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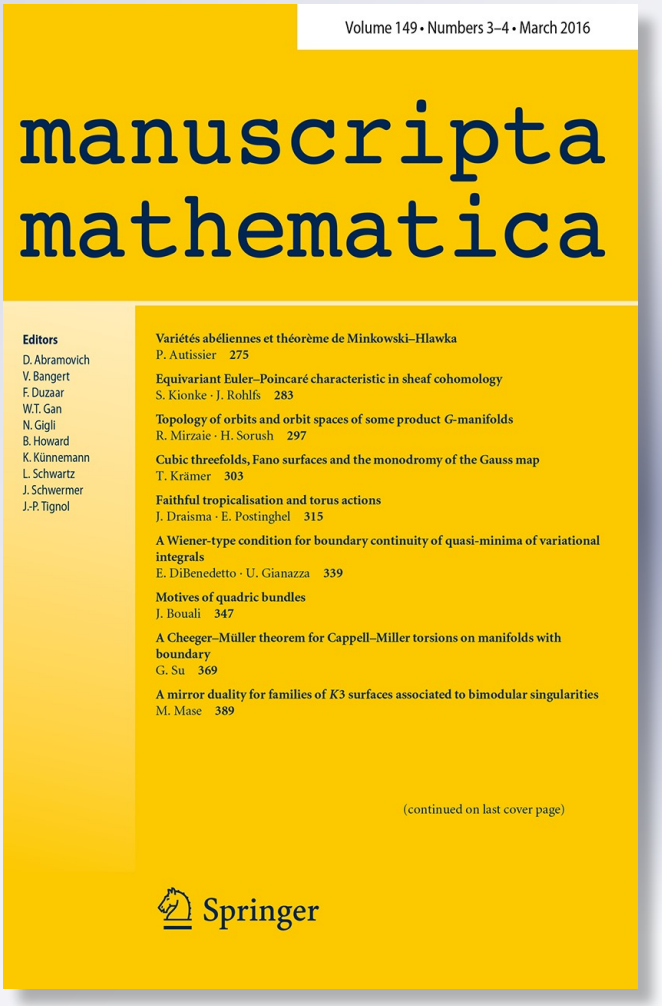

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Abstract. In this paper we compute the values of the p-adic multiple polylogarithms of depth two at roots of unity. Our method is to solve the fundamental differential equation satisfied by the crystalline frobenius morphism using rigid analytic methods. The main result could be thought of as a computation in the p-adic theory of higher cyclotomy. We expect the result to be useful in proving non-vanishing results since it gives quite explicit formulas.

1. Introduction

Let $M \geq 1$, ζ a primitive Mth root of unity and $E := \mathbb{Q}(\zeta)$. Let $V_M := \mathbb{G}_{m,E} \backslash \mu_{M,E}$ be the complement of the group of Mth roots of unity μ_M in the multiplicative group $\mathbb{G}_m$ over E. The unipotent completion of the fundamental group of V_M has a motivic interpretation. The case of $M = 1$ was studied in detail in [3], where the unipotent completion of $\pi_1(V_M, \cdot)$ was defined in different cohomology theories and comparison isomorphisms were given between them. The periods of the Betti–de Rham comparison isomorphism give multi-zeta values. The algebra that they generate has arithmetic significance since it is related to the Hopf algebra of the motivic Galois group through a conjecture of Grothendieck. The periods of the crystalline-de Rham comparison isomorphism give p-adic multi-zeta values which we studied in [11].

Choosing appropriate basepoints Deligne and Goncharov define the unipotent motivic fundamental group $\pi_1^{mot}(V_M, \cdot)$ of V_M whose ring of functions is an ind-object in the tannakian category of mixed Tate motives over $\mathcal{O}_E[M^{-1}]$ [6, §5]. Moreover, for $M = 1$, F. Brown showed that $\pi_1^{mot}(V_M, \cdot)$ generates this category as a tannakian category [1]. Similar results for $M = 2, 4, 6, 8$ were shown by Deligne [5]. As a consequence of this, the periods of mixed Tate motives over these rings are $\mathbb{Q}$-linear combinations of periods of the fundamental group $\pi_1^{mot}(V_M, \cdot)$. Therefore studying these periods have important arithmetic consequences. It is of special importance to prove linear independence or transcendence statements for the p-adic periods and for this, one would want to have as explicit a description as one can have for these values. This is what we aim to do for the p-adic periods in depth less than or equal to two. We describe the problem in more detail below.

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The comparison between the Betti and de Rham realization of $\pi_1(V_M, \cdot)$ is completely described by the cyclotomic versions of multi-zeta values [6, Proposition 5.17]. Namely, fix an imbedding of E in $\mathbb{C}$. In the de Rham theory of the fundamental group, there is a canonical fiber functor denoted by ω_{dR} (Sect. 2.1.2). Moreover, for any basepoint x, there is a canonical isomorphism between the fiber functor at x, and ω_{dR} (2.1.1). In the following, when we refer to the de Rham fundamental group without specifying the fiber functor, we always mean to use ω_{dR}. The Lie algebra of the de Rham fundamental group of V_M over $\mathbb{C}$ is the free pro-nilpotent Lie algebra with generators $\{e_i\}_{0\leq i\leq M}$, where e_0 (resp. e_i) are the functionals which send a unipotent connection to its residues at 0 (resp. ζ^i) (Sect. 2.1.3). Hence the set of $\mathbb{C}$-valued points of the de Rham fundamental group is the set of group-like elements in the non-commutative formal power series ring $\mathbb{C}\langle\langle e_0, \ldots, e_M\rangle\rangle$. The image of the Betti path from the tangential basepoint 1 at 0 to the tangential basepoint -1 at 1 under the de Rham–Betti comparison isomorphism gives an element of the de Rham fundamental groupoid between the same tangential basepoints. Using the above identification of the fiber functors at these basepoints and ω_{dR}, we obtain an element of the de Rham fundamental group and hence a group-like element ${}_1\gamma_0$ in $\mathbb{C}\langle\langle e_0, \ldots, e_M\rangle\rangle$. By [6, Proposition 5.17], the coefficient of $e_0^{s_m-1}e_{i_m}\cdots e_0^{s_1-1}e_{i_1}$ in ${}_1\gamma_0$, where $M \geq i_m, \ldots, i_1 \geq 1$ and $s_m > 1$, is

$$(-1)^m \sum_{n_m>\cdots>n_1>0} \frac{\zeta^{i_m(n_{m-1}-n_m)+\cdots+(-i_1 n_1)}}{n_m^{s_m}\cdots n_1^{s_1}}.$$

The study of these numbers is the Hodge-theoretic analog of higher cyclotomy [9].

The main result below is the crystalline analog of the above for $m \leq 2$. We describe this in more detail. Letting p be a prime which does not divide M, $\pi_1^{mot}(V_M, \cdot)$ has good reduction modulo p, and hence one would expect a crystalline realization of this motive at p. This is completely described by the frobenius action on the de Rham fundamental group $\pi_{1,dR}(V_M, \cdot)$. Let X_M denote the base change of V_M to $K := \mathbb{Q}_p(\zeta)$ and let g_i denote the image under frobenius of the canonical de Rham path from the tangential basepoint 1 at 0 to the tangential basepoint 1 at $\zeta^{i/p}$ (c.f. Sect. 2.2.3) on X_M. As above g_i is naturally in $K\langle\langle e_0, \ldots, e_M\rangle\rangle$. The main result of the paper, Theorem 6.4.3 below, gives an explicit formula for the coefficient of $e_j e_0^{s-1} e_k e_0^{t-1}$ in g_i, in terms of iterated sums, exactly as above. Since g_i is group-like, this also determines the coefficients of terms of the form $e_0^{r-1}e_j e_0^{s-1}e_k e_0^{t-1}$. This might be thought of as the p-adic theory of higher cyclotomy in depth two.

The restriction to depth two is only for computational reasons. The coefficients and the contributions coming from the lower depth values that are to be added for the regularization get very complicated when the depth increases. However, if we are only interested in the $\mathbb{Q}$-vector space, hence algebra, of (cyclotomic) p-adic multi-zeta, we might show that this algebra is contained in an explicit algebra formed by the regularized iterated sums such as the ones that appear in the present paper. This is the content of a work in progress. The $M = 1$ case of this work is done in [12]. The corresponding result for the algebra of cyclotomic p-adic values

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will give a result less precise than the one in the present paper, but it will have the advantage that it will work in all depths and not just depth two.

Comparison with Furusho's definition. Furusho defines p-adic multiple-zeta values (as in the case $M = 1$ above) using Coleman's theory of iterated p-adic integrals instead of our use of Deligne's theory of the comparison isomorphism. However, the information that is contained in both of these definitions are precisely the same and can be transferred from one to another just by basic linear algebraic computations [8, Theorem 2.8, Examples 2.10]. One can similarly define a version of the cyclotomic p-adic multiple-zeta values using Coleman's theory and this will again have a similar relation to our definition. Therefore our explicit computations also give explicit computations for these Coleman integrals. We would like to emphasize that no other explicit expressions are known for the p-adic multiple-zeta values of Furusho ($M = 1$) and their higher cyclotomic analogs ($M > 1$).

We describe the contents of the paper. In Sect. 2, we review the de Rham and crystalline fundamental groups of a curve in a manner which will be suitable for our purposes. In particular, using the horizontality of the frobenius with respect to the canonical connection we arrive at the fundamental differential equation (2.2.9). At the end of this section, we fix the notation for what follows. In Sect. 3, we obtain a certain relation between the coefficients of the power series expansions of rigid analytic functions on $\mathcal{U}_M$, which is essential for the computations (Corollary 3.0.4). In Sect. 4, we compute cyclotomic p-adic multi-zeta values of depth one, which is fairly straightforward. Next there is a section on the type of iterated sums that appear in the computations. These functions will appear as coefficients of the power series expansions above and will satisfy the hypotheses of Corollary 3.0.4, so the inductive process will continue. In Sect. 6, we will proceed with the computation, and finish with the main result in Theorem 6.4.3.

2. The fundamental differential equation

In this section, we will recall the de Rham-crystalline isomorphism theorem through which we will define the cyclotomic p-adic multi-zeta values. Studying the variation of the fundamental torsor of paths on X_M with respect to the standard lift of the frobenius will give us a differential equation (2.2.10) which will be one of the main tools for computing these values.

We start by reviewing the de Rham fundamental group of a curve X/K over an arbitrary field K of characteristic 0 in Sect. 2.1. This is defined as the automorphism group of a fiber functor on the category $\mathrm{Mic}_{uni}(X/K)$ of vector bundles with unipotent connection on X.

In case, X has a compactification $\overline{X}$ such that $H^1(\overline{X}, \mathcal{O}) = 0$, this category has a canonical fiber functor called the de Rham fiber functor. The corresponding fundamental group will be denoted by $\pi_{1,dR}(X/K)$. The fundamental $\pi_{1,dR}(X/K)$-torsor $\mathcal{T}_{dR}$ on X whose fiber at a point $x \in X$ represents the isomorphisms between ω_x, the fiber functor at x, and ω_{dR} is endowed with a natural connection. This connection is explained in detail in Sect. 2.1.3.

Next we describe the crystalline fundamental group of a smooth variety Y/k over a perfect field k of characteristic p in Sect. 2.2. The definition is similar to the

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one above and is based on the category of unipotent overconvergent isocrystals on Y. The essential difference is that there is an action of frobenius on the crystalline fundamental group. When Y has a sufficiently nice lifting over W, the ring of Witt vectors of k, then there is a comparison isomorphism between the crystalline fundamental group of Y and the de Rham fundamental group of the generic fiber of the lifting. Using this isomorphism one defines a frobenius action on this de Rham fundamental group. Applying this to X_M we define the cyclotomic p-adic multi-zeta values in Sect. 2.2.3. The interplay between the canonical connection and the frobenius is summarized by the differential equation in Sect. 2.2.4, which is a restatement of the fact that frobenius is horizontal with respect to the canonical connection.

Fix a prime p, which does not divide M. Let X_M denote the base change of V_M to $K = \mathbb{Q}_p(\zeta)$. Let A_M denote the rings of regular functions on X_M. Finally, let $D_M := \overline{X}_M \setminus X_M$, where $\overline{X}_M = \mathbb{P}^1_K$ is the smooth compactification of X_M.

