is simply $\{f \in k((t)) \mid v(f) \geq 0\}$. We have a norm $|-|_v : k((t)) \to \mathbb{R}$ given by $|f|_v := e^{-v(f)}$. For any integer $M > 0$, we have the supremum norm on the vector space $k((t))^M$ given by $|(f_1, \cdots, f_n)|_v := \sup_{1 \leq i \leq n} |f_i|_v$. This gives the non-archimedean t-adic metric topology, which is finer than the Zariski topology on $k((t))^M = \mathbb{A}^M(k((t)))$. For any $\alpha_0 \in k((t))^M$, we let $\mathcal{B}_N(\alpha_0)$ be the open ball around α_0 of radius e^{-N}. Here $k[[t]]^M \subset k((t))^M$ is open, while $k[t]^M \subset k[[t]]^M$ is dense.

2.5. **Milnor K-groups.** Let R be a commutative local ring with unity. Recall that the Milnor K-ring $K_*^M(R)$ of R is the graded tensor algebra $T_{\mathbb{Z}}(R^\times)$ of $R^\times$ over $\mathbb{Z}$ modulo the two-sided ideal generated by the elements of the form $\{a \otimes (1-a) \mid a, 1-a \in R^\times\}$. Its degree n part is the n-th Milnor K-group $K_n^M(R)$.

3. Milnor range I: reciprocity

The goal of the paper is to prove Theorem 3.0.1 which computes $\mathrm{CH}^n((k_m, (t)), n)$ in the Milnor range. In the case of additive higher Chow groups over fields of characteristic not equal to 2, similar results were obtained by Bloch-Esnault [4] and Rülling [25] in the Milnor range.

Theorem 3.0.1. *Let k be a field of characteristic 0 and let $m, n \geq 1$ be integers. Then we have* $\mathrm{CH}^n((k_m, (t)), n) \simeq (\Omega^{n-1}_{k/\mathbb{Z}})^{\oplus(m-1)}$.

The proof of Theorem 3.0.1 is broken largely into two parts: the first of the arguments is to define regulator maps on cycles and to prove that they vanish on the boundaries, as done in Proposition 3.0.2 below. The second part, done later in §4 and §5, is to show that the regulator maps respect the mod t^m-equivalence. Here, we would like to emphasize that although we are in the Milnor range, our representatives are 1-cycles, unlike the additive Chow group versions of [4] or [25] where the representatives were 0-cycles. In our discussion, the argument of the first part follows a path similar to one paved in [23]:

Proposition 3.0.2. *Let k be a field of characteristic 0. For each $1 \leq i \leq m-1$, define $\Upsilon_i : z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n) \to \Omega^{n-1}_{k/\mathbb{Z}}$ as follows. Consider the rational form $\gamma_{i,n} := \frac{1}{t^i} \mathrm{dlog} y_1 \wedge \cdots \wedge \mathrm{dlog} y_n \in \Omega^n_{\overline{\square}^n_{\widehat{\mathcal{O}}}/\mathbb{Z}}(*\{t = 0\})(\log F)$. For each integral 1-cycle $Z \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$, let $\nu : \widetilde{Z} \to \overline{Z} \hookrightarrow \overline{\square}^n_{\widehat{\mathcal{O}}}$ be a normalization of the closure $\overline{Z}$ of Z in $\overline{\square}^n_{\widehat{\mathcal{O}}}$. Define $\Upsilon_i(Z) := \sum_{p \in \widetilde{Z}_s} \mathrm{Tr}_{k(p)/k} \mathrm{res}_p \nu^* \gamma_{i,n} \in \Omega^{n-1}_{k/\mathbb{Z}}$, and $\mathbb{Z}$-linearly extend Υ_i to all of $z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$. Then $\Upsilon_i(\partial W) = 0$ for $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n+1)$.*

Proof. It is enough to prove the statement for any integral $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n+1)$. Let $\overline{W} \subset \overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$ be the Zariski closure of W, which is also integral. For each $1 \leq \ell \leq n+1$ and $\epsilon \in \{0, \infty\}$, via the codimension 1 face map $\iota_{\ell,\epsilon} : \overline{\square}^n_{\widehat{\mathcal{O}}} \hookrightarrow \overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$, identify the Zariski closure of $\partial_\ell^\epsilon(W)$ in $\overline{\square}^n_{\widehat{\mathcal{O}}}$ with its image $\overline{\partial_\ell^\epsilon(W)}$ in $\overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$. Consider the divisor $D := \sum_{\ell,\epsilon} \{y_\ell = \epsilon\}$ on $\overline{\square}^{n+1}_{\widehat{\mathcal{O}}}$. We omit the proof of the following claim, which is easily deduced by a standard argument using a finite sequence of point blow-ups and [10, Exercise II-7.12, p.171]:

Claim: *There exists a sequence of blow-ups $\widetilde{\phi} : \widetilde{\overline{\square}^n_{\widehat{\mathcal{O}}}} \to \overline{\square}^n_{\widehat{\mathcal{O}}}$ centered at points, such that for the strict transform $\widetilde{W} \subset \widetilde{\overline{\square}^n_{\widehat{\mathcal{O}}}}$ of $\overline{W}$ and the restriction $\phi := \widetilde{\phi}|_{\widetilde{W}} : \widetilde{W} \to \overline{W} \hookrightarrow \overline{\square}^n_{\widehat{\mathcal{O}}}$, we have the following properties:* (1) *each irreducible component of the strict transform $\phi^!(\overline{\partial_\ell^\epsilon(W)})$ of the 1-cycle $\overline{\partial_\ell^\epsilon(W)}$ is regular. We let $\phi^!(D) := \sum_{\ell,\epsilon} \phi^!(\overline{\partial_\ell^\epsilon(W)})$, the strict* *transform of* $\overline{D \cap W}$; (2) *each closed point* $p \in \widetilde{W}_s = \phi^*\{t = 0\}$ *satisfies exactly one of the following three possibilities:*

(2-i) p *belongs to a unique irreducible component of* $\widetilde{W}_s$, *but does not meet* $\phi^!(D)$.
(2-ii) p *belongs to a unique irreducible component of* $\widetilde{W}_s$, *and belongs to precisely one irreducible component of* $\phi^!(D)$.
(2-iii) p *belongs to exactly two irreducible components of* $\widetilde{W}_s$, *but does not meet* $\phi^!(D)$.

Going back to the proof of the proposition, notice that the irreducible components of $\phi^!(\overline{\partial_\ell^\epsilon(W)})$ are all regular, and are in one-to-one correspondence with the irreducible components of $\overline{\partial_\ell^\epsilon(W)}$ via ϕ. Hence each component of $\phi^!(\overline{\partial_\ell^\epsilon(W)})$ gives a normalization of the Zariski closure in $\overline{\square}^n_{\mathcal{O}}$ of the corresponding component of $\partial_\ell^\epsilon(W)$. Express the special fiber $\widetilde{W}_s$ as the union of (not necessarily regular) irreducible projective curves $C_1, \cdots, C_M$.

We use the theory of Parshin-Lomadze residues associated to pseudo-coefficient fields (see [34, Definitions 4.1.1, 4.1.3]). For each generic point of C_j seen as a point of the scheme $\widetilde{W}$, choose a pseudo-coefficient field σ_j. Consider the Parshin-Lomadze residue $\Xi_{\sigma_j} := \mathrm{res}_{(\widetilde{W}, C_j), \sigma_j} \phi^*(\gamma_{i,n+1})$ along the chain $(\widetilde{W}, C_j)$ for the choice of σ_j. For each $1 \leq j \leq M$, this Ξ_{σ_j} is a rational absolute Kähler n-form on C_j.

Let $p \in \widetilde{W}_s$. By our construction of $\widetilde{W}$ in the above claim, for the point p, exactly one of (2-i), (2-ii), and (2-iii) holds.

If (2-i) holds for p, then let C_j be the unique component of $\widetilde{W}_s$ with $p \in C_j$. Since p does not lie over any face $\overline{\partial_\ell^\epsilon(W)}$ for $1 \leq \ell \leq n+1$, $\epsilon \in \{0, \infty\}$, from the shape of $\gamma_{i,n+1}$, the form $\Xi_{\sigma_j} = \mathrm{res}_{(\widetilde{W}, C_j), \sigma_j}(\phi^* \gamma_{i,n+1})$ is regular at p so that we have $\mathrm{res}_{p \in C_j}(\Xi_{\sigma_j}) = 0$.

If (2-iii) holds for p, then let $C_j, C_{j'}$ with $j \neq j'$ be the two distinct components of $\widetilde{W}_s$ such that $p \in C_j \cap C_{j'}$. Here, again p does not lie over any face $\overline{\partial_\ell^\epsilon(W)}$ for $1 \leq \ell \leq n+1$, $\epsilon \in \{0, \infty\}$, therefore by [34, Theorem 4.2.15-(a)], we have $\mathrm{res}_{p \in C_j}(\Xi_{\sigma_j}) + \mathrm{res}_{p \in C_{j'}}(\Xi_{\sigma_{j'}}) = 0$.

Now suppose (2-ii) holds for p. Thus there exist (i) a unique index $1 \leq j(p) \leq M$ with $p \in C_{j(p)}$, (ii) a unique pair (ℓ_0, ϵ_0) with $1 \leq \ell_0 \leq n+1$, $\epsilon_0 \in \{0, \infty\}$, and a unique irreducible component $G \subset \phi^!(\overline{\partial_{\ell_0}^{\epsilon_0}(W)})$ such that $p \in G$.

From the shape of $\gamma_{i,n+1}$, the form $\phi^* \gamma_{i,n+1}$ on $\widetilde{W}$ has a simple (or logarithmic) pole (see [34, Definition 4.2.10]) along G, so that the residue of $\phi^* \gamma_{i,n+1}$ along the chain $(\widetilde{W}, G)$ is independent of the choice of a pseudo-coefficient field for G by [34, Corollary 4.2.13]. On the other hand, by [34, Theorem 4.2.15-(a)], we have

$$\mathrm{res}_{p \in C_{j(p)}}(\mathrm{res}_{(\widetilde{W}, C_{j(p)}), \sigma_{j(p)}}(\phi^* \gamma_{i,n+1})) = -\mathrm{res}_{p \in G}(\mathrm{res}_{(\widetilde{W}, G)}(\phi^* \gamma_{i,n+1})). \tag{3.0.1}$$

From the shape of $\gamma_{i,n+1} = \frac{1}{t^i} \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{n+1}}{y_{n+1}}$ again, since $G \subset \phi^!(\overline{\partial_{\ell_0}^{\epsilon_0}(W)})$, we have

$$\mathrm{res}_{(\widetilde{W}, G)}(\phi^* \gamma_{i,n+1}) = (-1)^{\ell_0} \cdot \iota(G; \ell_0, \epsilon_0) \cdot \mathrm{sgn}(\epsilon_0) \phi^*(\gamma_{i,n+1}^{\ell_0})|_G, \tag{3.0.2}$$

where $\iota(G; \ell_0, \epsilon_0)$ is the intersection multiplicity of G in $\partial_{\ell_0}^{\epsilon_0}(W)$, $\mathrm{sgn}(0) := 1, \mathrm{sgn}(\infty) := -1$, and $\gamma_{i,n+1}^{\ell_0} := \frac{1}{t^i} \frac{dy_1}{y_1} \wedge \cdots \widehat{\frac{dy_{\ell_0}}{y_{\ell_0}}} \wedge \cdots \wedge \frac{dy_{n+1}}{y_{n+1}}$.

Now, by the definition of Υ_i, we have

$$\begin{aligned}(-1)^{\ell_0} \mathrm{sgn}(\epsilon_0) \Upsilon_i(\partial_{\ell_0}^{\epsilon_0}(W)) &= (-1)^{\ell_0} \mathrm{sgn}(\epsilon_0) \sum_G \iota(G; \ell_0, \epsilon_0) \sum_{p \in G_s} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in G} \phi^*(\gamma_{i,n+1}^{\ell_0})|_G \\ &=^\dagger \sum_G \sum_{p \in G_s} \mathrm{Tr}_{k(p)/k} \mathrm{res}_{p \in G}(\mathrm{res}_{(\widetilde{W}, G)}(\phi^* \gamma_{i,n+1}))\end{aligned}$$

$$=^{\ddagger} -\sum_G \sum_{p\in G_s} \mathrm{Tr}_{k(p)/k}\mathrm{res}_{p\in C_{j(p)}}(\mathrm{res}_{(\widetilde{W},C_{j(p)}),\sigma_{j(p)}}(\phi^*\gamma_{i,n+1}))$$

$$=^1 -\sum_G \sum_{p\in G_s} \mathrm{Tr}_{k(p)/k}\mathrm{res}_{p\in C_{j(p)}}(\Xi_{\sigma_{j(p)}}), \tag{3.0.3}$$

where $\sum_G$ is the sum over all irreducible components of $\phi^!(\overline{\partial_{\ell_0}^{\epsilon_0}(W)})$, † holds by (3.0.2), ‡ holds by (3.0.1), and $=^1$ holds by definition. Note that the set of all points $p \in G_s$ over all irreducible components G of $\phi^!(D)$ is precisely equal to the set $\widetilde{W}_s^{(2-\mathrm{ii})}$ of all points of $\widetilde{W}_s$ of type (2-ii) in the claim. Hence, taking the sum of (3.0.3) over all $1 \le \ell_0 \le n+1$ and $\epsilon_0 \in \{0,\infty\}$, we obtain

$$\Upsilon_i(\partial W) = -\sum_{\ell=1}^{n+1}\sum_{\epsilon\in\{0,\infty\}} (-1)^{\ell}\mathrm{sgn}(\epsilon)\Upsilon_i(\partial_\ell^\epsilon(W)) = \sum_{p\in \widetilde{W}_s^{(2-\mathrm{ii})}} \mathrm{Tr}_{k(p)/k}\mathrm{res}_{p\in C_{j(p)}}(\Xi_{\sigma_{j(p)}}). \tag{3.0.4}$$

On the other hand, for the points of $\widetilde{W}_s$ of type (2-i) and (2-iii), we saw previously that there is no contribution of residues from them. Hence continuing (3.0.4), we have

$$\Upsilon_i(\partial W) = \sum_{p\in \widetilde{W}_s^{(2-\mathrm{ii})}} \mathrm{Tr}_{k(p)/k}\mathrm{res}_{p\in C_{j(p)}}(\Xi_{\sigma_{j(p)}}) = \sum_{j=1}^{M}\sum_{p\in C_j} \mathrm{Tr}_{k(p)/k}\mathrm{res}_{p\in C_j}(\Xi_{\sigma_j}) =^{\dagger} 0,$$

where † holds by the residue theorem (see [34, Theorem 4.2.15-(b)]), i.e. the sum of all residues of a form over a projective curve $\widetilde{W}_s$ is 0. This shows $\Upsilon_i(\partial W) = 0$ as desired. □

The remaining part of the proof of Theorem 3.0.1 is to check that the regulator maps in Proposition 3.0.2 restricted to $z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ respect the mod t^m-equivalence. This requires further discussions, and the rest of the paper deals with it.

4. Some perturbation lemmas and the mod t^m moving lemma

In working with cycles over the complete local ring $\widehat{\mathcal{O}}$, it is often convenient if one can transfer some of the known results for cycles over $\mathcal{O}$ to cycles over $\widehat{\mathcal{O}}$. The completion ring homomorphism $\mathcal{O} \to \widehat{\mathcal{O}}$ induces a natural flat pull-back homomorphism $\xi^n : z^q_{\mathfrak{m}}(\mathcal{O}, n)^? \to z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^?$, for $? = c, pc, \emptyset$, given by $[Z] \mapsto [\widehat{Z} := \mathrm{Spec}\,(\widehat{\mathcal{O}}) \times_{\mathrm{Spec}\,(\mathcal{O})} Z]$, but in general, ξ^n is not surjective. The goal of §4 is to prove the "mod t^m moving lemma" in Theorem 4.3.2, which states that this natural homomorphism induces a surjection modulo t^m in the Milnor range with $? = c, pc$. In this case $z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c = z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc}$.

In this section we suppose k is any field unless specified otherwise. In §4.1, we discuss some preparatory results needed in what follows. In §4.2, we discuss some sort of general position results as in Lemmas 4.2.2, 4.2.4, 4.2.7, and 4.2.9, which are needed in the proof of the mod t^m moving lemma in §4.3. These results might appear to be related to the Artin approximation theorem [1], but they do not follow from it. The results are stated in terms of schemes over $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_{\mathbb{A}^1_k,0}$, but some of them might work for more general integral k-schemes with the method presented here. We leave such generalizations to the reader.

In what follows in §4, to ease the proof, via the automorphism $y \mapsto 1/(1-y)$ of $\mathbb{P}^1$, we identify $(\square, \{\infty, 0\})$ with $(\mathbb{A}^1, \{0,1\})$ so that $\square^n \simeq \mathbb{A}^n$, and the faces of $\square^n$ under this identification are given by a finite set of equations of the form $y_j = \epsilon_j$ with $\epsilon_j \in \{0,1\}$.

4.1. **Some preparatory lemmas.** We are interested in understanding "small changes" of a given integral closed subscheme $W \subseteq \square^n_{\widehat{\mathcal{O}}}$ when we "perturb" the coefficients of a generating set of the ideal of W. So, we introduce:

Definition 4.1.1. For a closed subscheme $W \subseteq \square^n_{\widehat{\mathcal{O}}}$, let $\{f_1, \cdots, f_r\} \subset \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ be a set of generators of the ideal of W. The *coefficient perturbation* of the set $\{f_1, \cdots, f_r\}$ is the set $\{F_1, \cdots, F_r\}$ of polynomials obtained as follows: for each nonzero monomial term of each of f_j for $1 \leq j \leq r$, we consider an indeterminate and a copy of $\mathbb{A}^1_{\widehat{\mathcal{O}}}$, and replace each nonzero coefficient by the corresponding indeterminate. Let M be the total number of them and let $F_1, \cdots, F_r \in \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$ be the so-obtained polynomials from $f_1, \cdots, f_r$, respectively. Let $V \subset \mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \square^n_{\widehat{\mathcal{O}}}$ be the closed subscheme defined by the ideal $(F_1, \cdots, F_r)$. We may also say V is the *coefficient perturbation of* W *with respect to the generators* $\{f_1, \cdots, f_r\}$. For each $\alpha \in \mathbb{A}^M_{\widehat{\mathcal{O}}}$, we let V_α be the fiber over α. If $\alpha_0 \in \widehat{\mathcal{O}}^M$ is the original sequence of coefficients of $\{f_1, \cdots, f_r\}$, we have $V_{\alpha_0} = W$.

Example 4.1.2. For $n = 2$, consider $\{f_1, f_2\} = \{3y_1y_2^2 + y_1 + 2y_2 + 1, -y_1^2y_2 - 5y_1 + 3\}$. Then the corresponding coefficient perturbation is given by $\{F_1, F_2\} = \{x_1y_1y_2^2 + x_2y_1 + x_3y_2 + x_4, x_5y_1^2y_2 + x_6y_1 + x_7\}$.

The coefficient perturbation depends on the choice of a generating set $\{f_1, \cdots, f_r\}$. If we make a "bad" choice, then we might end up having undesirable phenomena:

Example 4.1.3. For $n = 2$, consider $W \subseteq \square^2_{\widehat{\mathcal{O}}}$ defined by $f_1 := y_1 + 1, f_2 := y_2 + 1$. The ideal of W also contains $f_1f_2 = y_1y_2 + y_1 + y_2 + 1$. If we take the coefficient perturbation with respect to just $\{f_1, f_2\}$, then we have $F_1 = x_1y_1 + x_2$, $F_2 = x_3y_3 + x_4$. In particular, if $\alpha = (x_1, x_2, x_3, x_4)$ is in the open subset of $\mathbb{A}^4_{\widehat{\mathcal{O}}}$ given by $x_1 \neq 0$ and $x_3 \neq 0$, we have V_α given by $(y_1, y_2) = (-x_2/x_1, -x_4/x_3)$, so that $V_\alpha \neq \emptyset$.

However, this time take a redundant third generator $f_3 := f_1f_2$. Then with respect to $\{f_1, f_2, f_3\}$, the corresponding coefficient perturbation is given by $F_1 = x_1y_1 + x_2, F_2 = x_3y_2 + x_4, F_3 = x_5y_1y_2 + x_6y_1 + x_7y_2 + x_8$. While one might hope that the indeterminates $x_1, \cdots, x_8$ are algebraically independent, unfortunately for V_α to be nonempty, we need a necessary condition. Suppose for a choice $\alpha = (x_1, \cdots, x_8)$, we have $V_\alpha \neq \emptyset$. Then $F_1 = 0$ and $F_2 = 0$ give $y_1 = -x_2/x_1, y_2 = -x_4/x_3$ so that by plugging them into F_3, we obtain $x_2x_4x_5/(x_1x_3) - x_2x_6/x_1 - x_4x_7/x_3 + x_8 = 0$, i.e. we have an algebraic dependence $x_2x_4x_5 - x_2x_3x_6 - x_1x_4x_7 + x_1x_3x_8 = 0$ for $x_1, \cdots, x_8$. Hence we can expect to have a nonempty fiber V_α only over this proper closed subset of $\mathbb{A}^8_{\widehat{\mathcal{O}}}$, which is not desirable for our purposes.

A central result of our discussion in §4.1 is to show that when $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ is integral of relative dimension 0 in the Milnor range, it is possible to choose a "nice" generating set from which we can prove that a set of properties of W that we began with are preserved for each fiber V_α when α is "close" to α_0. We will make this precise in what follows.

Lemma 4.1.4. *Let* $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ *be a nonempty integral cycle. Then* (1) *the structure morphism* $f : W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ *is surjective, flat, and quasi-finite, and* (2) *the generic fiber* W_η *is the singleton given by the generic point* η_W *of* W*.*

Proof. The surjectivity of f was proven in Lemma 2.2.7. The morphism f is flat by [9, Proposition 14.5.6] (or [10, Proposition III-9.7, p.257]) because $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ is a regular scheme of dimension 1. Since W is of dimension 1, the morphism f is consequently quasi-finite. This proves (1).

Since dim $W = 1$, the integral scheme W is the union of the generic point η_W of W and its closed points. Here, all the closed points map to the unique closed point of $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, while η_W cannot map to the closed point of $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ for otherwise f would not be surjective, contradicting Lemma 2.2.7. Hence η_W is the unique point of the generic fiber W_η. This proves (2). □

Proposition 4.1.5. *Let $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral cycle. Then W is a complete intersection in $\square^n_{\widehat{\mathcal{O}}}$ defined by a subset $\{f_1, \cdots, f_n\} \subseteq \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ of precisely n polynomials of the triangular form*

$$\begin{cases} f_1(y_1), \\ f_2(y_1, y_2), \\ \quad \vdots \\ f_n(y_1, \cdots, y_n), \end{cases} \tag{4.1.1}$$

such that (1) $f_i(y_1, \cdots, y_i)$ has y_i-degree ≥ 1 for each $1 \leq i \leq n$, (2) the highest y_i-degree term of f_i does not involve any variable other than y_i, and (3) the constant term of each f_i is 1.

Proof. Let $K = \mathrm{Frac}(\widehat{\mathcal{O}}) = k((t))$ and take the base change via $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$. Here the generic fiber W_η is the generic point η_W of W by Lemma 4.1.4. Since $\eta_W \in \square^n_K$ is a closed point on the affine space $\square^n_K \simeq \mathbb{A}^n_K$ over the field K, in particular it is a complete intersection so that we can find n polynomials in $K[y_1, \cdots, y_n]$ that defines η_K.

In fact, we can choose such n polynomials with much better properties (this is inspired by [27, Lemma 2]): for each $1 \leq i \leq n$, let $\eta_W^{(i)} \in \square^i_K$ be the image of η_W under the projection $\square^n_K \to \square^i_K$, $(y_1, \cdots, y_n) \mapsto (y_1, \cdots, y_i)$. Let $\eta_W^{(0)} := \eta$, the generic point of $\mathrm{Spec}\,(\widehat{\mathcal{O}})$. The map $\eta_W = \eta_W^{(n)} \to \eta_W^{(0)}$ is finite, and in particular each $\eta_W^{(i)} \to \eta_W^{(j)}$ is finite for each pair $1 \leq j < i \leq n$ of indices. We claim that there exists a sequence of polynomials

$$\begin{cases} f_{1,K}(y_1) \in K[y_1], \\ f_{2,K}(y_1, y_2) \in K[y_1, y_2], \\ \quad \vdots \\ f_{n,K}(y_1, \cdots, y_n) \in K[y_1, \cdots, y_n], \end{cases} \tag{4.1.2}$$

such that $f_{1,K}(y_1)$ is an irreducible polynomial monic in y_1 of degree ≥ 1, for $2 \leq i \leq n$, the image of $f_{i,K}(y_1, \cdots, y_i)$ in $(K[y_1, \cdots, y_{i-1}]/(f_{1,K}, \cdots, f_{i-1,K}))\,[y_i]$ is an irreducible polynomial monic in y_i of degree ≥ 1, and the highest y_i-degree term of $f_{i,K}$ does not involve any variable other than y_i; and the ideal generated by $f_{1,K}, \cdots, f_{n,K}$ is the ideal of the point η_W.