2.1. The de Rham fundamental group of X_M

We review the theory of the de Rham fundamental group [3, 10.24–10.53, §12], [11, §4, §5]. This theory is valid for any geometrically connected, smooth variety X/K over a field K, of characteristic 0. For simplicity, we will assume that X is a curve and after Sect. 2.1.2, we will assume that X has a compactification $\overline{X}$ that satisfies $H^1(\overline{X}, \mathcal{O}) = 0$.

Suppose that $K \to \mathbb{C}$ is an imbedding and let X_{an} be the underlying topological space of $X_{\mathbb{C}}$. The category of unipotent $\mathbb{C}$-local systems on X_{an} correspond to, after fixing a point x on X_{an}, the unipotent complex representations of the topological fundamental group $\pi_1(X_{an}, x)$. By the Riemann–Hilbert correspondence this category has a completely algebraic description [3, §10.25]: it is equivalent to the category of vector bundles with unipotent integrable connection on $X_{\mathbb{C}}$. This latter category makes sense over an arbitrary field and is the basis of the de Rham theory of the fundamental group.

2.1.1. The fundamental torsor Let K be any field of characteristic 0 and X/K be a smooth and geometrically connected curve and $\mathrm{Mic}_{uni}(X/K)$ denote the category of vector bundles with integrable connection which are unipotent. The objects of this category are vector bundles with connection (E, ∇) on X for which there exist an increasing filtration $\{(F_i, \nabla)\}_i$ by sub-bundles with connection such that there exists an N with $F_i = 0$ for $i < -N$ or $i > N$, and for all i, $(F_i/F_{i-1}, \nabla)$ is isomorphic to either the zero bundle or to $(\mathcal{O}, d)$; and the morphisms are morphisms of vector bundles which commute with the connections. This category naturally forms a tensor category over K in the sense of [3, §5.2], [4].

Let S/K be a scheme over K and let Vec_S denote the category of locally free sheaves of finite rank on S and $\omega : \mathrm{Mic}_{uni}(X/K) \to Vec_S$ be a fiber functor [3, §5.9]. Then to ω there is associated a K-groupoid acting over S [4, §1.6] called the *fundamental groupoid* of X at ω and denoted by $\mathcal{P}_{dR}(X, \omega)$. The fundamental groupoid is faithfully flat and affine over $S \times_K S$ and represents the functor on the

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category of $S \times_K S$-schemes whose T-valued points for any $\pi : T \to S \times_K S$ is the set of $\otimes$-isomorphisms from $\pi^* p_2^* \omega$ to $\pi^* p_1^* \omega$, where $p_1,\ p_2 : S \times_K S \to S$ are the projections [4, §1.11, Théorème 1.12].

Taking the cartesian product of $\mathcal{P}_{dR}(X, \omega) \to S \times_K S$ with the diagonal $\Delta : S \to S \times_K S$ gives $\pi_{1,dR}(X, \omega)$, the *de Rham fundamental group* of X at the fiber functor ω.

Let $x \in S(K)$ then attaching $\mathcal{F}(x)$, the fiber of $\mathcal{F}$ at x, to $\mathcal{F} \in \mathrm{Vec}_S$ gives rise to a fiber functor

$$\omega_x : \mathrm{Mic}_{uni}(X/K) \to \mathrm{Vec}_K.$$

Assuming that $S = X$ and pulling back $\mathcal{P}_{dR}(X, \omega) \to X \times_K X$ via the inclusion $X \to X \times_K X$ that sends s to (s, x) we obtain a torsor $\mathcal{T}_{dR}(X, \omega)_x$ on X under the group scheme $\pi_{1,dR}(X, \omega_x)$. If M is a manifold and $x \in M$, the topological analog of this torsor is the $\pi_1(M, x)$-torsor on M whose fiber at a point y in M is the homotopy class of paths from y to x.

2.1.2. The de Rham fiber functor on X_M From now on we assume that the smooth projective model $\overline{X}$ of X is isomorphic to $\mathbb{P}^1$. In this case, there is a canonical fiber functor [3, §12]:

$$\omega_{dR} : \mathrm{Mic}_{uni}(X/K) \to \mathrm{Vec}_K$$

defined as follows.

For any $(E, \nabla) \in \mathrm{Mic}_{uni}(X/K)$ let (E_{can}, ∇) denote the unique vector bundle with connection on $\overline{X}$ that has logarithmic singularites with nilpotent residues at $\overline{X} \setminus X$. The pair (E_{can}, ∇) is called the canonical extension of the unipotent vector bundle with connection (E, ∇). Since $H^1(\overline{X}, \mathcal{O}) = 0$, the bundle E_{can} is trivial [3, Proposition 12.3] and the functor ω_{dR} defined as

$$\omega_{dR}(E, \nabla) := \Gamma(\overline{X}, E_{can})$$

is a fiber functor [3, §12.4]. For a subscheme Y of $\overline{X}$ let

$$\omega(Y) : \mathrm{Mic}_{uni}(X/K) \to \mathrm{Vec}_Y$$

denote the fiber functor that sends (E, ∇) to $E_{can}|_Y$. There are canonical isomorphisms

$$\omega_{dR} \otimes_K \mathcal{O}_Y \cong \omega(Y) \tag{2.1.1}$$

of fiber functors.

Let $\mathcal{P}_{dR} := \mathcal{P}_{dR}(X, \omega(X))$, $\mathcal{T}_{dR,x} := \mathcal{T}_{dR}(X, \omega(X))_x$, $\overline{\mathcal{P}}_{dR} := \mathcal{P}_{dR}(X, \omega(\overline{X}))$ and $\overline{\mathcal{T}}_{dR,x} := \mathcal{T}_{dR}(X, \omega(\overline{X}))_x$. Finally let $\mathcal{T}_{dR}$ and $\overline{\mathcal{T}}_{dR}$ denote the torsors $\mathcal{T}_{dR,x}$ and $\overline{\mathcal{T}}_{dR,x}$ after the identification (2.1.1) of ω_{dR} with $\omega(x)$. Thus they are (right) $\pi_{1,dR}(X) := \pi_{1,dR}(X, \omega_{dR})$ torsors which depend only on X.

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2.1.3. Connection on the fundamental torsor $\mathcal{T}_{dR}$ Let $\Delta_X \subseteq X \times_K X$ denote the diagonal and $\Delta_X^{(1)}$ its first infinitesimal neighborhood. Let $p_i^{(1)} : \Delta_X^{(1)} \to X$ denote the two projections. Let $\mathcal{T}$ be a right torsor on X under an algebraic group G. A *connection* on $\mathcal{T}$ is an isomorphism $\nabla : p_1^{(1)*}\mathcal{T} \xrightarrow{\sim} p_2^{(1)*}\mathcal{T}$ between the two pull-backs of $\mathcal{T}$ to $\Delta_X^{(1)}$ which reduce to the identity map on Δ_X. If α is a section of $\mathcal{T}$ on an open subset U of X, then $\nabla(p_1^{(1)*}\alpha)$ and $p_2^{(1)*}\alpha$, which abusing the notation we denote by $\nabla\alpha$ and α, denote two sections of $p_2^{(1)*}\mathcal{T}$ on $\Delta_U^{(1)}$. Then $\alpha^{-1}\nabla(\alpha)$ defines a map from $\Delta_U^{(1)}$ to G, whose restriction to Δ_U is the constant map with value the identity element e of G. Giving such a map is equivalent to giving a K-linear map $\mathfrak{m}_{G,e}/\mathfrak{m}_{G,e}^2 \to \Gamma(U, \Omega^1_{U/K})$ and hence an element of $\mathrm{Lie}(G) \otimes_K \Gamma(U, \Omega^1_{U/K})$. Abusing the notation, let us denote the corresponding element in $\mathrm{Lie}(G) \otimes_K \Gamma(U, \Omega^1_{U/K})$, by $\alpha^{-1}\nabla(\alpha)$. For g a morphism from $U \to G$, for any $u \in U$ let $(dg)_u$ denote the linear map from $T^*_{g(u)}G$ to $T^*_u U$. If we identify $T^*_{g(u)}G$ with $T^*_e G = (\mathrm{Lie}(G))^\vee$, the dual of $\mathrm{Lie}(G)$, via multiplication by $g(u)$ for every u, we get a linear map $g^{-1}dg$ from $(\mathrm{Lie}(G))^\vee$ to $\Gamma(U, \Omega^1_{U/K})$. Again abusing the notation, we denote the corresponding element in $\mathrm{Lie}(G) \otimes_K \Gamma(U, \Omega^1_{U/K})$ by $g^{-1}dg$. Note that αg denotes another section of the torsor $\mathcal{T}$ over U, and with the notation above, we have the following formula

$$(\alpha g)^{-1}\nabla(\alpha g) = g^{-1}dg + g^{-1}(\alpha^{-1}\nabla(\alpha))g.$$

In case $\mathcal{T}$ is the trivial G-torsor $G \times_K X$ then a connection on $\mathcal{T}$ is determined by $-e^{-1}\nabla(e) \in \mathrm{Lie}(G) \otimes_K \Gamma(X, \Omega^1_{X/K})$, where e is the identity section. If $-e^{-1}\nabla(e) = \Omega$, then for every section g of $\mathcal{T}$,

$$g^{-1}\nabla(g) = g^{-1}dg - g^{-1}\Omega g. \tag{2.1.2}$$

By the definition of $\mathcal{P}_{dR}$, the sections of its restriction to $\Delta_X^{(1)}$ are $\otimes$-isomorphisms from $p_1^{(1)*}\omega(X)$ to $p_2^{(1)*}\omega(X)$ where $p_i^{(1)} : \Delta_X^{(1)} \to X$ are the two projections. If $(E, \nabla) \in \mathrm{Mic}_{uni}(X/K)$, then $p_i^{(1)*}\omega(X)(E, \nabla) = p_i^{(1)*}(E)$ and the connection ∇ induces an isomorphisms from $p_1^{(1)*}(E)$ to $p_2^{(1)*}(E)$ reducing to the identity on the diagonal. This in turn induces a canonical isomorphism between the above fiber functors, and hence a section of $\mathcal{P}_{dR}|_{\Delta_X^{(1)}}$ over $\Delta_X^{(1)}$ which is the identity section when restricted to Δ_X. This defines an isomorphism between the two pull-backs of $\mathcal{T}_{dR}$ to $\Delta_X^{(1)}$, which, by definition, is a connection on the $\pi_{1,dR}(X)$-torsor $\mathcal{T}_{dR}$.

Because of the canonical isomorphism $\mathcal{T}_{dR} \cong \pi_{1,dR}(X) \times_K X$ (2.1.1) a connection on $\mathcal{T}_{dR}$ is completely determined by $e^{-1}\nabla(e) \in \mathrm{Lie}\,\pi_{1,dR}(X) \otimes_K \Gamma(X, \Omega^1_{X/K})$ as above. Let $\Omega_{can} \in H^1_{dR}(X)^\vee \otimes_K H^1_{dR}(X)$ denote the canonical element, where $H^1_{dR}(X)^\vee$ denotes the dual of $H^1_{dR}(X)$. Since X is affine, $H^1(X, \mathcal{O}_X) = 0$ and $\Gamma(X, \Omega^1_{X/K}) \xrightarrow{\sim} H^1_{dR}(X)$. Below we will describe an imbedding

$$H^1_{dR}(X)^\vee \to \mathrm{Lie}\,\pi_{1,dR}(X).$$

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We continue to denote the image of Ω_{can} in $\mathrm{Lie}\,\pi_{1,dR}(X)\otimes_K \Gamma(X,\Omega^1_{X/K})$, via the map induced by this imbedding, by the same notation. Then by Deligne [3, §12.12], we have

$$e^{-1}\nabla(e) = -\Omega_{can} \tag{2.1.3}$$

in $\mathrm{Lie}\,\pi_{1,dR}(X)\otimes_K \Gamma(X,\Omega^1_{X/K})$.