We prove the claim by induction. Since $\eta_W^{(1)} \to \eta_W^{(0)}$ is finite, there exists an irreducible monic polynomial $f_{1,K}(y_1) \in K[y_1]$ that defines $\eta_W^{(1)} \in \square^1_{\widehat{\mathcal{O}}}$. When $n = 1$, this answers the claim. If $n \geq 2$, suppose we have constructed $f_{1,K}, \cdots, f_{i,K}$ for some $1 \leq i < n$ that satisfy the above properties. Since $K[y_1, \cdots, y_i]/(f_{1,K}, \cdots, f_{i,K})$ is a finite extension of K, and since $\eta_W^{(i+1)} \to \eta_W^{(i)}$ is finite, there exists an irreducible polynomial in $(K[y_1, \cdots, y_i]/(f_{1,K}, \cdots, f_{i,K}))\,[y_{i+1}]$ monic in y_{i+1}, that defines $\eta_W^{(i+1)}$. Choose any lifting of this polynomial in $K[y_1, \cdots, y_{i+1}]$ such that the coefficient of the highest y_{i+1} degree term does not involve any variable other than y_{i+1}, and call it $f_{i+1,K}(y_1, \cdots, y_{i+1})$. Hence the claim follows by induction.

For the generators in (4.1.2) for η_W, the nonzero coefficients of the terms of $f_{i,K}$ are all of the form $\frac{a}{b}$ for some nonzero $a, b \in \widehat{\mathcal{O}}$. Since the only irreducible element in the ring $\widehat{\mathcal{O}}$ up to multiplications of units is t, for each $1 \leq i \leq n$ there exists a unique integer $n_i \geq 0$ such that $f_i := t^{n_i} f_{i,K}$ is in $\widehat{\mathcal{O}}[y_1, \cdots, y_n]$ and t does not divide f_i in $\widehat{\mathcal{O}}[y_1, \cdots, y_n]$. This new collection $\{f_1, \cdots, f_n\} \subset \widehat{\mathcal{O}}[y_1, \cdots, y_n]$, seen as a subset of $K[y_1, \cdots, y_n]$ also generates the ideal of the closed point η_W in $\square^n_K$. In other words, η_W has a generating set consists of n polynomials in $\widehat{\mathcal{O}}[y_1, \cdots, y_n]$. On the other hand, the Zariski closure of η_W in $\square^n_{\widehat{\mathcal{O}}}$ is W, and since W is integral in $z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$, the closed subscheme of $\square^n_{\widehat{\mathcal{O}}}$ defined by the ideal generated by $f_1, \cdots, f_n$ is precisely W. Since $f_{1,K}, \cdots, f_{n,K}$ take the triangular form as in (4.1.2), the new polynomials $f_1, \cdots, f_n$ take the triangular form as in (4.1.1). This satisfies (1) and (2) by construction.

On the other hand, the given condition $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ implies that (see Corollary 2.2.4) its intersection with any proper face is empty. In particular, the constant term of $f_1(y_1)$ is a unit in $\widehat{\mathcal{O}}$. If the constant term of f_i is not a unit in $\widehat{\mathcal{O}}$ for some $2 \leq i \leq n$, then replace f_i by $f_i + f_1$. This procedure does not disturb the triangular shape of (4.1.1), nor the properties (1) and (2), and does not change the ideal of W, so that after this procedure we may assume the constant term of each f_i is a unit in $\widehat{\mathcal{O}}$. Now replacing each f_i by f_i divided by its unit constant term, we achieve that the constant term of each f_i is 1, without disturbing the triangular shape in (4.1.1), the properties (1) and (2) and without changing the ideal of W. This proves (3), hence proves the proposition. □

Corollary 4.1.6. *Let $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral cycle. For a defining set $\{f_1, \cdots, f_n\} \subseteq \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ of W in Proposition 4.1.5, consider the corresponding coefficient perturbation $V \subset \mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \square^n_{\widehat{\mathcal{O}}}$ given by $\{F_1, \cdots, F_n\}$ of $\{f_1, \cdots, f_n\}$ as in Definition 4.1.1. Then* (1) *the codimension of V in $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \square^n_{\widehat{\mathcal{O}}}$ is n and* (2) *V intersects each codimension 1 face of $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \square^n_{\widehat{\mathcal{O}}}$ properly.*

Proof. For $1 \leq i \leq n$, let V_i be the closed subscheme of $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \square^i_{\widehat{\mathcal{O}}}$ given by $(F_1, \cdots, F_i)$. We prove that the codimension of V_i in $B_i := \mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \square^i_{\widehat{\mathcal{O}}}$ is i by induction on i.

When $i = 1$, this is obvious because V_1 is given by a single polynomial $F_1(y_1)$ and $\deg_{y_1} F_1(y_1) \geq 1$, in particular $F_1(y_1) \neq 0$. Suppose the statement holds for $i \geq 1$. Then $V_i \subset B_i$ has codimension i so that $V_i \times_{\widehat{\mathcal{O}}} \square^1_{\widehat{\mathcal{O}}} \subset B_i \times_{\widehat{\mathcal{O}}} \square^1_{\widehat{\mathcal{O}}} = B_{i+1}$ has codimension i. On the other hand, V_{i+1} is given in $V_i \times_{\widehat{\mathcal{O}}} \square^1_{\widehat{\mathcal{O}}}$ by $F_{i+1}(y_1, \cdots, y_{i+1})$, and $\deg_{y_{i+1}} F_{i+1} \geq 1$, so that the codimension of V_{i+1} in $V_i \times_{\widehat{\mathcal{O}}} \square^1_{\widehat{\mathcal{O}}}$ is 1. Hence the codimension of V_{i+1} in B_{i+1} is $i + 1$, thus by induction the statement holds for all $1 \leq i \leq n$, proving (1).

For (2), let F be the codimension 1 face given by $\{y_i = \epsilon\}$ for some $1 \leq i \leq n$ and $\epsilon \in \{0, 1\}$. Since this is a divisor, we just need to show that $V \not\subseteq \mathbb{A}^n_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} F$. Suppose not, i.e. $V \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} F$. Then specializing at $x = \alpha_0 \in \mathbb{A}^M_{\widehat{\mathcal{O}}}$, which corresponds to W, we have $W \subseteq F$. But this is impossible because W intersects F properly. (N.B. Actually, $W \cap F = \emptyset$ by Corollary 2.2.4.) □

4.2. **Perturbation lemmas.** We now discuss several perturbation lemmas that play essential roles in the proof of the mod t^m-moving lemma in §4.3.

4.2.1. *Nonemptiness of fibers.* Here is the basic situation we consider:

Situation ($\star$)**:** Let $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral cycle and choose a triangular generating set $\{f_1, \cdots, f_n\} \subset \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ of the form (4.1.1) using Proposition 4.1.5. Consider the coefficient perturbation V of W with respect to $\{f_1, \cdots, f_n\}$ given by

$$(F_1, \cdots, F_n) \subset \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$$

as in Definition 4.1.1. Let $\alpha_0 \in \widehat{\mathcal{O}}^M$ be the coefficient vector corresponding to the generating set $\{f_1, \cdots, f_n\}$ of W. By Lemma 2.2.7, W is closed in $\overline{\square}^n_{\widehat{\mathcal{O}}}$. We regard $y_i = Y_{i1}/Y_{i0}$ and use $((Y_{10}; Y_{11}), \cdots, (Y_{n0}; Y_{n1})) \in \overline{\square}^n_{\widehat{\mathcal{O}}}$ as the projective coordinates. By homogenizing each f_j, we obtain $\bar{f}_j \in \widehat{\mathcal{O}}[\{Y_{10}, Y_{11}\}, \cdots, \{Y_{n0}, Y_{n1}\}]$. Here $\overline{W} = W$ in $\overline{\square}^n_{\widehat{\mathcal{O}}}$ is given by the ideal $(\bar{f}_1, \cdots, \bar{f}_n)$. Similarly, the homogenization $(\bar{F}_1, \cdots, \bar{F}_n)$ of $(F_1, \cdots, F_n)$ defines the Zariski closure $\overline{V} \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}}$ of V.

Let $\overline{pr} : \overline{V} \to \mathbb{A}^M_{\widehat{\mathcal{O}}}$ and $pr : V \to \mathbb{A}^M_{\widehat{\mathcal{O}}}$ be the restrictions of the projections $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times \overline{\square}^n_{\widehat{\mathcal{O}}} \to \mathbb{A}^M_{\widehat{\mathcal{O}}}$ and $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times \square^n_{\widehat{\mathcal{O}}} \to \mathbb{A}^M_{\widehat{\mathcal{O}}}$, respectively. For each $\alpha \in \mathbb{A}^M_{\widehat{\mathcal{O}}}$, let $\overline{V}_\alpha := \overline{pr}^{-1}(\alpha)$. We have $V_\alpha = \overline{V}_\alpha \cap \square^n_{\widehat{\mathcal{O}}} = pr^{-1}(\alpha)$, while $\overline{V}_{\alpha_0} = \overline{W} = W = V_{\alpha_0}$.

Proposition 4.2.1. *Under the Situation* ($\star$)*, there exists a nonempty open neighborhood* $U_{\rm ne} \subset \mathbb{A}^M_{\widehat{\mathcal{O}}}$ *of* α_0 *such that for each* $\alpha \in U_{\rm ne}$*, the fiber* V_α *is nonempty. Furthermore, this open set contains* $\mathbb{G}^M_{m,\widehat{\mathcal{O}}}$.

Proof. By Proposition 4.1.5, the coefficient perturbation V is given by polynomials

$$\begin{cases} F_1(y_1) \in \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1], \\ F_2(y_1, y_2) \in \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, y_2], \\ \quad \vdots \\ F_n(y_1, \cdots, y_n) \in \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]. \end{cases}$$

Here $\deg_{y_1} F_1 \geq 1$ and the coefficient in $\widehat{\mathcal{O}}[x_1, \cdots, x_M]$ of the highest y_1-degree term is a variable x_{ℓ_1} for some $1 \leq \ell_1 \leq M$. For the open subset $U_1 \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}}$ given by $x_{\ell_1} \neq 0$, y_1 is algebraic over $K(x_1, \cdots, x_M)$, and there is a solution y_1 in an algebraic extension of $K(x_1, \cdots, x_M)$. Plug this solution y_1 into the second equation. Since $\deg_{y_2} F_2 \geq 1$, and the coefficient of the highest y_2-degree term is x_{ℓ_2} for some $1 \leq \ell_2 \leq M$ with $\ell_2 \neq \ell_1$. Thus, for the open set $U_2 \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}}$ given by $x_{\ell_1} \neq 0$ and $x_{\ell_2} \neq 0$, y_2 is algebraic over $K(x_1, \cdots, x_M)$, and in particular there is a solution y_2 in an algebraic extension of $K(x_1, \cdots, x_M)$. Continuing this way, the coefficient of the highest y_n-degree term of F_n is x_{ℓ_n} for some $1 \leq \ell_n \leq M$ with $\ell_n \neq \ell_1, \cdots, \ell_{n-1}$, and for the open set $U_n \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}}$ given by $\{x_{\ell_1} \neq 0, \cdots, x_{\ell_n} \neq 0\}$ we have a system of solutions $y_1, \cdots, y_n$ in an algebraic extension of $K(x_1, \cdots, x_M)$. In other words, for each $\alpha \in U_{\rm ne} := U_n$, the fiber V_α is nonempty. By construction $U_{\rm ne}$ is given by the product of $\mathbb{A}^1_{\widehat{\mathcal{O}}}$ for each x_i with $i \notin \{\ell_1, \cdots, \ell_n\}$ and $\mathbb{G}_{m,\widehat{\mathcal{O}}}$ for each x_i with $i \in \{\ell_1, \cdots, \ell_n\}$, so that the second statement follows. That $\alpha_0 \in U_{\rm ne}$ follows immediately. □

4.2.2. *Empty intersection with faces.* Recall from Corollary 2.2.4 that for any proper face $F \subsetneq \square^n_{\widehat{\mathcal{O}}}$, we have $W \cap F = \emptyset$, which is stronger than having proper intersection with the face. We assert that this is an open condition in the following sense:

Lemma 4.2.2. *We are under the Situation* ($\star$)*. Then for each proper face* $F \subsetneq \square^n_{\widehat{\mathcal{O}}}$*, there exists an open neighborhood* $U_F \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}}$ *of* α_0 *such that for each* $\alpha \in U_F$*, we have* $V_\alpha \cap F = \emptyset$*. In particular, for each* $\alpha \in U_{\rm pi} := \bigcap_F U_F \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}}$*, where the intersection is taken over all proper faces* F*, the closed subscheme* V_α *intersects with no proper face at all.*

Proof. By Lemma 2.2.6, we know that $W \cap F = \emptyset$ if and only if $\overline{W} \cap \overline{F} = \emptyset$. So, we want to achieve the stronger assertion that $\overline{V}_\alpha \cap \overline{F} = \emptyset$ for each α in an open neighborhood of α_0. We use the projectivized system $\{\bar{F}_1, \cdots, \bar{F}_n\}$ of equations for $\overline{V}$.

When $\overline{F}$ is a codimension 1 face of $\overline{\square}^n_{\widehat{\mathcal{O}}}$, it is given by $\{y_{i_1} = \epsilon_1\}$ for some $1 \leq i_1 \leq n$ and $\epsilon_1 \in \{0, 1\}$. Here, the scheme $\overline{V}_\alpha$ *does* intersect with the face $\overline{F}$ if and only if the scheme given by $\{\bar{F}_1, \cdots, \bar{F}_n, y_{i_1} - \epsilon_1\}$ has a point lying over α. Here, the system $\{\bar{F}_1, \cdots, \bar{F}_n, y_{i_1} - \epsilon_1\}$ defines a closed subscheme of $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}}$ of dimension $\leq \dim (\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}}) - (n+1) = M + n + 1 - (n+1) = M$ by Corollary 4.1.6. Thus its image C_F under the projective morphism $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}} \to \mathbb{A}^M_{\widehat{\mathcal{O}}}$ is a closed subscheme of dimension $\leq M$. In particular, $C_F \subsetneq \mathbb{A}^M_{\widehat{\mathcal{O}}}$ is a proper closed subscheme since $\dim (\mathbb{A}^M_{\widehat{\mathcal{O}}}) = M + 1$. Hence $\overline{V}_\alpha$ *does not* intersect with $\overline{F}$ if and only if $\alpha \in U_F := \mathbb{A}^M_{\widehat{\mathcal{O}}} \setminus C_F$. By construction we have $\alpha_0 \in U_F$. Here, $\overline{V}_\alpha \cap \overline{F} = \emptyset$ implies that $V_\alpha \cap F = \emptyset$. Since every proper face is contained in some codimension 1 face, this answers the lemma. □

Corollary 4.2.3. *Under the Situation* ($\star$)*, for every sufficiently large integer* $N > 0$*, there exists an open ball* $\mathcal{B}_N(\alpha_0) \subseteq k[[t]]^M$ *in the non-archimedean t-adic sup-norm, such that* (1) $\mathcal{B}_N(\alpha_0) \cap (k[t]^M)$ *is nonempty,* (2) *for every* $\alpha \in \mathcal{B}_N(\alpha_0) \cap (k[t]^M)$*, the closed subscheme* V_α *does not intersect any face* $F \subsetneq \overline{\square}^n_{\widehat{\mathcal{O}}}$ *at all, and* (3) *these so obtained polynomials* $f_{1,\alpha}, \cdots, f_{n,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$ *of* V_α *satisfy* $f_{j,\alpha} \equiv f_j \bmod t^m$*, for each* $1 \leq j \leq n$*.*

Proof. Since the induced non-archimedean t-adic topology is finer than the Zariski topology on $\mathbb{A}^M_{\widehat{\mathcal{O}}}$ and $\alpha_0 \in \widehat{\mathcal{O}}^M = k[[t]]^M$, for every sufficiently large integer $N > 0$, the open ball $\mathcal{B}_N(\alpha_0) \subseteq k[[t]]^M$ of radius e^{-N} centered at α_0 is contained in the open subset $U_{\rm pi} \subset \mathbb{A}^{M-r}_{\widehat{\mathcal{O}}}$ of Lemma 4.2.2. We may assume $N > m$. But $k[t]^M \subseteq k[[t]]^M$ is dense in the non-archimedean topology, so $\mathcal{B}_N(\alpha_0) \cap (k[t]^M) \neq \emptyset$, proving (1). Since $\mathcal{B}_N(\alpha_0) \subseteq U_{\rm pi}$, we have (2). On the other hand, $\alpha \in \mathcal{B}_N(\alpha_0) \Leftrightarrow |\alpha - \alpha_0| < e^{-N} \Leftrightarrow$ for each $1 \leq j \leq n$, we have $f_{j,\alpha} \equiv f_j \bmod t^N$. In particular, since $N > m$, $f_{j,\alpha} \equiv f_j \bmod t^m$, proving (3). □

4.2.3. *Properness over* $\widehat{\mathcal{O}}$*.*

Lemma 4.2.4. *We are under the Situation* ($\star$)*. Then there exists an open neighborhood* $U_{\rm pr} \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}}$ *of* α_0 *such that* V_α *is a proper scheme over* ${\rm Spec}\,(\widehat{\mathcal{O}})$ *for each* $\alpha \in U_{\rm pr}$*.*

Proof. Let F^∞ be the divisor associated to $\overline{\square}^n_{\widehat{\mathcal{O}}} \setminus \square^n_{\widehat{\mathcal{O}}}$. By Lemma 2.2.6, to make V_α proper over ${\rm Spec}\,(\widehat{\mathcal{O}})$, it is enough to require that $\overline{V}_\alpha \cap F^\infty = \emptyset$. Here $F^\infty = \sum_{i=1}^n \{y_i = \infty\} = \sum_{i=1}^n \{Y_{i0} = 0\}$ so that $\overline{V}_\alpha \cap F^\infty = \emptyset$ if and only if $\overline{V}_\alpha \cap \{Y_{i0} = 0\} = \emptyset$, for all $1 \leq i \leq n$. Recall we have $y_i = Y_{i1}/Y_{i0}$ for the projective coordinate $(Y_{i0}; Y_{i1}) \in \overline{\square}_{\widehat{\mathcal{O}}} = \mathbb{P}^1_{\widehat{\mathcal{O}}}$.

To see which open subset of $\mathbb{A}^M_{\widehat{\mathcal{O}}}$ would do this job, we use an argument similar to the one in the proof of Lemma 4.2.2. The scheme $\overline{V}_\alpha$ *does* intersect $\{Y_{i0} = 0\}$ if and only if the scheme given by $\{\bar{F}_1, \cdots, \bar{F}_n, Y_{i0}\}$ has a point over α. The system $\{\bar{F}_1, \cdots, \bar{F}_n, Y_{i0}\}$ defines a closed subscheme of $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}}$ of dimension $\leq M + n + 1 - (n+1) = M$. Thus its image C_i under the projective morphism $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}} \to \mathbb{A}^M_{\widehat{\mathcal{O}}}$ is a closed subscheme of dimension $\leq M$, thus $C_i \subsetneq \mathbb{A}^M_{\widehat{\mathcal{O}}}$ is a proper closed subscheme. Hence $\overline{V}_\alpha$ *does not* intersect with F^∞ if and only if $\alpha \in U_{\rm pr} := \bigcap_{i=1}^n (\mathbb{A}^M_{\widehat{\mathcal{O}}} \setminus C_i)$. By construction, we have $\alpha_0 \in U_{\rm pr}$. This proves the lemma. □

Corollary 4.2.5. *Under the Situation* ($\star$)*, for every sufficiently large integer* $N > 0$*, there exists an open ball* $\mathcal{B}_N(\alpha_0) \subseteq k[[t]]^M$ *in the non-archimedean t-adic sup-norm, such that* (1) $\mathcal{B}_N(\alpha_0) \cap (k[t]^M)$ *is nonempty,* (2) *for every* $\alpha \in \mathcal{B}_N(\alpha_0) \cap (k[t]^M)$*, the closed subscheme* V_α *is proper over Spec*$(\widehat{\mathcal{O}})$*, and* (3) *these so obtained polynomials* $f_{1,\alpha}, \cdots, f_{n,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$ *of* V_α *satisfy* $f_{j,\alpha} \equiv f_j \bmod t^m$*, for each* $1 \leq j \leq n$*.*

Proof. The proof is almost identical to that of Corollary 4.2.3, where we use Lemma 4.2.4 instead of Lemma 4.2.2, so we omit it. □

4.2.4. *Flat stratum.* Let $pr : V \to \mathbb{A}^M_{\widehat{\mathcal{O}}}$ be the restriction of the projection $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \square^n_{\widehat{\mathcal{O}}}$. By Proposition 4.2.1, we know that the restriction $pr_{U_{\rm ne}} : pr^{-1}(U_{\rm ne}) \to U_{\rm ne}$ is surjective, but we do not know whether this is flat. By the generic flatness theorem of [8, Théorème 6.9.1], there is a nonempty open subset of $U_{\rm ne}$ over which $pr_{U_{\rm ne}}$ is flat, but this theorem does not tell us whether this open set contains α_0. This causes a small inconvenience. On the other hand, by the flattening stratification theorem of [8, Corollaire 6.9.3], we do know that there is a stratification partition $\{S_i\}$ of $U_{\rm ne}$ by locally closed subsets such the restriction of pr over the inverse image of each S_i is flat, and some stratum S_{i_0} must contain α_0. We will construct explicitly in Lemma 4.2.6 a locally closed subset of $U_{\rm ne}$

containing α_0 over which a more general collection of coherent sheaves are flat. This result will be used in §4.2.5 and §4.2.6.

Here is the situation updated from Situation ($\star$):

> **Situation** ($\star'$)**:** Let $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral cycle, and choose a triangular generating set $\{f_1, \cdots, f_n\} \subset \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ of the form (4.1.1) using Proposition 4.1.5. Let V be the coefficient perturbation of W given by $\{F_1, \cdots, F_n\} \subset \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$ as in Situation ($\star$). By renaming the variables x_i, we may assume that $x_{M-n+1}, \cdots, x_M$ corresponds to the constant terms ($= 1$) of $f_1, \cdots, f_n$. By Lemma 2.2.7, W is closed in $\overline{\square}^n_{\widehat{\mathcal{O}}}$ and it is given by $(\bar{f}_1, \cdots, \bar{f}_n)$ as in Situation ($\star$), with its coefficient perturbation $\overline{V} \subseteq \mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}}$ given by $(\bar{F}_1, \cdots, \bar{F}_n)$.
>
> Let $B := \mathbb{A}^{M-n}_{\widehat{\mathcal{O}}} \times \mathbf{1} \subset \mathbb{A}^M$, and $\overline{pr}_B : \overline{pr}^{-1}(B) \to B$ and $pr_B : pr^{-1}(B) \to B$ be the restrictions of $\overline{pr}$ and pr, respectively. Here, $\alpha_0 = \beta_0 \times \mathbf{1} \in B$.

Lemma 4.2.6. *Under the Situation* ($\star'$)*, denote* $\mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}}$ *by* X*. For each face* $\overline{F} \subseteq \overline{\square}^n_{\widehat{\mathcal{O}}}$*, including the case* $\overline{F} = \overline{\square}^n_{\widehat{\mathcal{O}}}$*, consider the coherent sheaf* $\mathcal{F}_{\overline{F}} := \mathcal{O}_X/(\mathcal{I}_{\overline{V}} + \mathcal{I}_{\overline{F}})$*, where* $\mathcal{I}_{\overline{V}}$ *is the ideal sheaf of* $\overline{V} \subseteq X$ *and* $\mathcal{I}_{\overline{F}}$ *is the pull-back to* X *of the ideal sheaf of* $\overline{F}$*. Then* $\mathcal{F}_{\overline{F}}$ *restricted to* $\overline{pr}^{-1}(B)$ *is* $\overline{pr}_B$*-flat. In particular, its restriction to* $pr^{-1}(B)$ *is* pr_B*-flat as well.*

Proof. Fix a face $\overline{F}$, and denote $\mathcal{F}_{\overline{F}}$ by $\mathcal{F}$. Let $X' := B \times_{\widehat{\mathcal{O}}} \overline{\square}^n_{\widehat{\mathcal{O}}} = \overline{pr}^{-1}(B)$, which is closed in X. Let $\mathcal{F}'$ be the restriction of $\mathcal{F}$ to X'. For each open chart $U' \subseteq X'$ from an affine cover of X' and each $x \in U'$, we need to show that the stalk $\mathcal{F}'_x$ is a flat $\mathcal{O}_{B, pr_B(x)}$-module. We prove it for the chart $U' := B \times_{\widehat{\mathcal{O}}} \square^n_{\widehat{\mathcal{O}}}$ of X', which is obtained from the open chart $U := \mathbb{A}^M_{\widehat{\mathcal{O}}} \times_{\widehat{\mathcal{O}}} \square^n_{\widehat{\mathcal{O}}}$ of X via $U' = U \cap X'$.

Now, $\mathcal{F}|_U = \mathcal{O}_U/(\mathcal{I}_V + \mathcal{I}_F)$ is given by the quotient of $\widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$ by $(F_1, \cdots, F_n) + (\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\})$, where $\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\}$ for some $\epsilon_j \in \{0, 1\}$, is the set of equations of the face $F = \overline{F} \cap \square^n_{\widehat{\mathcal{O}}}$.