From now on we let $X = X_M$ and $K = \mathbb{Q}_p(\zeta)$. For any $x \in \overline{X}\setminus X$ and (E,∇), we have the residue endomorphism

$$\mathrm{res}_x : E_{can}(x) \to E_{can}(x),$$

induced by the map that sends the local section u of E_{can} near x, to $(\nabla(u), t\frac{\partial}{\partial t})$, where t is a uniformizer at x. The residue endomorphism is independent of the choice of a uniformizer and satisfies,

$$\mathrm{res}_x((E_1,\nabla_1)\otimes(E_2,\nabla_2)) = 1\otimes \mathrm{res}_x(E_2,\nabla_2) + \mathrm{res}_x(E_1,\nabla_1)\otimes 1.$$

Hence $\mathrm{res}_x \in \mathrm{Lie}\,\pi_{1,dR}(X,\omega(x)) \simeq \mathrm{Lie}\,\pi_{1,dR}(X)$, under the identification (2.1.1). If Ext^i_X denotes the extension groups in the category of modules with integrable connection on X then we have $H^1_{dR}(X) = \mathrm{Ext}^1_X((\mathcal{O},d),(\mathcal{O},d))$. Using this, res_x defines an element in $H^1_{dR}(X)^\vee$ as follows. Given $\omega \in H^1_{dR}(X)$ we get a vector bundle with connection (E_ω,∇) in the above extension group. The fiber at x gives an extension $0\to K\to E_{\omega,can}(x)\to K\to 0$. Let f_1 be the image of $1\in K$ in $E_{\omega,can}(x)$ and let f_2 be any lift of $1\in K$ in $E_{\omega,can}(x)$. Then $\mathrm{res}_x(f_2) = \lambda f_1$ for some $\lambda\in K$, and the map that sends ω to λ defines an element of $H^1_{dR}(X)^\vee$. Viewed in this manner the subspace that res_x, with $x\in\overline{X}\setminus X$, generate in $\mathrm{Lie}\,\pi_{1,dR}(X)$ is precisely $H^1_{dR}(X)^\vee$. For $1\le i\le M$, we let $e_i\in \mathrm{Lie}\,\pi_{1,dR}(X_M)$ denote res_{ζ^i} and e_0 denote res_0. If we also put $\omega_0 := dlog z$ and $\omega_i := dlog(z-\zeta^i)$, for $1\le i\le M$, then

$$\Omega_{can} = \sum_{0\le i\le M} e_i\omega_i. \tag{2.1.4}$$

The de Rham fundamental group of X_M has a simple description. For any K-algebra A, denote the associative (non-commutative) algebra of formal power series in $\{e_i | 0\le i\le m\}$ over A by $A\langle\langle e_0,\dots,e_M\rangle\rangle$ and let

$$\mathcal{U}_{dR}(A) := A\langle\langle e_0,\dots,e_M\rangle\rangle.$$

Then the universal enveloping algebra of $\pi_{1,dR}(X_M)$ is $\mathcal{U}_{dR}(X_M)(K)$. The co-product of the Hopf algebra structure on $\mathcal{U}_{dR}(A)$ is induced by the fact that e_i are primitive elements: $\Delta(e_i) = 1\otimes e_i + e_i\otimes 1$, for $1\le i\le M$. The A-valued points of $\pi_{1,dR}(X_M)$ then correspond to the group-like elements in $\mathcal{U}_{dR}(A)$, i.e. elements g satisfying $\Delta(g) = g\hat{\otimes}g$ and with constant term equal to 1. For any g let $\underline{g}$ denote the image of g under the Hopf algebra automorphism of $\mathcal{U}_{dR}(A)$ that sends e_i to $p^{-1}e_i$, for all i.

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The canonical connection on $\mathcal{T}_{dR} = \pi_{1,dR}(X_M) \times X_M$ can be described as follows. A section of $\mathcal{T}_{dR}$ over X_M is given by a group-like element $\alpha(z) \in \mathcal{U}_{dR}(A_M)$, where z denotes the parameter on Spec $A_M = X_M \subseteq \mathbb{A}^1_K$. Let

$$d : \mathcal{U}_{dR}(A_M) \to \mathcal{U}_{dR}(A_M)\hat{\otimes}_{A_M}\Omega^1_{A_M/K}$$

denote the continuous differential extending the canonical differential $A_M \to \Omega^1_{A_M/K}$ such that $d(e_i) = 0$, for $0 \leq i \leq M$. In other words, applying d to an element $\alpha(z)$ amounts to applying d to each coefficient of $\alpha(z)$. Note that the identity section e of the trivial torsor $\mathcal{T}_{dR}$ is given by the element $1 \in \mathcal{U}_{dR}(A_M)$. By (2.1.3) and (2.1.4), the action of the canonical connection on the identity section e of the trivial torsor $\mathcal{T}_{dR}$ is described by the formula:

$$e^{-1}\nabla(e) = -\Omega_{can} = -\sum_{0\leq i\leq M} e_i\omega_i. \tag{2.1.5}$$

The action of the connection on arbitrary sections of $\mathcal{T}_{dR}$ was described by Eq. (2.1.2). Therefore for any group-like element $\alpha(z) \in \mathcal{U}_{dR}(A_M)$, we have:

$$\alpha(z)^{-1}\nabla(\alpha(z)) = \alpha(z)^{-1}d\alpha(z) - \alpha(z)^{-1}\left(\sum_{0\leq i\leq M} e_i\omega_i\right)\alpha(z) \tag{2.1.6}$$

in Lie $\pi_{1,dR}(X_M)\hat{\otimes}\Omega^1_{A_M/K}$.

2.2. *Crystalline fundamental group of X_M*

We review the theory of the crystalline fundamental group as described in [3, §11] and [11, §2.4]. The crystalline fundamental group can be defined for any smooth variety Y/k over a perfect field k, and if it has a smooth compactification $\overline{Y}$ such that $\overline{Y} \setminus Y$ is a simple normal crossings divisor in $\overline{Y}$, it agrees with the appropriately defined crystalline fundamental group of the log scheme $\overline{Y}_{log}$. This will be essential when constructing a comparison isomorphism between the de Rham fundamental group of a lifting of Y and the crystalline fundamental group of Y. The comparison theorem between the crystalline and de Rham fundamental groups will give us the frobenius map on the de Rham fundamental group which will play a central role.

2.2.1. The de Rham-crystalline comparison Let k be a perfect field of characteristic p, with W the ring of Witt vectors and K its field of fractions. For a smooth variety Y/k, we have $\mathrm{Isoc}^{\dagger}_{uni}(Y/W)$, the category of unipotent overconvergent isocrystals on Y/W [11, §2.4.1]. If ω is a fiber functor on $\mathrm{Isoc}^{\dagger}_{uni}(Y/W)$, the automorphisms of ω is represented by *the crystalline fundamental group* $\pi^{\dagger}_{1,crys}(Y,\omega)$ of Y [11]. Now suppose that Y has a smooth compactification $\overline{Y}/k$ such that $D := \overline{Y} \setminus Y$ is a simple normal crossings divisor in $\overline{Y}$, and let $\overline{Y}_{\log}$ denote the canonical log structure on $\overline{Y}$ associated to the divisor D. Shiho's theorem [10] implies that the restriction functor

$$\mathrm{Isoc}^{c}_{uni}(\overline{Y}_{log}/W) \to \mathrm{Isoc}^{\dagger}_{uni}(Y/W), \tag{2.2.1}$$

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from the category of unipotent convergent log isocrystals on $\overline{Y}_{log}$ to $\mathrm{Isoc}^{\dagger}_{uni}(Y/W)$, is an equivalence of categories [11, Lemma 2].

This is in complete analogy with the situation over the field K of characteristic 0. If X/K is a smooth variety with a smooth compactification $\overline{X}/K$ and with simple normal crossings divisor $E := \overline{X} \setminus X$ in $\overline{X}$ then the restriction

$$\mathrm{Mic}_{uni}(\overline{X}_{\log}/K) \to \mathrm{Mic}_{uni}(X/K) \tag{2.2.2}$$

gives an equivalence of categories [2, II.5.2].

The de Rham-crystalline comparison can be described as follows. Suppose that $\overline{\mathcal{Z}}/W$ is a smooth, projective scheme with geometrically connected fibers and with $\mathcal{F} \subseteq \overline{\mathcal{Z}}$ a relative simple normal crossings divisor. Let $\mathcal{Z} := \overline{\mathcal{Z}} \setminus \mathcal{F}$, and let $(\overline{X}, X, E)$ and $(\overline{Y}, Y, D)$ denote the corresponding data over the generic and special fibers respectively. The canonical functor

$$\mathrm{Mic}_{uni}(\overline{X}_{\log}/K) \to \mathrm{Isoc}^{c}_{uni}(\overline{Y}_{log}/W)$$

is an equivalence which, when combined with (2.2.2) and (2.2.1) gives the equivalence

$$\mathrm{Mic}_{uni}(X/K) \to \mathrm{Isoc}^{\dagger}_{uni}(Y/W). \tag{2.2.3}$$

Choosing a (tangential) basepoint z on $\mathcal{Z}$, we get an isomorphism

$$\pi^{\dagger}_{1,crys}(Y, y) \xrightarrow{\sim} \pi_{1,dR}(X, x),$$

where x and y are the generic and special fibers of z.

Let $\sigma : W \to W$ denote the lifting of the p-power frobenius map on k, and let $\mathcal{Z}^{(p)}$, denote the base change of $\mathcal{Z}/W$ via σ and $X^{(p)}$, $Y^{(p)}$ etc. the corresponding fibers of $\mathcal{Z}^{(p)}$. The relative frobenius morphism induces a $\otimes$-functor $F^* : \mathrm{Isoc}^{\dagger}_{uni}(Y^{(p)}/W) \to \mathrm{Isoc}^{\dagger}_{uni}(Y/W)$, and hence a map $F_* : \pi^{\dagger}_{1,crys}(Y, y) \to \pi^{\dagger}_{1,crys}(Y^{(p)}, y^{(p)})$. This, together with the above isomorphism, gives a morphism

$$F_* : \pi_{1,dR}(X, x) \to \pi_{1,dR}(X^{(p)}, x^{(p)}). \tag{2.2.4}$$

Similarly, for a pair of (tangential) basepoints z_1 and z_2 we obtain a morphism

$$F_* : {}_{x_2}\mathcal{P}_{dR}(X)_{x_1} \to {}_{x_2^{(p)}}\mathcal{P}_{dR}(X^{(p)})_{x_1^{(p)}}. \tag{2.2.5}$$

2.2.2. Tangential basepoints in the crystalline case The definitions of the crystalline and de Rham fundamental groups look similar. However, a major difference is that one does not have a canonical fiber functor on $\mathrm{Isoc}^{\dagger}_{uni}(Y/W)$ analogous to ω_{dR}. Therefore, in order to make use of the comparison isomorphism one cannot simply use ω_{dR} and one needs to choose specific basepoints on the variety which might increase the places of bad reduction for the motivic fundamental group. In the case of V_M/E, which is the important case for us, this takes the following form. Let $S := \mathrm{Spec}\, \mathcal{O}_E[M^{-1}]$, and let $\mathcal{V}_M := \mathbb{G}_{m,S} \setminus \mu_{M,S}$ be the standard model of V_M over S, with $\overline{\mathcal{V}}_M := \mathbb{P}^1_S$. Then if we let $x \in V_M(E)$, the fundamental group

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$\pi_1^{mot}(V_M, x)$ will have bad reduction at those primes $\mathfrak{p} \in S$ for which x does not have finite reduction on $\mathcal{V}_M \subseteq \overline{\mathcal{V}}_M$ at $\mathfrak{p}$. Deligne overcomes this problem by introducing special *tangential* basepoints at the missing points of the variety and thereby avoids increasing the places of bad reduction [3, §15]. In the crystalline case, we gave a detailed exposition of the tangential basepoints in [11, §3]. Below we will only recall the definition.

Let $\mathcal{Z}/W$ be as above with relative dimension 1, for simplicity, and let $z \in (\overline{\mathcal{Z}} \setminus \mathcal{Z})(W)$ with fibers x and y. Let $T_z^\times(\overline{\mathcal{Z}})/W$ denote the tangent space of $\overline{\mathcal{Z}}$ at z with the zero section removed. It is (non-canonically) isomorphic to $\mathbb{G}_m/W$. Fix $w \in T_z^\times(\overline{\mathcal{Z}})(W)$, with fibers $v \in T_y^\times(\overline{Y})(k)$ and $u \in T_x^\times(\overline{X})(K)$. The *crystalline tangential basepoint at* v is a fiber functor

$$\omega_v : \mathrm{Isoc}^\dagger_{uni}(Y/W) \to \mathrm{Vec}_K.$$

Corresponding to the lifting $\mathcal{Z}$, z and w and the identification of $\mathrm{Isoc}^\dagger_{uni}(Y/W)$ with $\mathrm{Mic}_{uni}(X/K)$ described above this fiber functor corresponds to the fiber functor

$$\omega_x : \mathrm{Mic}_{uni}(X/K) \to \mathrm{Vec}_K$$

which associates to (E, ∇) the fiber $E_{can}(x)$ of its canonical extension at x. In this description it looks as if the fiber functor ω_v depends only on y and not on the tangent vector. However, we would like to emphasize that the identification of ω_v with ω_x *depends on* the choice of an integral model $(\overline{\mathcal{Z}}, z)$ as above and in order to define canonical isomorphisms between the fiber functors which correspond to different choices of integral models one *needs* to fix a tangent vector at y. Since it has no direct consequence for what follows, we refer the reader to [11, §3] for details on the effect of choosing different integral models and the importance of fixing a tangent vector.

2.2.3. Cyclotomic p-adic multi-zeta values Let t_0 denote the tangent vector 1 at 0 and t_i denote the tangent vector 1 at ζ^i, for $1 \leq i \leq M$. In the following, we identify the tangent space at a point x in $\mathbb{A}^1$ with $\mathbb{A}^1$ itself. Therefore, if z denotes the coordinate function on $\mathbb{A}^1$, then t_i is the tangent vector that satisfies $dz(t_i) = 1$. For $1 \leq i \leq M$, let $\bar{i}$ denote the unique integer such that $1 \leq \bar{i} \leq M$ and $M|(\bar{i} - pi)$. Similarly, let $\underline{i}$ denote the unique integer such that $1 \leq \underline{i} \leq M$ and $M|(i - p\underline{i})$ and let $\underline{0} = \bar{0} = 0$. By (2.1.1), for (tangential) basepoints x_i on X_M, there are canonical isomorphisms between ω_{x_i}. This gives a canonical element ${}_{x_2}\gamma_{x_1}$ of ${}_{x_2}\mathcal{P}_{dR}(X_M)_{x_1}$, which we call the *canonical de Rham path* from x_1 to x_2.

For any $1 \leq i \leq M$, we have elements ${}_{t_0}\gamma_{t_i} \cdot F_*({}_{t_{\underline{i}}}\gamma_{t_0}) \in \pi_{1,dR}(X_M, t_0)(K)$, with $K = \mathbb{Q}_p(\zeta)$. Identifying ω_{t_0} with ω_{dR} using (2.1.1), we obtain elements

$$g_i \in \pi_{1,dR}(X_M)(K) \subseteq K\langle\langle e_0, \ldots, e_M\rangle\rangle,$$

for $1 \leq i \leq M$. Let $g := g_M$. We denote the coefficient of the monomial $e_{i_1} \cdots e_{i_n}$ in g by $g[e_{i_1} \cdots e_{i_n}]$ and call it a *cyclotomic p-adic multi-zeta value*. In analogy with [11, Definition 3], for $1 \leq i_1, \ldots, i_k \leq M$, and $1 \leq s_1, \ldots, s_k$, let us put

$$g[e_0^{s_k-1}e_{i_k} \cdots e_0^{s_1-1}e_{i_1}] = p^{\sum s_i}\zeta_p(s_k, \ldots, s_1; i_k, \ldots, i_1).$$

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We call k, the *depth* and $\sum s_i$, the *weight* of the multi-zeta value.

The p-adic cyclotomic multi-zeta values completely determine the frobenius action on $\pi_{1,dR}(X_M, t_0) \xrightarrow{\sim} \pi_{1,dR}(X_M)$ as follows.

First note that

$$F_*(e_0) = pe_0, \;\; F_*(e_i) = pg_{\bar{i}}^{-1} e_{\bar{i}} g_{\bar{i}}. \tag{2.2.6}$$

On the other hand, all the g_i are determined by g through functoriality. Let α_i denote the automorphism of X_M given by $\alpha_i(z) = \zeta^i z$. Then $\alpha_{i*}(e_0) = e_0$ and $\alpha_{i*}(e_j) = e_{i+j}$, where $i + j$ is between 1 and M computed modulo M. On the special fiber we have $F \circ \alpha_{\underline{i}} = \alpha_i \circ F$. By the functoriality of frobenius we have

$$\alpha_{i*}(g_j) = g_{i+j}. \tag{2.2.7}$$

2.2.4. The differential equation satisfied by the frobenius We will first recall the explicit description of the frobenius on $\mathrm{Mic}_{uni}(\overline{X}_{M,\log}/K)$, which was explained in detail in [11, §2.4.2].

Let $\overline{\mathcal{P}}_M/W$ denote the formal scheme obtained by completing $\mathbb{P}^1_W$ along its closed fiber. Note that $\mathbb{P}^1_W$ is an integral model of $\overline{X}_M = \mathbb{P}^1_K$. Let $\mathcal{D}_M$ denote the divisor on $\overline{\mathcal{P}}_M$ obtained by completing the Zariski closure of D_M on $\mathbb{P}^1_W$ along its closed fiber. Let $\{\overline{\mathcal{P}}_i\}_{1\le i\le n}$ be an open cover of $\overline{\mathcal{P}}_M$, and $\mathcal{F}_i : \overline{\mathcal{P}}_i \to \overline{\mathcal{P}}_i$ be a lifting of the frobenius such that $\mathcal{F}_i^*(\mathcal{D}_M \cap \overline{\mathcal{P}}_i) = p \cdot (\mathcal{D}_M \cap \overline{\mathcal{P}}_i)$. For a formal scheme $\mathcal{P}/W$, let $\mathcal{P}_K$ denote the associated rigid analytic space over K. Now given (E, ∇) in $\mathrm{Mic}_{uni}(\overline{X}_{M,\log})$, its pull-back via the frobenius is defined as the vector bundle with connection whose restriction to $\overline{\mathcal{P}}_{iK}$ is given by $\mathcal{F}_{i,K}^*(E, \nabla)|_{\overline{\mathcal{P}}_{i,K}}$. The isomorphisms between the different pull-backs are given by using the fact that the connections converge within a p-adic disk of radius one and that the different liftings of the frobenius lie in the same disk [11, §2.4.2].

The computation of the cyclotomic p-adic multi-zeta values would be an easier problem if there were a global lifting $\mathcal{F}_M$ of frobenius to $\overline{\mathcal{P}}_M/W$ which satisfies the property that $\mathcal{F}_M^*(\mathcal{D}_M) = p \cdot \mathcal{D}_M$. Unfortunately, such a global lifting does not exist even for $M = 1$. The standard way to proceed would be to choose several local liftings of frobenii and piece the information obtained together. However, it turns out that using several frobenii as above makes the computations too complicated for any practical use. Instead, what we do is to work with a single local lifting of frobenius $\mathcal{F}$ and study the associated differential equation in detail to relate the function $g_{\mathcal{F}}$ to cyclotomic p-adic multi-zeta values.

Let $\overline{\mathcal{P}}$ denote the completion of $X_M \cup \{0, \infty\}$ along the closed fiber and let $\mathcal{F}(z) = z^p$. Then $\mathcal{F} : \overline{\mathcal{P}} \to \overline{\mathcal{P}}$ is a lifting of frobenius that satisfies $\mathcal{F}^*(0) = p \cdot (0)$ and $\mathcal{F}^*(\infty) = p \cdot (\infty)$. We identify the $\pi_{1,dR}(X_M, t_0)$-torsor of paths that start at t_0 (Sect. 2.1.1) with $\overline{T}_{dR}$ (Sect. 2.1.3) by using the identification of $\omega(t_0)$ and ω_{dR} (Sect. 2.1.2). Recall that the notation ${}_x\gamma_{t_0}$ in the beginning of Sect. 2.2.3 denotes the canonical de Rham path from the tangential basepoint t_0 to the point x. If we let x vary we will obtain a section γ_{t_0} of the $\pi_{1,dR}(X_M, t_0)$-torsor of paths that start at t_0. We identify this torsor with $\overline{T}_{dR}$ as above and continue to denote the corresponding section by γ_{t_0}. The principal part of $\mathcal{F}$ sends t_0 to itself [11, §3.2.(ii)]. Then by

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the description of the frobenius map above, we obtain the following commutative diagram:

$$\begin{array}{ccc} \overline{\mathcal{T}}_{dR}|_{\mathcal{U}_M} & \xrightarrow{\mathcal{F}_*} & \mathcal{F}^*\overline{\mathcal{T}}_{dR}|_{\mathcal{U}_M} \\ \downarrow \nabla & & \downarrow \mathcal{F}^*\nabla \\ \mathrm{Lie}\,\pi_1(X_M)\hat{\otimes}\Omega^1_{\mathcal{U}_M}(log(0)) & \xrightarrow{\mathrm{Lie}F_*} & \mathrm{Lie}\,\pi_1(X_M)\hat{\otimes}\Omega^1_{\mathcal{U}_M}(log(0)), \end{array}$$

where $\mathcal{U}_M := \overline{\mathcal{P}}_K$ is the rigid analytic space associated to $\overline{\mathcal{P}}$. Let $\mathcal{A}_M$ denote the ring of rigid analytic functions on $\mathcal{U}_M$. Applying the lift $\mathcal{F}$ of frobenius to the section γ_{t_0} of $\overline{\mathcal{T}}_{dR}|_{\mathcal{U}_M}$, we obtain a section $\mathcal{F}_*(\gamma_{t_0})$ of $\mathcal{F}^*\overline{\mathcal{T}}_{dR}|_{\mathcal{U}_M}$. Using the canonical de Rham path to trivialize the torsor $\overline{\mathcal{T}}_{dR}|_{\mathcal{U}_M}$, γ_{t_0} corresponds to the identity element e of the trivial torsor or equivalently the element 1 in $\mathcal{U}_{dR}(\mathcal{A}_M)$ and $\mathcal{F}_*(\gamma_{t_0})$ corresponds to an element in $\mathcal{U}_{dR}(\mathcal{A}_M)$ which we will denote by $g_{\mathcal{F}}$. By the commutative diagram and the notation above, we have the equality

$$\mathrm{Lie}F_*(e^{-1}\nabla(e)) = g_{\mathcal{F}}^{-1}(\mathcal{F}^*\nabla(g_{\mathcal{F}})). \tag{2.2.8}$$

Let us first look at the left hand side of the equality. By Eq. (2.1.5), we have $e^{-1}\nabla(e) = -\sum_{0\le i\le M} e_i\omega_i$ and hence

$$\mathrm{Lie}F_*(e^{-1}\nabla(e)) = -\sum_{0\le i\le M} F_*(e_i)\omega_i.$$

Now looking at the right hand side, the value of the connection $\mathcal{F}^*\nabla$ on the identity element of the trivial torsor is described as follows:

$$e^{-1}\mathcal{F}^*\nabla(e) = \mathcal{F}^*(e^{-1}\nabla(e)) = \mathcal{F}^*\left(-\sum_{0\le i\le M} e_i\omega_i\right) = -\sum_{0\le i\le M} e_i\mathcal{F}^*\omega_i.$$

Then just as in (2.1.2) and (2.1.6), the value $g_{\mathcal{F}}^{-1}\mathcal{F}^*\nabla(g_{\mathcal{F}})$ is given by

$$\begin{aligned} g_{\mathcal{F}}^{-1}\mathcal{F}^*\nabla(g_{\mathcal{F}}) &= g_{\mathcal{F}}^{-1}dg_{\mathcal{F}} + g_{\mathcal{F}}^{-1}(e^{-1}\mathcal{F}^*\nabla(e))g_{\mathcal{F}} \\ &= g_{\mathcal{F}}^{-1}dg_{\mathcal{F}} - g_{\mathcal{F}}^{-1}\left(\sum_{0\le i\le M} e_i\mathcal{F}^*\omega_i\right)g_{\mathcal{F}}. \end{aligned}$$

Therefore by (2.2.8) we obtain the differential equation

$$-\sum_{0\le i\le M} F_*(e_i)\omega_i = g_{\mathcal{F}}^{-1}dg_{\mathcal{F}} - g_{\mathcal{F}}^{-1}\left(\sum_{0\le i\le M} e_i\mathcal{F}^*\omega_i\right)g_{\mathcal{F}}. \tag{2.2.9}$$

Putting $g_0 = 1$, and $\underline{0} = \overline{0} = 0$, and using (2.2.6), we can rewrite this as:

$$dg_{\mathcal{F}} = \left(\sum_{0\le i\le M} e_i\mathcal{F}^*\omega_i\right)g_{\mathcal{F}} - g_{\mathcal{F}}\left(\sum_{0\le i\le M} pg_i^{-1}e_ig_i\omega_{\underline{i}}\right). \tag{2.2.10}$$

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Even though it is clear from the notation, we would like to emphasize that $g_{\mathcal{F}}$ is a non-canonical, technical object, which *depends on* a lifting of the frobenius unlike the cyclotomic p-adic multi-zeta values. It could be thought of as an analog of the multiple polylogarithms on a disc which depend on the specific path chosen from 0 to a point in that disc.