Recall the constant term of each of $f_1, \cdots, f_n$ is 1. By the labeling convention of the Situation ($\star'$), x_{M-n+j} is the variable corresponding to the nonzero constant term of f_j for $1 \leq j \leq n$. So, we have $F_j = x_{M-n+j} + G_j$ for some $G_j \in \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][y_1, \cdots, y_n]$. Hence, the sections $(\mathcal{O}_U/\mathcal{I}_V)(U) = \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]/(F_1, \cdots, F_n)$ can be obtained from $\widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]$ by replacing each x_{M-n+j} by $-G_j$ for $1 \leq j \leq n$. Here each G_j does not involve any of the variables $x_{M-n+1}, \cdots, x_M$. Thus, $(\mathcal{O}_U/\mathcal{I}_V)(U) \simeq$

$$\widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}, -G_1, \cdots, -G_n][y_1, \cdots, y_n] = \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][y_1, \cdots, y_n],$$

which is a polynomial ring over $\widehat{\mathcal{O}}$ with the variables $\{x_1, \cdots, x_{M-n}\} \cup \{y_1, \cdots, y_n\}$. Now the further quotient

$$R_F := \widehat{\mathcal{O}}[x_1, \cdots, x_M][y_1, \cdots, y_n]/((F_1, \cdots, F_n) + (\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\}))$$

can be obtained from $(\mathcal{O}_U/\mathcal{I}_V)(U) \simeq \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][y_1, \cdots, y_n]$ by replacing each variable y_{i_u} by ϵ_u for $1 \leq u \leq s$, i.e.

$$\begin{aligned} R_F &\simeq \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][y_1, \cdots, y_n]/(\{y_{i_1} = \epsilon_1, \cdots, y_{i_s} = \epsilon_s\}) \\ &\simeq \widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}][\{y_\ell \mid 1 \leq \ell \leq n, \ell \neq i_1, \cdots, i_s\}], \end{aligned}$$

which is again a polynomial ring over $\widehat{\mathcal{O}}$ with the variables $\{x_1 \cdots, x_{M-n}\} \cup \{y_\ell \mid 1 \leq \ell \leq n,\ \ell \neq i_1, \cdots, i_s\}$. In particular, the natural map $\widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}] \to R_F$ induced by the projection $\overline{pr}$ is injective and it is flat. Here, we have $\mathrm{Spec}\,(\widehat{\mathcal{O}}[x_1, \cdots, x_{M-n}]) = \mathbb{A}^{M-n}_{\widehat{\mathcal{O}}} \simeq \mathbb{A}^{M-n}_{\widehat{\mathcal{O}}} \times \mathbf{1} = B$. Hence in particular, $\mathcal{F}' = \mathcal{F}|_{U'}$ is flat. The proof for other charts of X' is similar, so we omit it. $\square$

4.2.5. *Dominance.*

Lemma 4.2.7. *Under the Situation* $(\star')$*, recall that* $W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ *is dominant. Then there is an open neighborhood* $U_{\mathrm{dom}} \subseteq \mathbb{A}^{M-r}_{\widehat{\mathcal{O}}}$ *of* β_0 *such that for every* $\beta \in U_{\mathrm{dom}}$*, the associated closed subscheme* $V_\alpha \subseteq \square^n_{\widehat{\mathcal{O}}}$ *with* $\alpha := \beta \times \mathbf{1}$*, is dominant over* $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ *as well.*

Proof. Let $K := \mathrm{Frac}(\widehat{\mathcal{O}}) = k((t))$. Note that a morphism $Z \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is dominant if and only if the base change $Z_K \to \mathrm{Spec}\,(K)$ is a nonempty K-scheme. So, we consider the situation after the base change via $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$.

By Lemma 4.2.6, the morphism $pr_B : pr^{-1}(B) \to B$ is flat. For the open set U_{ne} of Proposition 4.2.1, we have $\alpha_0 \in B \cap U_{\mathrm{ne}}$ so that $B \cap U_{\mathrm{ne}} \neq \emptyset$, and this proposition shows that the restriction $pr_{B\cap U_{\mathrm{ne}}} : pr^{-1}(B \cap U_{\mathrm{ne}}) \to B \cap U_{\mathrm{ne}}$ is flat and surjective. Since U_{ne} contains $\mathbb{G}_m^M$, there exists a nonempty open neighborhood $U' \subset \mathbb{A}^{M-n}_{\widehat{\mathcal{O}}}$ of β_0 such that $U' \times \mathbf{1} \subseteq B \cap U_{\mathrm{ne}}$. Hence $pr_{U'} : pr^{-1}(U' \times \mathbf{1}) \to U' \times \mathbf{1}$ is flat and surjective. So, after base change via $\mathrm{Spec}\,(K) \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$, the new morphism $pr_{U'_K} : pr^{-1}(U' \times \mathbf{1})_K \to (U' \times \mathbf{1})_K$ is flat and surjective. We implicitly used [8, Proposition 2.1.4] several times. For this flat family, the dimensions of the fibers are all equal (see [8, Corollaire 6.1.2], or [10, Corollary III-9.6, p.257]). In particular, for every $\beta \in \mathbb{A}^{M-n}_K \cap U'$, we have $0 \leq \dim(V_{\alpha_0}) = \dim(V_\alpha)$ with $\alpha = \beta \times \mathbf{1}$. In particular $V_\alpha \neq \emptyset$. But, $\mathbb{A}^{M-n}_K$ is a nonempty open subset of $\mathbb{A}^{M-n}_{\widehat{\mathcal{O}}}$, so that we can take $U_{\mathrm{dom}} := \mathbb{A}^{M-n}_K \cap U'$ to finish the proof of the lemma. □

Corollary 4.2.8. *Under the assumptions of Lemma 4.2.7, for every sufficiently large integer* $N > 0$*, there exists an open ball* $\mathcal{B}_N(\beta_0) \subseteq k[[t]]^{M-n}$ *in the non-archimedean t-adic sup-norm, such that* (1) $\mathcal{B}_N(\beta_0) \cap (k[t]^{M-n})$ *is nonempty,* (2) *for every* $\beta \in \mathcal{B}_N(\beta_0) \cap (k[t]^{M-n})$*, the closed subscheme* V_α *for* $\alpha = \beta \times \mathbf{1}$*, is dominant over* $\mathrm{Spec}\,(\widehat{\mathcal{O}})$*, and* (3) *these so obtained polynomials* $f_{1,\alpha}, \cdots, f_{n,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$ *of* V_α *satisfy* $f_{j,\alpha} \equiv f_j \bmod t^m$ *for each* $1 \leq j \leq n$*.*

Proof. The proof is almost identical to that of Corollary 4.2.3, where we use Lemma 4.2.7 instead of Lemma 4.2.2, so we omit it. □

4.2.6. *Geometric integrality.* Although we began with an integral scheme W, this integrality may not necessarily be preserved under "small" perturbations of the coefficients. However, we will show that the geometrical integrality over k in the sense of [8, Définition 4.6.2] is better behaved. Later in Case 2 of the proof of Theorem 4.3.2, we will reduce the general integral situation to the geometrically integral situation.

Lemma 4.2.9. *Under the Situation* $(\star')$*, suppose further that* W *is geometrically integral over* k*. Then there exists an open neighborhood* $U_{\mathrm{gi}} \subseteq \mathbb{A}^{M-n}_{\widehat{\mathcal{O}}}$ *of* β_0 *such that for each* $\beta \in U_{\mathrm{gi}}$*, the fiber* V_α *with* $\alpha = \beta \times \mathbf{1}$*, is geometrically integral over* k*.*

Proof. Note that V_α is geometrically integral over k if and only if so is its Zariski closure $\overline{V}_\alpha$ in $\overline{\square}^n_{\widehat{\mathcal{O}}}$. Now, by Lemma 4.2.6 with $F = \overline{\square}^n_{\widehat{\mathcal{O}}}$, the morphism $\overline{pr}_B : \overline{pr}^{-1}(B) \to B = \mathbb{A}^{M-n}_{\widehat{\mathcal{O}}} \times \mathbf{1}$ is proper and flat. Hence by [9, Théorème 12.2.4(viii)], the set $U_{\mathrm{gi}} := \{\beta \in \mathbb{A}^{M-n}_{\widehat{\mathcal{O}}} \mid \overline{V}_\alpha$ with $\alpha = \beta \times \mathbf{1}$, is geometrically integral$\}$ is open in $\mathbb{A}^{M-n}_{\widehat{\mathcal{O}}}$. This U_{gi} is nonempty because $\beta_0 \in U_{\mathrm{gi}}$. But again, for each $\beta \in U_{\mathrm{gi}}$ with $\alpha = \beta \times \mathbf{1}$, we have that $\overline{V}_\alpha$ is geometrically integral over k if and only if so is V_α. This proves the lemma. □

Corollary 4.2.10. *Under the assumptions of Lemma 4.2.9, for every sufficiently large integer* $N > 0$*, there exists an open ball* $\mathcal{B}_N(\beta_0) \subseteq k[[t]]^{M-n}$ *in the non-archimedean t-adic sup-norm, such that* (1) $\mathcal{B}_N(\beta_0) \cap (k[t]^{M-n})$ *is nonempty,* (2) *for every* $\beta \in \mathcal{B}_N(\beta_0) \cap (k[t]^{M-n})$ *and* $\alpha := \beta \times \mathbf{1}$*, the closed subscheme* V_α *is geometrically integral over* k*, and* (3) *these so obtained polynomials* $f_{1,\alpha}, \cdots, f_{n,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$ *of* V_α *satisfy* $f_{j,\alpha} \equiv f_j \bmod t^m$ *for each* $1 \leq j \leq n$*.*

Proof. The proof is almost identical to that of Corollary 4.2.3, where we use Lemma 4.2.9 instead of Lemma 4.2.2, so we omit it. $\square$

4.3. **The mod t^m moving lemmas.** First observe:

Lemma 4.3.1. *Let T be an $\mathrm{Spec}(\widehat{\mathcal{O}})$-scheme of finite type. Let $W_1, W_2 \subseteq T$ be two integral closed subschemes, both surjective over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, such that we have the equality $W_{1,s} = W_{2,s}$ of the special fibers. Then $\dim W_1 = \dim W_2$.*

Proof. Let $d_i := \dim W_i$. The morphisms $W_i \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ for $i = 1, 2$ are flat (of relative dimension d_i-1) because they are surjective and $\mathrm{Spec}\,(\widehat{\mathcal{O}})$ is a regular scheme of dimension 1 (see [8, Corollaire 6.1.2] or [10, Proposition III-9.7, p.257]). Since $W_{1,s} = W_{2,s}$, we have $d_1 - 1 = d_2 - 1$. Hence $d_1 = d_2$. $\square$

We now prove the main result of §4:

Theorem 4.3.2. *For the completion homomorphism $\xi^n : z^n_{\mathfrak{m}}(\mathcal{O}, n)^c \to z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$, the composition $\xi^n_m : z^n_{\mathfrak{m}}(\mathcal{O}, n)^c \xrightarrow{\xi^n} z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c \to z^n(k_m, n)$ is a surjection.*

Proof. Let $W \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ be a nonempty integral closed subscheme of $\square^n_{\widehat{\mathcal{O}}}$. By Lemma 2.2.7, the structure morphism $W \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ is surjective.

Case 1: First consider the case when W is geometrically integral over k. Take the generators of the ideal of W given by $f_1, \cdots, f_n \in \widehat{\mathcal{O}}[y_1, \cdots, y_n]$ satisfying the Situation $(\star')$, i.e. of the form in (4.1.1) in Proposition 4.1.5.

By Corollaries 4.2.3, 4.2.5, 4.2.8 and 4.2.10, there exists a sufficiently large integer $N > m$ such that for every $\beta \in \mathcal{B}_N(\beta_0) \cap (k[t])^{M-n}$ with $\alpha := \beta \times \mathbf{1}$, the corresponding cycle $V_\alpha \subseteq \square^n_{\widehat{\mathcal{O}}}$ has empty intersection with all proper faces of $\square^n_{\widehat{\mathcal{O}}}$ (in particular, the intersections with all faces are proper), is proper and dominant (in particular surjective) over $\mathrm{Spec}\,(\widehat{\mathcal{O}})$, and is geometrically integral over k, and furthermore the defining ideal of V_α in $\widehat{\mathcal{O}}[y_1, \cdots, y_n]$ is given by polynomials $f_{j,\alpha} \in k[t][y_1, \cdots, y_n]$ satisfying $f_j \equiv f_{j,\alpha} \bmod t^m$ for all $1 \leq j \leq n$. Since both W and V_α are geometrically integral over k, they are integral, thus we have $W \sim_{t^m} V_\alpha$. By Lemma 4.3.1, this implies $\dim W = \dim V_\alpha$. Furthermore, for each proper face $F \subsetneq \square^n_{\widehat{\mathcal{O}}}$, we have $V_\alpha \cap F_s = \emptyset$ so that $\mathrm{codim}_F(V_\alpha \cap F_s) \geq n$, while for $F = \square^n_{\widehat{\mathcal{O}}}$ we have $\mathrm{codim}_F(V_\alpha \cap F_s) = \mathrm{codim}_F((V_\alpha)_s) = \mathrm{codim}_F(W_s) \geq n$, so that $V_\alpha \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$.

Note that V_α is given by the ideal generated by $\{f_{1,\alpha}, \cdots, f_{n,\alpha}\}$ in $\widehat{\mathcal{O}}[y_1, \cdots, y_n]$ with $f_{j,\alpha} \in k[t][y_1, \cdots, y_n] \subseteq \mathcal{O}[y_1, \cdots, y_n]$. So, if we let $Z \subseteq \square^n_{\mathcal{O}}$ be the closed subscheme given by the ideal generated by the same set $\{f_{1,\alpha}, \cdots, f_{n,\alpha}\}$, this time in $\mathcal{O}[y_1, \cdots, y_n]$, then we have $\widehat{Z} := Z \times_{\mathcal{O}} \widehat{\mathcal{O}} = V_\alpha$ by definition.