3. Rigid analytic functions on $\mathcal{U}_M$

Recall that in Sect. 2.2.4, we chose the lifting $\mathcal{F}$ of frobenius on $\overline{\mathcal{P}}$ and set $\mathcal{U}_M$ to be the rigid analytic space associated to $\overline{\mathcal{P}}$. Reinterpreting the action of $\mathcal{F}$ on the fundamental de Rham path naturally gave us $g_{\mathcal{F}}$ whose coefficients are rigid analytic functions on $\mathcal{U}_M$. For our purposes the interesting points are the roots of unity, all of which, unfortunately, lie outside $\mathcal{U}_M$. As a first step we will try to get a hold of $g_{\mathcal{F}}$ and in particular its value at ∞. The following fundamental proposition and its corollary will serve that purpose.

Let $D(a, r)$ and $D(a, r)^\circ$ denote the closed and open disks of radius r around a. Then $\mathcal{U}_M = \mathbb{P}^1_K \setminus \cup_{1\leq i\leq M} D(\zeta^i, 1)^\circ$. The following proposition, which is a generalization of (Prop. 2, [11]), describes rigid analytic functions on $\mathcal{U}_M$ in terms of their power series expansions around 0.

Proposition 3.0.1. *Let f be a rigid analytic function on $\mathcal{U}_M$ with $f(0) = 0$ and a power series expansion*

$$f(z) = \sum_{0<n} a_n z^n$$

around 0. Then the sequence of rational functions

$$f_N(z) := \frac{1}{1 - z^{Mp^N}} \sum_{0<n\leq Mp^N} a_n z^n$$

converge uniformly on $\mathcal{U}_M$ to f. The value of f at ∞ is given by

$$f(\infty) = -\lim_{N\to\infty} a_{Mp^N}.$$

Proof. Since f is rigid analytic on the affinoid $\mathcal{U}_M$, it is a uniform limit of rational functions with poles outside $\mathcal{U}_M$ (Sect. 2.2, [7]). We may also assume, without loss of generality, that these rational functions are 0 at 0.

Claim 3.0.2. If $r(z)$ is a rational function with poles outside $\mathcal{U}_M$, then $r(z)$ is a linear combination of functions of the form

$$\frac{z^i}{(1 - az^M)^k},$$

for some $0 \leq k$, $0 \leq i < M$, and $|1 - a| < 1$.

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Proof of the claim. By the method of partial fractions, $r(z)$ is a linear combination of rational functions of the form

$$\frac{1}{(1-bz)^t}, \tag{3.0.11}$$

with $|b-\zeta^i|<1$, for some $0\le i<M$, and $0\le t$. Therefore, we need to prove the statement only for rational functions as in (3.0.11). Note that

$$\frac{1}{(1-bz)^t}=\frac{p(z)}{(1-az^M)^t}=\sum_{0\le i\le M-1} z^i\frac{q_i(1-az^M)}{(1-az^M)^t},$$

for some polynomials $p(z)$ and $q_i(z)$, $0\le i\le M-1$ and $a=b^M$. Since $|1-a|<1$, and the left hand side does not have a pole at ∞, the right hand side is exactly as in the form stated in the claim. This proves the claim. □

Using the claim above, we will prove the following estimate on the coefficients of the Taylor expansion of f:

Claim 3.0.3. For $n\in\mathbb{N}:=\{1,2,3,\cdots\}$, let $n|_N$ denote the unique integer such that $0<n|_N\le Mp^N$, and Mp^N divides $n-n|_N$. If we let $c_N:=\sup_{n\in\mathbb{N}}|a_n-a_{n|_N}|$, then

$$\lim_{N\to\infty} c_N=0.$$

Proof of the claim. First we note that, for $1\le k$, $0\le i<M$ and $|1-a|<1$,

$$\frac{z^i}{(1-az^M)^k}=\sum_{0\le n}\binom{n+k-1}{k-1}a^n z^{i+Mn}=:\sum_{0\le n}a_n z^n;$$

satisfy the property in the claim. If $n\not\equiv i\,(mod\ M)$ then $a_n=0$ and hence

$$\begin{aligned} c_N&=\sup_{\substack{n\in\mathbb{N}\\ n\equiv i\,(mod\ M)}}|a_n-a_{n|_N}|\le\sup_{s,t\ge 0}|a_{i+M(t+sp^N)}-a_{i+Mt}|\\ &=\sup_{\substack{t\ge 0\\ s\ge 1}}|q(t+sp^N)a^{sp^N}-q(t)|=:d_N, \end{aligned}$$

where $q(t):=\binom{t+k-1}{k-1}$ is a polynomial of degree $k-1$ in t. Let α denote the maximum of the absolute value of the coefficients of $q(t)$, and $\beta:=|a-1|<1$. Since $|a^p-1|\le\max(\beta/p,\beta^p)$, choosing N_0 sufficiently large $|a^{p^{N_0}}-1|\le p^{-1}$, and hence for $N\ge N_0$, $|a^{p^N}-1|\le p^{-(N-N_0)}$. Then for $N\ge N_0$, $d_N\le\alpha(p^{-N}+p^{-(N-N_0)})$, and hence $\lim_{N\to\infty}d_N=\lim_{N\to\infty}c_N=0$.

Since any rational function $r(z)$, whose poles are outside $\mathcal{U}_M$, is a linear combination of functions as above (Claim 3.0.2), the statement is true for $r(z)$. Note that for any power series $g(z):=\sum_{0\le n}b_nz^n$, which is convergent on $D(0,1)^\circ$:

$$\sup_{0\le n}|b_n|\le\sup_{|z|<1}|g(z)|. \tag{3.0.12}$$

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Let (r_m) be a sequence of rational functions which are 0 at 0, have poles outside $\mathcal{U}_M$, and which converge, uniformly on $\mathcal{U}_M$, to f. Letting

$$r_m(z) := \sum_{0<n} a_n^{(m)} z^n,$$

and $c_N^{(m)} := \sup_{n\in\mathbb{N}} |a_n^{(m)} - a_{n|N}^{(m)}|$; we know that $\lim_{N\to\infty} c_N^{(m)} = 0$, for all m. By uniform convergence and (3.0.12), $\lim_{m\to\infty} \sup_{N\in\mathbb{N}} |c_N^{(m)} - c_N| = 0$. This implies the claim. □

Now, note that

$$f_{N+1}(z) - f_N(z) = \frac{1}{1 - z^{Mp^{N+1}}} \sum_{0<n\le Mp^{N+1}} (a_n - a_{n|N}) z^n.$$

Note that $z \in \mathcal{U}_M$ if and only if $1 \le |1 - z^M|$. Letting $0 < n \le Mp^{N+1}$,

$$\left|\frac{z^n}{1 - z^{Mp^{N+1}}}\right| = |z^n| < 1,$$

if $|z| < 1$; and

$$\left|\frac{z^n}{1 - z^{Mp^{N+1}}}\right| \le \frac{1}{|(1/z^M)^{p^{N+1}} - 1|} \le 1,$$

if $1 \le |z|$ and $z \in \mathcal{U}_M$. Therefore,

$$\sup_{z\in\mathcal{U}_M} |f_{N+1}(z) - f_N(z)| \le c_N,$$

and we conclude, by Claim 3.0.3, that (f_N) converges uniformly to a rigid analytic function on $\mathcal{U}_M$. To see that this function, indeed, is f, we note that for $|z| < 1$, $|f(z) - f_N(z)| \le c_N$. Then again Claim 3.0.3 implies the assertion. The last assertion follows from $f_N(\infty) = -a_{Mp^N}$. □

The following corollary will play a key role in the computations. Let q be the cardinality of the residue field of $K = \mathbb{Q}_p(\zeta)$. In the applications, the existence of the limit $\lim_{N\to\infty} lq^N a_{lq^N}$ will be shown by explicit computations.

Corollary 3.0.4. *Let $f(z) = \sum_{0<n} a_n z^n$ be as in Proposition 3.0.1, and $0 < l \le pM$ then*

$$\lim_{N\to\infty} |a_{lq^{N+1}} - a_{lq^N}| = 0.$$

If $\lim_{N\to\infty} lq^N a_{lq^N}$ exists then it is equal to 0.

Proof. Since $M|(q-1)$, $Mp^N|(lq^{N+1} - lq^N)$ and hence $|a_{lq^{(N+1)}} - a_{lq^N}| \le 2c_N$. Since $\lim_{N\to\infty} c_N = 0$, the first statement follows. Assume that $\lim_{N\to\infty} lq^N a_{lq^N} = \alpha$. Then

$$q\alpha = \lim_{N\to\infty} (lq^{N+1} a_{lq^N} + lq^{N+1}(a_{lq^{N+1}} - a_{lq^N})) = \lim_{N\to\infty} lq^{N+1} a_{lq^{N+1}} = \alpha.$$

Hence $\alpha = 0$. □

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4. Cyclotomic p-adic multi-zeta values of depth one

In this section, we give a series expression for cyclotomic p-adic multi-zeta values of depth one. For the reader who wants to get to the main idea behind the proof of Theorem 6.4.3 without the technical and notational complications, we would suggest them to first focus on the proof of Proposition 4.2.2. The idea is to use the differential equation (2.2.9) to relate $g_{\mathcal{F}}$ to the g_i's, and use Corollary 3.0.4 to relate the coefficients of the power series expansion of $g_{\mathcal{F}}$ back to the g_i's. Below we give two different expressions for $g_j[e_0^s e_i]$. The one in Sect. 4.1 is analogous to the expression for p-adic multi-zeta values in [11], but it does not carry over to higher weights. The other method in Sect. 4.2, which assumes that $i \neq j$, is representative of the method in depth two in the later sections. We would like to emphasize that the shape of the expression in Sect. 4.2 is much different than the one in Sect. 4.1, since there is not a p^N in the denominator in the limit in Sect. 4.2 which makes the computation of the limit much easier and hence the formula much more useful.

For any power series $f \in K[[z]]$, we let $f[w]$ denote the coefficient of z^w in f.

4.1. Computation of $g_j[e_0^s e_i]$

Let $e_\infty \in \mathrm{Lie}\,\pi_{1,dR}(X_M)$ denote the element which is obtained by res_∞, the residue at ∞, as in Sect. 2.1.3; and let t_∞ denote the tangent vector at ∞ that maps to the tangent vector t_0 under the map $\theta(z) = \frac{1}{z}$. Let g_∞ be defined by the action of frobenius on the canonical de Rham path from t_0 to t_∞, analogously to the g_i's as in Sect. 2.2.3.

Applying F_* to the identity

$$\sum_{0\leq i\leq M} e_i + e_\infty = 0$$

and using (2.2.6), we get

$$\sum_{0\leq i\leq M} g_i^{-1} e_i g_i + g_\infty^{-1} e_\infty g_\infty = 0,$$

which gives the fundamental identity

$$g_\infty \left(\sum_{0\leq i\leq M} g_i^{-1} e_i g_i \right) = \left(\sum_{0\leq i\leq M} e_i \right) g_\infty. \tag{4.1.1}$$

Noting that $\mathcal{F}$ is a lifting of frobenius on $\overline{\mathcal{P}}$, whose principal parts map t_0 and t_∞ to itself [11, §3.2.(ii)], we have

$$g_{\mathcal{F}}(\infty) = g_\infty. \tag{4.1.2}$$

Proposition 3.0.1 gives a method for computing $g_{\mathcal{F}}(\infty)$. This makes the last equality, together with (4.1.1), one of the main tools for the computations.