We need to show that $Z \in z^n_{\mathfrak{m}}(\mathcal{O}, n)^c$. Here for each face $F_{\mathcal{O}} \subseteq \square^n_{\mathcal{O}}$, we have $\dim\,(Z \cap F_{\mathcal{O}}) = \dim\,(\widehat{Z} \cap F_{\widehat{\mathcal{O}}})$, where $F_{\widehat{\mathcal{O}}}$ is the base change of $F_{\mathcal{O}}$. In particular, when $F_{\mathcal{O}} = \square^n_{\mathcal{O}}$, we have $\dim Z = \dim V_\alpha$, while when $F \subsetneq \square^n_{\mathcal{O}}$ is a proper face, Z intersects with F properly. Furthermore via the identification $\mathcal{O}/\mathfrak{m} = \widehat{\mathcal{O}}/\widehat{\mathfrak{m}}$, we have $Z_s = (V_\alpha)_s = W_s$ so that $\mathrm{codim}_{F_{\mathcal{O}}}(Z \cap F_{\mathcal{O},s}) \geq n$ for each face $F_{\mathcal{O}} \subseteq \square^n_{\mathcal{O}}$. Hence $Z \in z^n_{\mathfrak{m}}(\mathcal{O}, n)$. The structure morphism $Z \to \mathrm{Spec}\,(\mathcal{O})$ is proper by [8, Proposition 2.7.1(vii)], because its base change $Z \times_{\mathcal{O}} \widehat{\mathcal{O}} = V_\alpha \to \mathrm{Spec}\,(\widehat{\mathcal{O}})$ via the faithfully flat morphism $\mathrm{Spec}\,(\widehat{\mathcal{O}}) \to \mathrm{Spec}\,(\mathcal{O})$, is proper. Hence $Z \in z^n_{\mathfrak{m}}(\mathcal{O}, n)^c$ and $V_\alpha = \xi^n(Z)$. Combined with that $W \sim_{t^m} V_\alpha$, we thus have $W \in \mathrm{im}(\xi^n_m)$.

Case 2: Now we suppose that W is integral, but not geometrically integral over k. Let k' be an algebraic closure of k. We first claim that there is a finite extension $k \subset L$ contained in k' such that for $p_{L/k} : \mathrm{Spec}\,(L) \to \mathrm{Spec}\,(k)$ and its associated flat pull-back map $p^*_{L/k} : z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c \to z^n_{\widehat{\mathfrak{m}}'}(\widehat{\mathcal{O}}_L, n)^c$, the base-change $p^*_{L/k}(W) = W'$ is a closed subscheme of $\square^n_{\widehat{\mathcal{O}}_L}$, whose associated cycle is given by $\sum m_i W_i'$ for some closed subschemes $W_i' \subset \square^n_{\widehat{\mathcal{O}}_L}$ geometrically integral over L and integers $m_i \geq 1$. Here, $\widehat{\mathcal{O}}_L := \widehat{\mathcal{O}}_{\mathbb{A}^1_L, 0} \simeq L[[t]]$ and $\widehat{\mathfrak{m}}' := (t) \subset L[[t]]$.

Indeed, by [8, Corollaire 4.5.10], there is a finite extension $k \subset L$ such that the irreducible components $\widetilde{W}_i'$ of the base change W_L to L are all geometrically irreducible over L in the sense of [8, Définition 4.5.2]. These irreducible components $\widetilde{W}_i'$ may not be geometrically reduced over L, but yet when m_i is the length of $\widetilde{W}_i'$ over $W_i' := (\widetilde{W}_i')_{\mathrm{red}}$, the cycle associated to $W' = W_L$ is $\sum_i m_i W_i'$. If k is perfect, geometrically reducedness over L is equivalent to reducedness by [8, Corollaire 4.6.11], so that each W_i' is actually geometrically integral over L, so the claim holds in this perfect field case. If k is not perfect, then by [8, Proposition 4.6.6], there exists a finite radicial (i.e. purely inseparable) extension $L \subset L'$ such that for the further base change of W_i' to $(W_i')_{L'}$ from L to L', the scheme $((W_i')_{L'})_{\mathrm{red}}$ is geometrically reduced over L', and under this procedure, $((W_i')_{L'})_{\mathrm{red}}$ is still geometrically irreducible over L' by definition. Hence replacing the finite extension $k \subset L$ by the further extension $k \subset L \subset L'$, with m_i replaced by the lengths for the corresponding base change to L', we prove the claim.

Now, pick any W_i' of the claim. This is in $z^n_{\widehat{\mathfrak{m}}'}(\widehat{\mathcal{O}}_L, n)^c$ by construction. Since $p_{L/k}$ is proper and flat, we have $p_{L/k}(W_i') = W$. Note that we have the following commutative diagram, where $\mathcal{O}_L := \mathcal{O}_{\mathbb{A}^1_L, 0}$ and $\mathfrak{m}' := \mathfrak{m}_{\mathbb{A}^1_L, 0}$:

$$\begin{array}{ccccc} z^n_{\mathfrak{m}'}(\mathcal{O}_L, n)^c & \xrightarrow{\xi^n_L} & z^n_{\widehat{\mathfrak{m}}'}(\widehat{\mathcal{O}}_L, n)^c & \longrightarrow & z^n(L_m, n) \\ \downarrow{\scriptstyle p_{L/k}} & & \downarrow{\scriptstyle p_{L/k}} & & \downarrow{\scriptstyle p_{L/k}} \\ z^n_{\mathfrak{m}}(\mathcal{O}, n)^c & \xrightarrow{\xi^n} & z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c & \longrightarrow & z^n(k_m, n), \end{array} \tag{4.3.1}$$

where the left square is clearly commutative, while the right square is well-defined and commutative because $p_{L/k}$ is a Spec $(\widehat{\mathcal{O}})$-morphism of Spec $(\widehat{\mathcal{O}})$-schemes so that $p_{L/k}$ maps a pair of mod t^m-equivalent integral cycles to a pair of mod t^m-equivalent integral cycles.

Going back to the proof of Case 2, since W_i' is geometrically integral over L, by Case 1, there exists some $Z' \in z^n_{\mathfrak{m}'}(\mathcal{O}_L, n)^c$ such that $\xi^n_L(Z') = \widehat{Z}' \sim_{t^m} W_i'$. Hence we have $W = p_{L/k}(W_i') \sim^{\dagger}_{t^m} p_{L/k}\xi^n_L(Z') =^{\ddagger} \xi^n p_{L/k}(Z')$, where † and ‡ hold by the commutativity of the right and the left squares of the diagram (4.3.1), respectively. This shows that W lies in the image of ξ^n_m. This finishes the proof that ξ^n_m is surjective. □

Corollary 4.3.3. *The morphism $\xi^n_m : z^n_{\mathfrak{m}}(\mathcal{O}, \bullet)^{pc} \to z^n(k_m, \bullet)$ of complexes induces a surjective group homomorphism $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to \mathrm{CH}^n(k_m, n)$.*

Proof. Let $K_\bullet := \ker(\xi^n_m)$ and $I_\bullet := \mathrm{im}(\xi^n_m)$ so that we have a short exact sequence $0 \to K_\bullet \to z^n_{\mathfrak{m}}(\mathcal{O}, \bullet)^{pc} \to I_\bullet \to 0$ of homological complexes. From the morphisms $z^n_{\mathfrak{m}}(\mathcal{O}, \bullet)^{pc} \to I_\bullet \hookrightarrow z^n(k_m, \bullet)$ of complexes, we have homomorphisms

$$\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to \mathrm{H_n}(I_\bullet) \to \mathrm{CH}^n(k_m, n). \tag{4.3.2}$$

Here, by Remark 2.2.3, we have $z^n_{\mathfrak{m}}(\mathcal{O}, n-1)^{pc} = 0$ so that $K_{n-1} = 0$, while we have $K_j = 0$ for all $j \leq n-1$ due to dimension reason. In particular, $\mathrm{H}_{n-1}(K_\bullet) = 0$ and we have part of the associated long exact sequence $\cdots \to \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to \mathrm{H_n}(I_\bullet) \to \mathrm{H}_{n-1}(K_\bullet) = 0$ so that the first map $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to \mathrm{H_n}(I_\bullet)$ of (4.3.2) is surjective.

On the other hand, by Theorem 4.3.2, we have $I_n = z^n(k_m, n)$, while $I_j = 0$ for all $j \leq n-1$ by Remark 2.3.6 and dimension reason. Hence

$$\mathrm{H}_n(I_\bullet) = \frac{z^n(k_m, n)}{\partial(\xi^n_m(z^n_{\mathfrak{m}}(\mathcal{O}, n+1)^{pc}))}, \quad \mathrm{CH}^n(k_m, n) = \frac{z^n(k_m, n)}{\partial(z^n(k_m, n+1))}$$

with $\partial(\xi_m^n(z^n_{\mathfrak{m}}(\mathcal{O}, n+1)^{pc})) \subseteq \partial(z^n(k_m, n+1))$ in $z^n(k_m, n)$, so that the second map $\mathrm{H}_n(I_\bullet) \to \mathrm{CH}^n(k_m, n)$ in (4.3.2) is the surjective quotient map. Hence the composite in (4.3.2) is surjective, as desired. $\square$

Remark 4.3.4. One may wonder whether Theorem 4.3.2 extends beyond the Milnor range, i.e. when $q < n$, whether the composite $z^q_{\mathfrak{m}}(\mathcal{O}, n)^{pc} \to z^q_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \to z^q(k_m, n)$ is surjective. To test if this question is affirmatively answerable, concentrate only on the subset of integral effective cycles. Since the cycles considered are flat over $\mathrm{Spec}(\widehat{\mathcal{O}})$, such effective cycles may be, under mild additional assumptions, represented by (a locally closed subset of) a Hilbert scheme H, and there exists a (nonconstant) morphism $\mathrm{Spec}\,(\widehat{\mathcal{O}}) \to H$ of schemes. On the other hand, if the surjectivity assertion mod t^m would hold for those integral effective cycles, then it implies that for the fpqc cover $\mathrm{Spec}\,(\widehat{\mathcal{O}}) \to \mathrm{Spec}\,(\mathcal{O})$, the morphism $\mathrm{Spec}\,(\widehat{\mathcal{O}}) \to H$ should give an fpqc descent to a morphism $\mathrm{Spec}\,(\mathcal{O}) \to H$. However, this means that there exists a nonconstant rational map $\mathbb{A}^1 \to H$, which imposes a restrictive condition on H. Thus, we do not expect an extension of Theorem 4.3.2 to cycles of arbitrary dimension.

5. Milnor range II: mod t^m-equivalence and conclusion

In §5.1, we use the mod t^m moving lemma of Theorem 4.3.2 to transport the main theorem of [5, Theorem 3.4] (or equivalently, [6], [15]) to our situation of cycles over $\widehat{\mathcal{O}}$ modulo t^m. This allows a significant simplification of the generators of our relative cycle group $\mathrm{CH}^n((k_m, (t)), n)$, and helps in finally proving in §5.2 that the regulators Υ_i defined in Proposition 3.0.2 of §3 descend to the cycle classes mod t^m-equivalence. Using this, the proof of the main theorem of the article, Theorem 3.0.1, is finished in §5.3.

5.1. **The graph cycles.** Recall that for each integral k-domain R of finite Krull dimension, and a sequence $a_1, \cdots, a_n \in R^\times$ of units, we have its associated closed subscheme $\Gamma_{(a_1,\cdots,a_n)} \subset \square^n_R$ given by the set of equations $\{y_1 = a_1, \cdots, y_n = a_n\}$. This closed subscheme is called the *graph cycle* of the sequence, and this is geometrically integral over k. In case R is local with the maximal ideal $\mathfrak{m}$, actually $\Gamma_{(a_1,\cdots,a_n)} \in z^n_{\mathfrak{m}}(R, n)$, and we get the graph homomorphism $gr : K^M_n(R) \to \mathrm{CH}^n_{\mathfrak{m}}(R, n)$. This was proven in [5, Lemma 2.1] for a ring R essentially of finite type over k, but exactly the same argument proves it for the general case. By construction, the Zariski closure $\overline{\Gamma}$ of Γ in $\overline{\square}^n_R$ is equal to Γ, so that in particular Γ is closed in $\overline{\square}^n_R$ as well. Furthermore, one sees immediately that $\partial_i^\epsilon(\Gamma) = 0$ for each $1 \leq i \leq n$ and $\epsilon \in \{0, \infty\}$. In our situation of $R = \mathcal{O}, \widehat{\mathcal{O}}$, the graph cycles lie in $z^n_{\mathfrak{m}}(\mathcal{O}, n)^c$ and $z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$, respectively.

Definition 5.1.1. For $R = \mathcal{O}$ or $\widehat{\mathcal{O}}$, let $z^n_{gr}(R, n)$ be the subgroup generated by the images of the graph cycles $\Gamma_{(a_1,\cdots,a_n)}$ over all sequences $a_1, \cdots, a_n \in R^\times$. For the well-defined homomorphism $z^n_{gr}(\widehat{\mathcal{O}}, n) \to \mathrm{CH}^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \to \mathrm{CH}^n(k_m, n)$, define $\mathrm{CH}^n_{gr}(k_m, n)$ to be the image of $z^n_{gr}(\widehat{\mathcal{O}}, n)$ in $\mathrm{CH}^n(k_m, n)$.

Lemma 5.1.2. *Let k be an infinite field. The composite $K^M_n(\widehat{\mathcal{O}}) \stackrel{gr}{\to} \mathrm{CH}^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \to \mathrm{CH}^n(k_m, n)$ is surjective. In particular, the group $\mathrm{CH}^n(k_m, n)$ is generated by the graph cycles $\Gamma_{(a_1,\cdots,a_n)}$ for sequences $a_1, \cdots, a_n \in \widehat{\mathcal{O}}^\times$, and the natural homomorphism $\mathrm{CH}^n_{gr}(k_m, n) \to \mathrm{CH}^n(k_m, n)$ is an isomorphism.*

Proof. We have a commutative diagram

$$\begin{array}{ccccc} K^M_n(\mathcal{O}) & \xrightarrow{gr_{\mathcal{O}}} & \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} & \overset{*}{\searrow} & \\ \downarrow & & \downarrow \xi & & \\ K^M_n(\widehat{\mathcal{O}}) & \xrightarrow{gr_{\widehat{\mathcal{O}}}} & \mathrm{CH}^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} & \longrightarrow & \mathrm{CH}^n(k_m, n), \end{array}$$

where the left square commutes by [5, Proposition 2.3]. The map $gr_{\mathcal{O}}$ is surjective since the map $K_n^M(\mathcal{O}) \to \mathrm{CH}^n(\mathcal{O}, n)$ is surjective by [5, Theorem 3.4] (or [6], [15]; this uses the assumption that k is infinite), and by [5, Lemma 3.11], every cycle in $\mathrm{CH}^n(\mathcal{O}, n)$ is represented by a cycle in the group called $\mathrm{CH}^n_{\mathsf{sfs}}(\mathcal{O}, n)$, which is some subgroup of $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)$, where each irreducible component is finite (in particular, proper) surjective over Spec $(\mathcal{O})$. (See [5] for its precise definition.) In particular, $\mathrm{CH}^n_{\mathsf{sfs}}(\mathcal{O}, n)$ is a subgroup of $\mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc}$. Thus we have $\mathrm{CH}^n_{\mathsf{sfs}}(\mathcal{O}, n) = \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n)^{pc} = \mathrm{CH}^n_{\mathfrak{m}}(\mathcal{O}, n) = \mathrm{CH}^n(\mathcal{O}, n)$. The sloped map $*$ is surjective by Corollary 4.3.3. By diagram chasing, the map $K_n^M(\widehat{\mathcal{O}}) \to \mathrm{CH}^n(k_m, n)$ is surjective. The second assertion follows immediately from the first one. □

Lemma 5.1.3. *Let k be an infinite field. The surjection $K_n^M(\widehat{\mathcal{O}}) \to \mathrm{CH}^n(k_m, n)$ of Lemma 5.1.2 induces a surjection $K_n^M(k_m) \to \mathrm{CH}^n(k_m, n)$.*

Proof. There is a natural surjection $K_n^M(\widehat{\mathcal{O}}) \to K_n^M(\widehat{\mathcal{O}}/(t^m)) = K_n^M(k_m)$. So, for any Milnor symbol $\{a_1, \cdots, a_n\} \in K_n^M(k_m)$ with $a_i \in k_m^\times$, we choose any liftings $\tilde{a}_1, \cdots, \tilde{a}_n \in \widehat{\mathcal{O}}^\times = k[[t]]^\times$ and send the symbol $\{\tilde{a}_1, \cdots, \tilde{a}_n\} \in K_n^M(\widehat{\mathcal{O}})$ to the cycle class in $\mathrm{CH}^n(k_m, n)$ of the graph cycle $\Gamma_{(\tilde{a}_1, \cdots, \tilde{a}_n)} \subset \square^n_{\widehat{\mathcal{O}}}$. To prove that this map is well-defined, choose another sequence of liftings $\tilde{a}'_1, \cdots, \tilde{a}'_n \in \widehat{\mathcal{O}}^\times$ of the sequence $a_1, \cdots, a_n \in k_m^\times$, and here $\tilde{a}_i - \tilde{a}'_i \in t^m k[[t]]$. By definition, we have $\Gamma_{(\tilde{a}_1, \cdots, \tilde{a}_n)} \sim_{t^m} \Gamma_{(\tilde{a}'_1, \cdots, \tilde{a}'_n)}$, so that the map $K_n^M(k_m) \to \mathrm{CH}^n(k_m, n)$ is well-defined. The surjectivity of this map now follows from the surjectivity of $K_n^M(\widehat{\mathcal{O}}) \to \mathrm{CH}^n(k_m, n)$ of Lemma 5.1.2. □

5.2. The graph cycles over $\widehat{\mathcal{O}}$ mod t^m.

For graph cycles, it is easy to describe mod t^m equivalence:

Lemma 5.2.1. *Let $Z_1, Z_2 \in z^n_{gr}(\widehat{\mathcal{O}}, n)^c$ be two integral graph cycles, represented by*

$$Z_1 : \{y_1 = a_1, \cdots, y_n = a_n\}, \quad Z_2 : \{y_1 = b_1, \cdots, y_n = b_n\}, \tag{5.2.1}$$

where $a_j, b_j \in \widehat{\mathcal{O}}^\times$ for $1 \le j \le n$. Then the following are equivalent:

1. $Z_1 \sim_{t^m} Z_2$
2. *For each $1 \le j \le n$, we have $a_j \equiv b_j$ in $\widehat{\mathcal{O}}/(t^m)$.*
3. *For each $1 \le j \le n$, there exists $c_j \in \widehat{\mathcal{O}}$ such that $a_j = b_j(1 + c_j t^m)$ in $\widehat{\mathcal{O}}$.*

Proof. The equivalence (1) ⇔ (2) and the implication (3) ⇒ (2) are obvious. For the implication (2) ⇒ (3), note that $a_j \equiv b_j$ in $\widehat{\mathcal{O}}/(t^m)$ implies that $a_j b_j^{-1} \equiv 1$ in $\widehat{\mathcal{O}}/(t^m)$ so that $a_j b_j^{-1} = 1 + c_j t^m$ in $\widehat{\mathcal{O}}$ for some $c_j \in \widehat{\mathcal{O}}$. This proves (3). □

Proposition 5.2.2. *Let k be a field of characteristic 0. Let $Z_1, Z_2 \in z^n_{gr}(\widehat{\mathcal{O}}, n)^c$ be two integral graph cycles such that $Z_1 \sim_{t^m} Z_2$. Then $\Upsilon_i(Z_1) = \Upsilon_i(Z_2)$ for each $1 \le i \le m-1$.*

Proof. For Z_1 and Z_2, express them by the equations as in (5.2.1). By Lemma 5.2.1, the assumption that $Z_1 \sim_{t^m} Z_2$ implies that we have $a_j = b_j(1 + c_j t^m)$ for some $c_j \in \widehat{\mathcal{O}}$ for each $1 \le j \le n$. Notice that the common special fiber $(Z_1)_s = (Z_2)_s$ is given by a single closed point $\mathfrak{p}$ whose coordinates are $\bar{a}_1 = \bar{b}_1, \cdots, \bar{a}_n = \bar{b}_n$, where the bars denote the images in the residue field k of $\widehat{\mathcal{O}}$ mod (t). For each j, we have

$$d\mathrm{log} a_j - d\mathrm{log} b_j = d\mathrm{log}(1 + c_j t^m) = \frac{t^m dc_j + c_j m t^{m-1} dt}{1 + c_j t^m} \in t^{m-1} \Omega^1_{\widehat{\mathcal{O}}/\mathbb{Z}}.$$

Hence by expanding out, we immediately have $d\mathrm{log} y_1 \wedge \cdots \wedge d\mathrm{log} y_n|_{Z_1} - d\mathrm{log} y_1 \wedge \cdots \wedge d\mathrm{log} y_n|_{Z_2} = d\mathrm{log} a_1 \wedge \cdots \wedge d\mathrm{log} a_n - d\mathrm{log} b_1 \wedge \cdots \wedge d\mathrm{log} b_n \in t^{m-1} \Omega^n_{\widehat{\mathcal{O}}/\mathbb{Z}}$. Thus for each $1 \le i \le m-1$, we have $\frac{1}{t^i} d\mathrm{log} a_1 \wedge \cdots \wedge d\mathrm{log} a_n - \frac{1}{t^i} d\mathrm{log} b_1 \wedge \cdots \wedge d\mathrm{log} b_n \in t^{m-1-i} \Omega^n_{\widehat{\mathcal{O}}/\mathbb{Z}} \subset \Omega^n_{\widehat{\mathcal{O}}/\mathbb{Z}}$ so that the residue at $t = 0$ (which is the residue at the unique point $\mathfrak{p}$ of the common special fiber) of the difference vanishes. In other words, $\Upsilon_i(Z_1) = \Upsilon_i(Z_2)$ for $1 \le i \le n$. □

Corollary 5.2.3. *Let k be a field of characteristic* 0*. For $1 \leq i \leq m-1$, the map Υ_i of Proposition 3.0.2 induces a homomorphism $\Upsilon_i : \mathrm{CH}^n((k_m,(t)),n) \to \Omega^{n-1}_{k/\mathbb{Z}}$.*

Proof. By Proposition 3.0.2, the map Υ_i descends to $\Upsilon_i : \mathrm{CH}^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} \to \Omega^{n-1}_{k/\mathbb{Z}}$. Since $\mathrm{CH}^n_{gr}(k_m,n) = \mathrm{CH}^n(k_m,n)$ by Lemma 5.1.2, we may consider only the graph cycles. For all the pairs of mod t^m equivalent integral graph cycles, by Proposition 5.2.2, the maps Υ_i respect the mod t^m equivalence, so that we have the induced map $\Upsilon_i : \mathrm{CH}^n(k_m,n) \to \Omega^{n-1}_{k/\mathbb{Z}}$. Now, since $\mathrm{CH}^n(k_m,n) = \mathrm{CH}^n((k_m,(t)),n) \oplus \mathrm{CH}^n(k,n)$, by restriction we have the desired homomorphism. □

Remark 5.2.4. In fact, $\Upsilon_i|_{\mathrm{CH}^n(k,n)} = 0$ for $1 \leq i \leq m-1$. Indeed, by the theorem of Nesterenko-Suslin [21] and Totaro [27], we have an isomorphism $K^M_n(k) \simeq \mathrm{CH}^n(k,n)$ so that it is enough to check that for the graph cycles Γ given by the system of the equations of the form $\{y_1 = a_1, \cdots, y_n = a_n\}$ considered as a closed subscheme of $\square^n_{\widehat{\mathcal{O}}}$, with $a_1, \cdots, a_n \in k^\times$, we have $\Upsilon_i(\Gamma) = 0$. The form is $\frac{1}{t^i} d\log y_1 \wedge \cdots \wedge d\log y_n|_\Gamma = \frac{1}{t^i} d\log a_1 \wedge \cdots \wedge d\log a_n$ with each $a_j \in k^\times$ so that there is no term with dt anywhere in the form. Thus its residue along $t = 0$ is 0, i.e. $\Upsilon_i(\Gamma) = 0$.