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From the Eq. (2.2.10), we obtain

$$dg_{\mathcal{F}}[e_0] = \mathcal{F}^*\omega_0 - p\omega_0 = 0,$$

and hence that

$$g_{\mathcal{F}}[e_0] = 0, \tag{4.1.3}$$

since $g_{\mathcal{F}}(0) = 1$.

Similarly, for $1 \leq i \leq M$, the Eq. (2.2.10) gives $dg_{\mathcal{F}}[e_i] =$

$$\mathcal{F}^*\omega_i - p\omega_{\underline{i}} = p\frac{z^{p-1}dz}{z^p - \zeta^i} - p\frac{dz}{z - \zeta^{\underline{i}}} = p\left(\frac{1}{1-\zeta^{-\underline{i}}z} - \frac{(\zeta^{-\underline{i}}z)^{p-1}}{1-(\zeta^{-\underline{i}}z)^p}\right)d(\zeta^{-\underline{i}}z),$$

which implies that

$$g_{\mathcal{F}}(z)[e_i] = p\sum_{\substack{1\leq n\\ p\nmid n}} \frac{(\zeta^{-\underline{i}}z)^n}{n}, \tag{4.1.4}$$

for $z \in D(0,1)^\circ$. Since $g_{\mathcal{F}}[e_i]$ is a rigid analytic function on $\mathcal{U}_M$, we can use Proposition 3.0.1 to compute its value at ∞. Then using Eq. (4.1.2) we obtain

$$g_\infty[e_i] = g_{\mathcal{F}}(\infty)[e_i] = -\lim_{N\to\infty} g_{\mathcal{F}}[e_i][Mp^N] = 0, \tag{4.1.5}$$

since $g_{\mathcal{F}}[e_i][Mp^N] = 0$, for all N.

Comparing the coefficients of e_ie_0 in both sides of (4.1.1) implies that

$$g_\infty[e_i] + g_i[e_0] = g_\infty[e_0].$$

Using (4.1.5), we obtain $g_i[e_0] = 0$.

Lemma 4.1.1. *Suppose that α is a group-like element of $K\langle\langle e_0, \ldots, e_M\rangle\rangle$, that is the constant term of α is 1 and $\Delta(\alpha) = \alpha\otimes\alpha$. If $\alpha[e_0] = 0$, then $\alpha[e_0^n] = 0$ for all $n \geq 1$ and*

$$\alpha[e_0^a e_i e_0^b] = (-1)^b\binom{a+b}{a}\alpha[e_0^{a+b}e_i], \tag{4.1.6}$$

for $a, b \geq 0$.

Proof. Let $e_{i_1}\cdots e_{i_s}$ and $e_{j_1}\cdots e_{j_t}$ be any two monomials in $K\langle\langle e_0, \ldots, e_M\rangle\rangle$. Then comparing the coefficients of $(e_{i_1}\cdots e_{i_s})\otimes(e_{j_1}\cdots e_{j_t})$ on both sides of the equality $\Delta(\alpha) = \alpha\otimes\alpha$, using $\Delta(e_i) = 1\otimes e_i + e_i\otimes 1$, implies that

$$\alpha[e_{i_1}\cdots e_{i_s}]\alpha[e_{j_1}\cdots e_{j_t}] = \sum_{(k_1,\ldots,k_{s+t})} \alpha[e_{k_1}\cdots e_{k_{s+t}}],$$

where in the sum $(k_1, \ldots, k_{s+t})$ ranges over all shuffles of $(i_1, \ldots, i_s)$ and $(j_1, \ldots, j_t)$.

Using the last identity, we obtain $0 = \alpha[e_0^{n-1}]\alpha[e_0] = n\alpha[e_0^n]$, which proves the first claim.

To prove the second identity, note that

$$0 = \alpha[e_0^a e_i e_0^b]\alpha[e_0] = (b+1)\alpha[e_0^a e_i e_0^{b+1}] + (a+1)\alpha[e_0^{a+1}e_i e_0^b].$$

The result then follows by induction on b. □

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Since $g_i[e_0] = 0$ the above lemma implies

$$g_i[e_0^n] = 0, \tag{4.1.7}$$

for all $0 < n$, and $0 \leq i \leq M$.

Let

$$S(s;i)(z) := p^s \sum_{\substack{0<n \\ p\nmid n}} \frac{\zeta^{-\underline{i}n} z^n}{n^s}.$$

Then we have the following expression for $g_{\mathcal{F}}[e_0^{s-1}e_i]$.

Lemma 4.1.2. *For $z \in D(0,1)^\circ$, and $1 \leq i \leq M$,*

$$g_{\mathcal{F}}(z)[e_0^{s-1}e_i] = S(s;i)(z). \tag{4.1.8}$$

Proof. Let us compare the coefficients of $e_0^{s-1}e_i$ on both sides of Eq. (2.2.10). Since $g_{\mathcal{F}}[e_0] = 0$ by (4.1.3), Lemma 4.1.1 implies that $g_{\mathcal{F}}[e_0^n] = 0$ for all n. This and (4.1.7) then give that

$$dg_{\mathcal{F}}(z)[e_0^{s-1}e_i] = pg_{\mathcal{F}}(z)[e_0^{s-2}e_i]\frac{dz}{z}.$$

Since $g_{\mathcal{F}}[e_i] = S(1;i)$ by (4.1.4), the statement follows by induction on s. □

Since $g_{\mathcal{F}}[e_0] = 0$ by (4.1.3), Lemma 4.1.1 and (4.1.8) give

$$g_{\mathcal{F}}[e_0^a e_i e_0^b] = (-1)^b \binom{a+b}{a} g_{\mathcal{F}}[e_0^{a+b}e_i] = (-1)^b \binom{a+b}{a} S(a+b+1;i).$$

This helps us compute the following coefficients of g_∞:

Corollary 4.1.3. *For all $a, b \geq 0$,*

$$g_\infty[e_0^a e_i e_0^b] = 0. \tag{4.1.9}$$

Proof. We have seen that $g_\infty = g_{\mathcal{F}}(\infty)$ in (4.1.2). We also know that $g_{\mathcal{F}}[e_0^a e_i e_0^b]$ is a rigid analytic function on $\mathcal{U}_M$ with the power series expansion around 0 given in terms of $S(a+b+1;i)$ above. Then by Proposition 3.0.1, we only need to prove that $\lim_{N\to\infty} S(s;i)[Mp^N] = 0$. The statement then follows from the observation that $S(s;i)[Mp^N] = 0$, for any s, i and N. □

Using (2.2.10) and (4.1.7) we obtain that $dg_{\mathcal{F}}[e_i e_0^{s-1} e_j] =$

$$\mathcal{F}^*\omega_i g_{\mathcal{F}}[e_0^{s-1}e_j] - p\big(g_i[e_0^{s-1}e_j]\omega_{\underline{i}} + g_{\mathcal{F}}[e_i e_0^{s-1}]\omega_{\underline{j}} + g_j^{-1}[e_i e_0^{s-1}]\omega_{\underline{j}}\big).$$

From (4.1.9) and the fact that the above differential is regular at ∞, we get

$$g_j^{-1}[e_i e_0^{s-1}] = -g_i[e_0^{s-1}e_j].$$

Using this and solving the differential equation we obtain that

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$$g_{\mathcal{F}}[e_i e_0^{s-1} e_j] = -S(s,1;j,i;d_2) + (-1)^{s-1} S(s,1;i,j) + g_i[e_0^{s-1} e_j](T(1;i) - T(1;j)), \tag{4.1.10}$$

where

$$T(s;i) := p^s \sum_{0<n} \frac{\zeta^{-\underline{i}n} z^n}{n^s},$$

$$S(s_1,s_2;j,i;d_2) := p^{s_1+s_2} \sum_{\substack{0<n_1<n_2 \\ p\nmid n_1, p\mid(n_2-n_1)}} \frac{\zeta^{(\underline{i}-\underline{j})n_1 - \underline{i}n_2} z^{n_2}}{n_1^{s_1} n_2^{s_2}},$$

and

$$S(s_1,s_2;j,i) := p^{s_1+s_2} \sum_{\substack{0<n_1<n_2 \\ p\nmid n_1}} \frac{\zeta^{(\underline{i}-\underline{j})n_1 - \underline{i}n_2} z^{n_2}}{n_1^{s_1} n_2^{s_2}}.$$

This gives that

$$g_\infty[e_i e_0^{s-1} e_j] = (-1)^s p^{s+1} \lim_{N\to\infty} \frac{1}{Mp^N} \sum_{\substack{0<n<Mp^N \\ p\nmid n}} \frac{\zeta^{(\underline{j}-\underline{i})n}}{n^s}.$$

Using this we find a formula for $g_j[e_0^s e_i]$, with $s \geq 1$, as follows. First upon comparing the coefficients of $e_0 e_i e_0^{s-1} e_j$ in (4.1.1) and using (4.1.7) and (4.1.9) we find that

$$g_\infty[e_i e_0^{s-1} e_j] = g_j^{-1}[e_0 e_i e_0^{s-1}].$$

Again by (4.1.7), $g_j^{-1}[e_0 e_i e_0^{s-1}] = -g_j[e_0 e_i e_0^{s-1}]$ and by (4.1.6), $g_j[e_0 e_i e_0^{s-1}] = (-1)^{s-1} s g_j[e_0^s e_i]$. Combining these we get the following expression.

Proposition 4.1.4. *For $s \geq 1$,*

$$g_j[e_0^s e_i] = \frac{p^{s+1}}{s} \lim_{N\to\infty} \frac{1}{Mp^N} \sum_{\substack{0<n<Mp^N \\ p\nmid n}} \frac{\zeta^{(\underline{j}-\underline{i})n}}{n^s}.$$

4.2. An alternative expression for $g_j[e_0^{s-1} e_i]$ when $i \neq j$

Let $F(s;i)$ be the function defined by

$$F(s;i)(n) := p^s \sum_{\substack{0<n_1<n \\ p\nmid n_1}} \frac{\zeta^{\underline{i}n_1}}{n_1^s}.$$

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Then note that the expression (4.1.10) for $g_{\mathcal{F}}[e_i e_0^{s-1} e_j]$ in Sect. 4.1 gives the following identity for the coefficients of its power series expansion around 0, for $0 < l \leq pM$,

$$lq^N g_{\mathcal{F}}[e_i e_0^{s-1} e_j][lq^N] = p((-1)^{s-1}\zeta^{-\underline{jl}} F(s; j-i)(lq^N) + g_i[e_0^{s-1} e_j](\zeta^{-\underline{il}} - \zeta^{-\underline{jl}})).$$

Claim 4.2.1. $\lim_{N\to\infty} lq^N g_{\mathcal{F}}[e_i e_0^{s-1} e_j][lq^N] = 0$.

Proof. We know that the coefficients of $g_{\mathcal{F}}$ are rigid analytic functions on $\mathcal{U}_M$. Therefore in order to show that the above limit is 0 we only need to show that the above limit exists by Corollary 3.0.4. More explicitly, the above expression for $lq^N g_{\mathcal{F}}[e_i e_0^{s-1} e_j][lq^N]$ can be written as

$$p((-1)^{s-1}\zeta^{-\underline{jl}} p^s \sum_{\substack{0<n_1<lq^N \\ p\nmid n_1}} \frac{\zeta^{\underline{(j-i)n_1}}}{n_1^s} + g_i[e_0^{s-1} e_j](\zeta^{-\underline{il}} - \zeta^{-\underline{jl}})).$$

Therefore the fact that the limit $\lim_{N\to\infty} lq^N g_{\mathcal{F}}[e_i e_0^{s-1} e_j][lq^N]$ exists follows from the observation that

$$\sum_{\substack{0<n<lq^{N+1} \\ p\nmid n}} \frac{\zeta^n}{n^s} - \sum_{\substack{0<n<lq^N \\ p\nmid n}} \frac{\zeta^n}{n^s} = \sum_{1\leq t\leq q-1} \sum_{\substack{0<n<lq^N \\ p\nmid n}} \frac{\zeta^{n+tlq^N}}{(n+tlq^N)^s}$$

is congruent modulo q^N to

$$\sum_{1\leq t\leq q-1} \zeta^{lt} \cdot \sum_{\substack{0<n<lq^N \\ p\nmid n}} \frac{\zeta^n}{n^s}.$$

The term on the left is equal to 0 if $\zeta^l \neq 1$. If $\zeta^l = 1$, then the term on the right converges to 0. This last statement can be seen, for example, by the existence of the limit in Proposition 4.1.4. □

Let $\mathcal{X}(s; i) := \lim_{N\to\infty} F(s; i)(q^N)$. Letting $l = 1$ in the expression before Claim 4.2.1 and taking the limit as $N \to \infty$, we obtain the expression for $g_i[e_0^{s-1} e_j]$ we were looking for:

Proposition 4.2.2. *For $i \neq j$, we have*

$$g_i[e_0^{s-1} e_j] = \frac{(-1)^{s-1}}{1 - \zeta^{\underline{j-i}}} \mathcal{X}(s; j-i).$$

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4.3. Computation of $g_j[e_i]$

By (2.2.7), $g_j[e_i] = (\alpha_{j*}(g))[e_i] = g[e_{i-j}]$. Let $\iota : X_M \to X_1$ denote the inclusion. If $i = j$ then using the functoriality of frobenius with respect to ι we see that $g[e_{i-j}]$ computed on X_M is equal to $g[e_1]$ computed on X_1. But this last expression is 0 by [11, §5.6]. Suppose now that $i \neq j$. Then $g_j[e_i] = \alpha_{i*}(g_{j-i})[e_i] = g_{j-i}[e_M]$. Then as above, by the functoriality of frobenius for ι, $g_{j-i}[e_M]$, which is computed on X_M, is equal to

$$({}_{t_0}\gamma_{z'} \cdot F_*({}_z\gamma_{t_0}))[e_1], \tag{4.3.1}$$

which is computed on X_1. Here $z = \zeta^{\underline{j-i}}$ and $z' = \zeta^{j-i}$. Note that $\mathcal{F}$ is good lifting of frobenius on $\mathcal{U}_1 \subseteq X_1$. Since $i \neq j$, $z \in \mathcal{U}_1$, and since $\mathcal{F}(z) = z^p = z'$, we see that (4.3.1) is equal to $g_{\mathcal{F}}[\zeta^{\underline{j-i}}][e_1]$. The last expression is computed by Proposition 3.0.1 to be

$$\lim_{N\to\infty} \frac{p}{1-\zeta^{(\underline{j-i})p^N}} \sum_{\substack{0<n<p^N \\ p\nmid n}} \frac{\zeta^{(\underline{j-i})n}}{n}.$$

Therefore we have the following expression for $g_j[e_i]$.

Proposition 4.3.1. *If $i = j$ then $g_j[e_i] = 0$. Otherwise*

$$g_j[e_i] = \log \frac{1-\zeta^{j-i}}{(1-\zeta^{\underline{j-i}})^p}.$$

5. M-power series functions

In order to compute the higher depth part of the frobenius action, we first study the type of functions that appear in these computations which we call M-power series functions. From an M-power series function f we will construct another M-power series function $f^{(s)}$ which could be thought of as a regularized version of $\frac{f(k)}{k^s}$. The main result is Proposition 5.0.5 below which will help us continue the inductive process in the proof.

Definition 5.0.2. Let $n \in \mathbb{N}$ and let $f : \mathbb{N}_{\geq n} \to \mathbb{Q}_p[\zeta]$ be any function. We say that f is an *M-power series function*, if there exist power series $p_i(x) \in \mathbb{Q}_p[\zeta][[x]]$, which converge on $D(0, r_i)$ for some $r_i > |p|$, for $0 < i \leq pM$, such that $f(a) = p_i(a-i)$, for all $a \geq n$ and $pM|(a-i)$. We define the absolute value of a power series around 0 to be the supremum of the absolute values of its coefficients and the absolute value of the M-power series function f to be the maximum of the absolute values of the p_i.

Remark 5.0.3. (i) By the Weierstrass preparation theorem, the power series p_i in the above definition are unique.

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(ii) Fix $0 < l \leq pM$, and let f be as above. Then there is a power series $p(x) \in \mathbb{Q}_p[\zeta][[x]]$ which converges on some $D(0, r)$ with $r > |p|$ and

$$f(lq^N) = p(lq^N),$$

for N sufficiently large.

Example 5.0.4. (i) Let $s \in \mathbb{Z}$ and $f(k) := \zeta^{ik}k^s$, for $p \not| k$ and $f(k) = 0$ for $p|k$. Then f is an M-power series function.

(ii) Clearly the sums and products of M-power series functions are M-power series functions.

(iii) Let f be an M-power series function. For any $0 < l \leq pM$, with $p|l$ let

$$f_l := \lim_{\substack{n \to 0 \\ pM|(n-l)}} f(n),$$

with n ranging over positive integers such that $pM|(n-l)$, and tending to 0 in the p-adic metric.

Let $f^{[1]}$ be defined by

$$f^{[1]}(k) = \frac{f(k) - f_l}{k},$$

if $p|k$ and $pM|(k-l)$; and $f^{[1]}(k) = 0$, if $p \not| k$. We then see that $f^{[1]}$ is an M-power series function. In fact, if $p|l$, and p is a power series around 0 such that $f(n) = p(n)$ for all $pM|(n-l)$ then $f^{[1]}(n) = q(n)$, for all $pM|(n-l)$, where

$$q(x) = \frac{p(x) - p(0)}{x}.$$

Inductively, we let $f^{[k+1]} := (f^{[k]})^{[1]}$.

(iv) Using the notation as above, let $f^{(1)}$ be defined by $f^{(1)}(k) := f^{[1]}(k)$, if $p|k$; and $f^{(1)}(k) = \frac{f(k)}{k}$, if $p \not| k$. Then $f^{(1)}$ is also an M-power series function.

Proposition 5.0.5. *Let $f : \mathbb{N}_{\geq n_0} \to \mathbb{Q}_p[\zeta]$ be an M-power series function. If we define $F : \mathbb{N}_{\geq n_0} \to \mathbb{Q}_p[\zeta]$ by*

$$F(n) := \sum_{n_0 \leq k \leq n} f(k)$$

then F is also an M-power series function.

Proof. Note that f is uniquely extended to an M-power series function $\tilde{f}$ which is defined on all $\mathbb{N}$. Then since $\tilde{F}(n) := \sum_{1 \leq k \leq n} \tilde{f}(k) = \tilde{F}(n_0 - 1) + F(n)$ for all $n \geq n_0$, that $\tilde{F}$ is an M-power series function implies the same for F. Therefore, without loss of generality, we will assume that $n_0 = 1$.

For $1 \leq t \leq pM$, let

$$F_t(n) := \sum_{\substack{1 \leq k \leq n \\ pM|(k-t)}} f(k) = \sum_{\substack{0 \leq \alpha \leq n-t \\ pM|\alpha}} p_t(\alpha),$$

for $t \leq n$ and $F_t(n) = 0$ otherwise.

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Since $F(n) = \sum_{1\le t\le pM} F_t(n)$, it suffices to prove that each F_t is an M-power series function. Fix $1 \le i, t \le pM$ and suppose first that $t \le i$. Let $p_t(x) = \sum_{0\le j} a_j x^j$. By assumption there is an $\varepsilon > 0$ such that $\lim_{n\to\infty} a_j p^{j(1-\varepsilon)} = 0$.

Recall the formula for the sum of the j-th powers:

$$\sum_{1\le m\le n} m^j = \frac{1}{j+1} \sum_{0\le k\le j} \binom{j+1}{k} (-1)^k B_k n^{j+1-k}$$

where B_k are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{0\le k} \frac{B_k x^k}{k!}.$$

The Von Staudt–Clausen theorem gives the bound $|B_k| \le p$.

Then for $n \ge 0$

$$\begin{aligned} F_t(i + npM) &= \sum_{0\le k\le j} \frac{a_j (pM)^j}{j+1} \binom{j+1}{k} (-1)^k B_k n^{j+1-k} \\ &= \sum_{\substack{1\le l \\ 0\le k}} \frac{a_{l+k-1}(pM)^{k-1}}{l+k} \binom{l+k}{k} (-1)^k B_k (npM)^l. \end{aligned}$$

Therefore letting $q_t(x) = \sum_{1\le l} b_l x^l$, with

$$b_l = \sum_{0\le k} \frac{a_{l+k-1}(pM)^{k-1}}{l+k} \binom{l+k}{k} (-1)^k B_k,$$

we have $F_t(i + npM) = q_t(npM)$.

Note that

$$|b_l p^{l(1-\varepsilon/2)}| \le p^{1+\varepsilon} \max_k |a_{l+k-1} p^{(l+k-1)(1-\varepsilon)} \frac{p^{(l/2+k)\varepsilon}}{l+k}|.$$

Since $\lim_{l\to\infty} a_{l+k-1} p^{(l+k-1)(1-\varepsilon)} = 0$ and

$$\lim_{l\to\infty} |\frac{p^{(l/2+k)\varepsilon}}{l+k}| \le \lim_{l\to\infty} |\frac{p^{(l+k)\frac{\varepsilon}{2}}}{l+k}| = 0,$$

we see that $\lim_{l\to\infty} b_l p^{l(1-\varepsilon/2)} = 0$.

On the other hand if $i < t$, then we have

$$F_t(i + npM) = F_t(t + npM) - f(t + npM) = q_t(npM) - p_t(npM).$$

This proves that F_t is an M-power series function as desired. □

Corollary 5.0.6. *If $f_1, f_2, \ldots, f_k$ are M-power series functions, then the function G defined by*

$$G(n_k) := \sum_{0<n_1<n_2<\cdots<n_k} f_1(n_1) f_2(n_2) \cdots f_k(n_k)$$

is an M-power series function.

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6. Computation of the higher depth part

In this section, we will concentrate on multi-zeta values of depth two. The proof of the series expression for these values is a somewhat lengthy argument. The main idea of the proof is contained in the simplest case of $g_i[e_j e_k]$, which correspond to $\zeta_p(1,1;a,b)$. The expression for these values is given in Proposition 6.1.4. The main idea is to use Corollary 3.0.4 to show that $\lim_{N\to\infty} lq^N g_{\mathcal{F}}[e^I][lq^N] = 0$ for a monomial e^I in the e_i's. In order to be able to use this corollary we need to first show that the limit exists. This is done by rearranging the terms in the expression for $g_{\mathcal{F}}[e^I]$'s in such a way that each term is an M-power series function. The last step is to deduce from the knowledge of this limit the expression for the multi-zeta values.

Notation. We fix the notation for the series which appear as summands in the power series expansion of $g_{\mathcal{F}}$ around 0. Most of these series themselves do not extend to rigid analytic functions on $\mathcal{U}_M$, however certain linear combinations of their regularized versions do. Next we will fix the notation for these regularized versions, which we described in Sect. 5.