5.3. **Proof of Theorem 3.0.1.** Finally, we prove the main theorem of the paper. We show that $\bigoplus_{i=1}^{m-1} \Upsilon_i : \mathrm{CH}^n((k_m,(t)),n) \to \bigoplus_{i=1}^{m-1} \Omega^{n-1}_{k/\mathbb{Z}}$ is an isomorphism. Recall from Lemma 5.1.3 that we had a surjection $K^M_n(k_m) \to \mathrm{CH}^n(k_m,n)$. This induces a surjective map $K^M_n(k_m,(t)) \to \mathrm{CH}^n((k_m,(t)),n)$, where $K^M_n(k_m,(t)) := \ker(K^M_n(k_m) \overset{\mathrm{ev}_{t=0}}{\to} K^M_n(k))$. We know from Proposition 5.4.2 in the appendix §5.4 below that we have an isomorphism

$$K^M_n(k_m,(t)) \overset{\sim}{\to} \Omega^{n-1}_{k_m,(t)/\mathbb{Z}} / d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}} \overset{\sim}{\leftarrow} \bigoplus_{1 \leq i \leq m-1} t^i \Omega^{n-1}_{k/\mathbb{Z}},$$

given by $\{a_1, \cdots, a_n\} \mapsto \log(a_1) d\log(a_2) \wedge \cdots \wedge d\log(a_n)$, where $a_1 \in 1 + tk_m$ and $\Omega^i_{k_m,(t)/\mathbb{Z}} := \ker(\Omega^i_{k_m/\mathbb{Z}} \overset{\mathrm{ev}_{t=0}}{\to} \Omega^i_{k/\mathbb{Z}})$. Then, looking at the $k^\times$-weight i parts, we obtain the maps

$$\Omega^{n-1}_{k/\mathbb{Z}} \overset{\sim}{\to} t^i \Omega^{n-1}_{k/\mathbb{Z}} \hookrightarrow K^M_n(k_m,(t)) \twoheadrightarrow \mathrm{CH}^n((k_m,(t)),n) \overset{\Upsilon_i}{\to} \Omega^{n-1}_{k/\mathbb{Z}}, \tag{5.3.1}$$

where $r_1 dr_2 \wedge \cdots \wedge dr_n \in \Omega^{n-1}_{k/\mathbb{Z}}$ is mapped to $\{e^{rt^i}, r_2, \cdots, r_n\} \in K^M_n(k_m,(t))$, where $r := r_1 \cdots r_n$. Let $\Gamma \in z^n_{\widehat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ denote the graph of this Milnor element. The composition (5.3.1) then sends $r_1 dr_2 \wedge \cdots \wedge dr_n$ to $\Upsilon_i(\Gamma) = i r_1 dr_2 \wedge \cdots \wedge dr_n$ by a straightforward calculation. Since $i \neq 0$, the composition (5.3.1) is an isomorphism. In particular, the composite

$$\bigoplus_{i=1}^{m-1} \Omega^{n-1}_{k/\mathbb{Z}} \simeq K^M_n(k_m,(t)) \twoheadrightarrow \mathrm{CH}^n((k_m,(t)),n) \overset{\bigoplus_i \Upsilon_i}{\to} \bigoplus_{i=1}^{m-1} \Omega^{n-1}_{k/\mathbb{Z}} \tag{5.3.2}$$

is an isomorphism. Therefore, the above map $K^M_n(k_m,(t)) \to \mathrm{CH}^n((k_m,(t)),n)$ is injective, hence an isomorphism. Since the composite (5.3.2) is an isomorphism, this implies that $\bigoplus_i \Upsilon_i$ is an isomorphism, as desired. □

5.4. **Appendix.** In the middle of the proof of Theorem 3.0.1 in §5.3, we used the following Proposition 5.4.2. This is probably well-known to the experts, and with some effort it should follow from e.g. [7]. However, since the Milnor K-groups are given by the concrete Milnor symbols we sketch a direct argument as follows, partly due to the fact that the authors could not find a suitable reference. We first have:

Lemma 5.4.1. *Let k be a field. Then $K^M_n(k_m,(t))$ is generated by the Milnor symbols $\{a_1, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ and $a_2, \cdots, a_n \in k_m^\times$.*

Proof. Let $G \subseteq K_n^M(k_m)$ be the subgroup generated by the Milnor symbols $\{a_1, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ (which is contained in $k_m^\times$) and $a_2, \cdots, a_n \in k_m^\times$. Certainly under the evaluation map $\mathrm{ev}_{t=0} : K_n^M(k_m) \to K_n^M(k)$, we have $\{a_1|_{t=0}, a_2|_{t=0}, \cdots, a_n|_{t=0}\} = \{1, a_2|_{t=0}, \cdots, a_n|_{t=0}\} = 0$ in $K_n^M(k)$. Hence each such generator $\{a_1, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ is contained in $\ker(\mathrm{ev}_{t=0}) = K_n^M(k_m, (t))$, thus $G \subseteq K_n^M(k_m, (t))$.

Every $a \in k_m^\times$ can be written as the product $a = c \cdot b$ with $c \in k^\times$ and $b \in 1+tk_m$. Hence by the multi-linearity and the anti-commutativity of $K_n^M(k_m)$, every symbol $\{a_1, \cdots, a_n\}$ with $a_i \in k_m^\times$ can be written as a sum of symbols in G (type I) and symbols $\{c_1, \cdots, c_n\}$ such that $c_i \in k^\times$ (type II). Here the splitting ring homomorphisms $k \to k_m \overset{\mathrm{ev}_{t=0}}{\to} k$ induce the splitting $K_n^M(k_m) = K_n^M(k) \oplus K_n^M(k_m, (t))$. The type II symbols are definitely in $K_n^M(k)$, while the symbols of type I generate G. Hence $K_n^M(k_m, (t)) = G$. □

Proposition 5.4.2. *Let k be a field of characteristic 0 and let $m \geq 2$ be an integer. Then we have an isomorphism $\phi_n : K_n^M(k_m, (t)) \simeq \Omega^{n-1}_{k_m,(t)/\mathbb{Z}}/d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}}$ given by $\{a_1, \cdots, a_n\} \mapsto \log(a_1) d\log(a_2) \wedge \cdots \wedge d\log(a_n)$, where $a_1 \in 1+tk_m$, where $\log(a_1)$ makes sense in k_m. The isomorphism can be rewritten as $K_n^M(k_m, (t)) \simeq \bigoplus_{i=1}^{m-1} t^i \Omega^{n-1}_{k/\mathbb{Z}}$, where the map $t^i\Omega^{n-1}_{k/\mathbb{Z}} \to K_n^M(k_m, (t))$ is given by sending $r_1 dr_2 \wedge \cdots dr_n$ to $\{e^{r_1 r_2 \cdots r_n t^i}, r_2, \cdots, r_n\} \in K_n^M(k_m, (t))$.*

Proof. By Lemma 5.4.1, $K_n^M(k_m, (t))$ is generated by $\{a_1, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ and $a_2, \cdots, a_n \in k_m^\times$. We define $\psi_n : \Omega^{n-1}_{k_m,(t)/\mathbb{Z}}/d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}} \to K_n^M(k_m, (t))$ by sending $r_1 dr_2 \wedge \cdots \wedge dr_n$, where $r_1, \cdots, r_\ell \in (t)$ and $r_{\ell+1}, \cdots, r_n \in k_m^\times$, to $\{e^{r_1 r_{\ell+1} \cdots r_n}, e^{r_2}, \cdots, e^{r_\ell}, r_{\ell+1}, \cdots, r_n\}$ in $K_n^M(k_m, (t))$.

One can check by induction that ϕ_n and ψ_n are well-defined group homomorphisms. We omit the proof as they follow from elementary but tedious arguments. Let's check that ϕ_n and ψ_n are inverse to each other. Indeed, for $x = r_1 dr_2 \wedge \cdots \wedge dr_n \in \Omega^{n-1}_{k_m,(t)/\mathbb{Z}}/d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}}$ with $r_1, \cdots, r_\ell \in (t)$ and $r_{\ell+1}, \cdots, r_n \in k_m^\times$, we have

$$\begin{aligned} & (\phi_n \circ \psi_n)(x) = \phi_n\{e^{r_1 r_{\ell+1} \cdots r_n}, e^{r_2}, \cdots, e^{r_\ell}, r_{\ell+1}, \cdots, r_n\} \\ = \ & \log(e^{r_1 r_{\ell+1} \cdots r_n}) d\log(e^{r_2}) \wedge \cdots \wedge d\log(e^{r_\ell}) \wedge d\log(r_{\ell+1}) \wedge \cdots \wedge d\log(r_n) \\ = \ & r_1 r_{\ell+1} \cdots r_n dr_2 \wedge \cdots \wedge dr_\ell \wedge \frac{dr_{\ell+1}}{r_{\ell+1}} \wedge \cdots \wedge \frac{dr_n}{r_n} = r_1 dr_2 \wedge \cdots \wedge dr_n = x, \end{aligned}$$

so that $\phi_n \circ \psi_n = \mathrm{Id}$. On the other hand, for $y = \{a_1, a_2, \cdots, a_n\}$ with $a_1 \in 1 + tk_m$ and $a_i \in k_m^\times$ for $2 \leq i \leq n$, we have $(\psi_n \circ \phi_n)(y) =$

$$\psi_n(\log(a_1) d\log(a_2) \wedge \cdots \wedge d\log(a_n)) = \psi_n\left(\frac{\log(a_1)}{a_2 \cdots a_n} da_2 \wedge \cdots \wedge da_n\right). \tag{5.4.1}$$

Here $a_1 \in 1 + tk_m$ so that $\log(a_1) \in (t)$, hence $\log(a_1)/(a_2 \cdots a_n) \in (t)$. Hence (5.4.1) equals to $\{e^{\frac{\log(a_1)}{a_2 \cdots a_n} \cdot a_2 \cdots a_n}, a_2, \cdots, a_n\} = \{a_1, \cdots, a_n\} = y$, i.e. $\psi_n \circ \phi_n = \mathrm{Id}$. The second statement follows from Lemma 5.4.3 below. □

We used the following elementary and nice lemma in the middle of the proof of Proposition 5.4.2, which we learned from the proof of [12, Lemma 6.2]:

Lemma 5.4.3. *Let k be a field of characteristic 0. Let $m \geq 2$ be an integer. Then for $n \geq 2$, we have $\Omega^{n-1}_{k_m,(t)/\mathbb{Z}}/d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}} \simeq d\Omega^{n-1}_{k_m/\mathbb{Z}}/d\Omega^{n-1}_{k/\mathbb{Z}} \underset{d}{\overset{\simeq}{\leftarrow}} tk_m \otimes_k \Omega^{n-1}_{k/\mathbb{Z}} = \bigoplus_{i=1}^{m-1} t^i \Omega^{n-1}_{k/\mathbb{Z}}$.*

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{H}^{n-1}(\Omega^\bullet_{k_m/\mathbb{Z}}) & \longrightarrow & \Omega^{n-1}_{k_m/\mathbb{Z}}/d\Omega^{n-2}_{k_m/\mathbb{Z}} & \xrightarrow{d} & d\Omega^{n-1}_{k_m/\mathbb{Z}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \downarrow{\scriptstyle \mathrm{ev}_{t=0}} & & \\ 0 & \longrightarrow & \mathrm{H}^{n-1}(\Omega^\bullet_{k/\mathbb{Z}}) & \longrightarrow & \Omega^{n-1}_{k}/d\Omega^{n-2}_{k/\mathbb{Z}} & \xrightarrow{d} & d\Omega^{n-1}_{k/\mathbb{Z}} & \longrightarrow & 0, \end{array}$$

where the vertical maps are all split surjections. Furthermore, by the Poincaré lemma in [33, Corollary 9.9.3], we have $\mathrm{H}^{n-1}(\Omega^\bullet_{k_m,(t)/\mathbb{Z}}) = 0$ so that the left vertical map is actually an isomorphism. Hence the snake lemma gives an isomorphism $\Omega^{n-1}_{k_m,(t)/\mathbb{Z}}/d\Omega^{n-2}_{k_m,(t)/\mathbb{Z}} \simeq d\Omega^{n-1}_{k_m/\mathbb{Z}}/d\Omega^{n-1}_{k/\mathbb{Z}}$. The second isomorphism $d\Omega^{n-1}_{k_m/\mathbb{Z}}/d\Omega^{n-1}_{k/\mathbb{Z}} \underset{d}{\overset{\simeq}{\leftarrow}} tk_m \otimes_k \Omega^{n-1}_{k/\mathbb{Z}} = \bigoplus_{i=1}^{m-1} t^i\Omega^{n-1}_{k/\mathbb{Z}}$ is obvious. □

5.5. **Final remarks.** We have two remarks on strengthening Theorem 3.0.1.

Remark 5.5.1. We could have defined $z^q(k_m, n)$ in Definition 2.3.5 as $z^q_{\hat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)/\sim_{t^m}$ using the complex $z^q_{\hat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$, but then part of the perturbation results in §4 may not be easy to establish. If one can prove the guess that "*every integral cycle $Z \in z^n_{\hat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)$ is equivalent to a cycle in $z^n_{\hat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^{pc} = z^n_{\hat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)^c$ modulo the boundary of a cycle in $z^n_{\hat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n+1)$*", then we can still prove a stronger version of Theorem 3.0.1 for the cycles using $z^q_{\hat{\mathfrak{m}}}(\widehat{\mathcal{O}}, n)/\sim_{t^m}$. In fact, it is easy to see that when $n = 1$, every integral cycle in $z^1_{\hat{\mathfrak{m}}}(\widehat{\mathcal{O}}, 1)$ is automatically proper over Spec $(\widehat{\mathcal{O}})$. However for $n \geq 2$, we could yet find neither a proof nor a counterexample to the guess, so we gave this version of the definition in Definition 2.3.5.

Remark 5.5.2. Reflecting on the main theorem of [25], it is desirable to remove the assumption that the base field k is of characteristic 0 in Theorem 3.0.1. The right hand side $(\Omega^{n-1}_{k/\mathbb{Z}})^{\oplus(m-1)}$ of the isomorphism of Theorem 3.0.1 should be replaced by the big de Rham-Witt forms $\mathbb{W}_{m-1}\Omega^{n-1}_k$ for a general base field k. To proceed further, we need to understand whether there exists a Parshin-Lomadze residue for the big de Rham-Witt complexes when the base field is of positive characteristic, and especially when it is imperfect. This is not trivial and may require serious works. A more minor problem is to give an explicit description of the relative Milnor K-groups of the ring of truncated polynomials over a field of positive characteristic in terms of the big de Rham-Witt complexes. This would improve Proposition 5.4.2 for a field of positive characteristic. We leave these as future tasks to finish.

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