Suppose that $\underline{s} = (s_1, \ldots, s_k)$, where s_j are positive integers and $\underline{i} := (i_1, \ldots, i_k)$, where $1 \leq i_j \leq M$. For each $i \in \mathbb{N}$, let f_i and d_i be symbols and let $\underline{\alpha} := \{\alpha_1, \ldots, \alpha_r\} \subseteq \{f_i | 1 \leq i \leq k\} \cup \{d_i | 1 \leq i \leq k\}$. Then we let

$$S(\underline{s};\underline{i};\underline{\alpha})(z) := p^{\sum s_i} \sum \frac{\zeta^{(\underline{i_2}-\underline{i_1})n_1+(\underline{i_3}-\underline{i_2})n_2+\cdots+(-\underline{i_k})n_k}}{n_1^{s_1}\cdots n_k^{s_k}} z^{n_k},$$

where the sum is taken over all $0 < n_1 < \cdots < n_k$, which satisfy:

(i) $p \not| n_1$
(ii) $p|n_i$, if $f_i \in \underline{\alpha}$, and
(iii) $p|(n_i - n_{i-1})$, if $d_i \in \underline{\alpha}$,

If we take the sum over all $0 < n_1 < \cdots < n_k$ which satisfy (ii) and (iii) [and not necessarily (i)] then we denote the resulting series by $T(\underline{s};\underline{i};\underline{\alpha})(z)$. We use $\underline{S}(\cdot)$ and $\underline{T}(\cdot)$ to denote $S(\cdot)$ and $T(\cdot)$ without the $p^{\sum s_i}$ factors. Let

$$F(\underline{s};\underline{i};\underline{\alpha})(n) := p^{\sum s_i} \sum \frac{\zeta^{\underline{i_1}n_1+\cdots+\underline{i_k}n_k}}{n_1^{s_1}\cdots n_k^{s_k}},$$

where the sum is over all $0 < n_1 < \cdots < n_k < n$ that satisfy (i), (ii) and (iii) above. We denote the function obtained by taking the sum over $0 < n_1 < \cdots < n$ that satisfy (ii) and (iii) [and not necessarily (i)], by $G(\underline{s},\underline{i},\underline{\alpha})(n)$. Similarly, let $\underline{F}$ and $\underline{G}$ be the versions without the p-power factor. Clearly,

$$\underline{S}(\underline{s};\underline{i})[n] = \frac{\zeta^{-\underline{i_k}n}}{n^{s_k}} \underline{F}(\underline{s}',\underline{i}')(n),$$

where $\underline{s}' = (s_1, \ldots, s_{k-1})$ and $\underline{i}' = (i_2 - i_1, \ldots, i_k - i_{k-1})$.

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Using the definition of $L^{(k)}$ for an M-power series function L in Example 5.0.4 (iv), we define $F(s_1, (s_2); i, j)$ as follows. Noting that

$$F(s_1, s_2; i, j)(n) = p^{s_2} \sum_{0<k<n} \frac{F(s_1; i)(k)}{k^{s_2}} \zeta^{\underline{jk}},$$

we put

$$F(s_1, (s_2); i, j)(n) = p^{s_2} \sum_{0<k<n} F^{(s_2)}(s_1; i)(k) \zeta^{\underline{jk}}.$$

We define $F(s_1, (s_2); i, j; \underline{\alpha})$ analogously, and we let $F^{(\cdot)}(\cdot) := F(\cdot)^{(\cdot)}$. When the limit exists, we let $\mathcal{X}^{(\cdot)}(\cdot) := \lim_{N\to\infty} F^{(\cdot)}(\cdot)(q^N)$.

If we put $\underline{i} = (i, j, k)$, then we define $S(a, (b), c; \underline{i})$ and $S(a, (b), (c); \underline{i})$ as follows:

$$S(a, (b), c; \underline{i})[n] = \frac{p^c \zeta^{-\underline{kn}}}{n^c} F(a, (b); \underline{i}')(n)$$

and

$$S(a, (b), (c); \underline{i})[n] = p^c \zeta^{-\underline{kn}} F^{(c)}(a, (b); \underline{i}')(n).$$

We define $S(a, b, (c); \underline{i}; \underline{\alpha})$ and $S(a, (b), c; \underline{i}; \underline{\alpha})$ and $S(a, (b), (c); \underline{i}; \underline{\alpha})$ similarly.

6.1. Computation of $g_i[e_j e_k]$

In this section we will get two expressions relating $g_i[e_j e_k]$ and $g_k^{-1}[e_i e_j]$. The first one will be obtained by looking at residues in the differential equation (2.2.10). The second one will be obtained using Claim 6.1.3 below. Combining these two will give the main result.

Since $g_{\mathcal{F}}$ is regular at ∞ computing the residues at ∞ in the differential equation (2.2.10) gives the equality:

$$-g_\infty[e_j e_k] + g_\infty[e_i e_j] + g_k^{-1}[e_i e_j] + g_j^{-1}[e_i] g_j[e_k] + g_i[e_j e_k] = 0. \tag{6.1.1}$$

Since by Sect. 4.1,

$$g_\infty[e_i e_j] = -g_j[e_0 e_i] = \frac{\mathcal{X}(2; i-j)}{1 - \zeta^{\underline{i}-\underline{j}}}$$

for $i \neq j$, we have, for i, j, k all distinct, $g_k^{-1}[e_i e_j] =$

$$\frac{\mathcal{X}(2; j-k)}{1 - \zeta^{\underline{j}-\underline{k}}} - \frac{\mathcal{X}(2; i-j)}{1 - \zeta^{\underline{i}-\underline{j}}} + g_j[e_i] g_j[e_k] - g_i[e_j e_k]. \tag{6.1.2}$$

Recall the expression for $g_{\mathcal{F}}[e_l e_m]$ from Sect. 4.1:

$$g_{\mathcal{F}}[e_l e_m] = -S(1, 1; m, l; d_2) + S(1, 1; l, m) + g_l[e_m](T(1; l) - T(1; m)).$$

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6.1.1. Computation of $g_i[e_je_i]$ First we deal with the degenerate case when $i = k$. Using (6.1.1) with $k = i$, the fact that $g_{\mathcal{F}}$ and g_l are group-like we obtain:

$$g_i[e_je_i] = \frac{1}{2}g_j[e_i]^2 - g_\infty[e_ie_j].$$

Then Proposition 4.3.1 gives the following expression for $g_i[e_je_i]$.

Proposition 6.1.1. *Assuming that $i \neq j$, we have*

$$g_i[e_je_i] = \frac{1}{2}\left(\frac{\mathcal{X}(1; i-j)}{1-\zeta^{\underline{i}-\underline{j}}}\right)^2 - \frac{\mathcal{X}(2; j-i)}{1-\zeta^{\underline{j}-\underline{i}}}.$$

6.1.2. Computation of $g_i[e_je_k]$ Using the differential equation above gives that

$$\begin{aligned}
&g_{\mathcal{F}}[e_ie_je_k]\\
&= S(1,1,1;k,j,i;d_2,d_3) - S(1,1,1;j,k,i;d_3) - S(1,1,1;j,i,k;d_2)\\
&\quad + S(1,1,1;i,j,k) - g_j[e_k](T(1,1;j,i;d_2) - T(1,1;k,i;d_2))\\
&\quad + g_i[e_j](T(1,1;i,k) - T(1,1;j,k)) - g_k[e_j]S(1,1;i,k) + g_k^{-1}[e_ie_j]T(1;k)\\
&\quad + g_j[e_k]S(1,1;i,j) - g_j[e_i]g_j[e_k]T(1;j) + g_i[e_je_k]T(1;i).
\end{aligned}$$

For a fixed $0 < l \leq pM$, we are interested in showing that $\lim_{N\to\infty} lq^N g_{\mathcal{F}}[e_ie_je_k][lq^N]$ is 0. In order to do this, first we group the terms in the limit as follows so that each summand has a limit.

(i) $lq^N S(\underline{s};\underline{i};d_2,d_3)[lq^N] = 0$, for any $\underline{s} := (s_1,s_2,s_3)$ and $\underline{i} := (i_1,i_2,i_3)$.
(ii) $lq^N \underline{T}(1;i)[lq^N] = \zeta^{-\underline{i}l}$
(iii) $lq^N \underline{S}(s,1;i,j)[lq^N] = \zeta^{-\underline{j}l}\underline{F}(s;j-i)(lq^N)$. Hence the limit exists and for $l = 1$ it is

$$\lim_{N\to\infty} q^N \underline{S}(s,1;i,j)[q^N] = \zeta^{-\underline{j}}\underline{\mathcal{X}}(s;j-i).$$

(iv) $lq^N \underline{S}(s_1,s_2,1;j,i,k;d_2)[lq^N] = \zeta^{-\underline{k}l}\underline{F}(s_1,s_2;i-j,k-i;d_2)(lq^N)$. Again the limit exists and $\lim_{N\to\infty} q^N \underline{S}(s_1,s_2,1;j,i,k;d_2)[q^N] = \zeta^{-\underline{k}}\underline{\mathcal{X}}(s_1,s_2;i-j,k-i;d_2)$.
(v) $lq^N(\underline{S}(1,1,1;j,k,i;d_3) + \underline{g}_j[e_k](\underline{T}(1,1;j,i;d_2) - \underline{T}(1,1;k,i;d_2)))[lq^N]$ is equal to

$$\begin{aligned}
&\zeta^{-\underline{i}l}(\underline{F}(1,1;k-j,i-k;f_2)(lq^N)\\
&\quad + \underline{g}_j[e_k](\underline{G}(1;i-j;f_1) - \underline{G}(1;i-k;f_1))(lq^N)))
\end{aligned}$$

Let us rewrite the same expression as

$$\zeta^{-\underline{i}l}\sum_{\substack{0<n_2<lq^N\\ p|n_2}} \frac{\zeta^{n_2(\underline{i}-\underline{k})}}{n_2}\left(\underline{g}_j[e_k](\zeta^{(\underline{k}-\underline{j})n_2} - 1) + \underline{F}(1;k-j)(n_2)\right).$$

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Lemma 6.1.2. *Let $0 < l \leq pM$ such that $p|l$, and $1 \leq t$ then*

$$\lim_{\substack{n\to 0\\ pM|(n-l)}} F(t;i)(n) = (-1)^{t-1} g[e_0^{t-1}e_i](1-\zeta^{\underline{il}}),$$

with n ranging over positive integers such that $pM|(n-l)$, and tending to 0 in the p-adic metric.

Proof. By Corollary 5.0.6, $F(t;i)$ is an M-power series function. Note that if f is an M-power series function and $p|l$ then

$$\lim_{\substack{n\to 0\\ pM|(n-l)}} f(n) = \lim_{N\to\infty} f(lq^N),$$

where in the first limit n goes to 0 in the p-adic metric and in the second one N goes to ∞ in the archimedean metric. Therefore the limit in the statement of the lemma exists and is equal to

$$\lim_{N\to\infty} F(t;i)(lq^N) = p^t \lim_{N\to\infty} \sum_{\substack{0<n_1<lq^N\\ p\nmid n_1}} \frac{\zeta^{\underline{in_1}}}{n_1^t} = p^t \sum_{0\leq a<l} \zeta^{\underline{ia}} \lim_{N\to\infty} \sum_{\substack{0<n<q^N\\ p\nmid n}} \frac{\zeta^{\underline{in}}}{n^t},$$

which is equal to $g[e_0^{t-1}e_i](1-\zeta^{\underline{il}})$, if $0 < i < M$, by Proposition 4.2.2. On the other hand, if $i = M$, then the equality follows since both sides of the expressions are 0 [11, §5.11]. □

The above expression then can be written as

$$\zeta^{-\underline{il}} \sum_{\substack{0<n_2<lq^N\\ p|n_2}} \zeta^{n_2(\underline{i}-\underline{k})} F^{(1)}(1;k-j)(n_2) = \zeta^{-\underline{il}} F(1,(1);k-j,i-k;f_2)(lq^N).$$

By Example 5.0.4 (iii), $F^{(1)}(1;k-j;f_1)$ is an M-power series function. Then by Proposition 5.0.5, and Remark 5.0.3 (ii), the limit exists as $N \to \infty$ and is equal to $\zeta^{-\underline{i}}\underline{\mathcal{X}}(1,(1);k-j,i-k;f_2)$, if $l = 1$.

(vi) $lq^N(\underline{S}(1,1,1;i,j,k) + \underline{g}_i[e_j](\underline{T}(1,1;i,k) - \underline{T}(1,1;j,k)))[lq^N]$

$$= \zeta^{-\underline{kl}} \sum_{0<n_2<lq^N} \frac{\zeta^{(\underline{k}-\underline{j})n_2}}{n_2} (\underline{g}_i[e_j](\zeta^{(\underline{j}-\underline{i})n_2} - 1) + \underline{F}(1;j-i))$$

$$= \zeta^{-\underline{kl}}(\underline{F}(1,(1);j-i,k-j) + \underline{g}_i[e_j](\underline{F}(1;k-i) - \underline{F}(1;k-j)))(lq^N).$$

As above the limit of the main expression as $N \to \infty$ exists and is equal to $\zeta^{-\underline{k}}(\underline{\mathcal{X}}(1,(1);j-i,k-j) + \underline{g}_i[e_j](\underline{\mathcal{X}}(1;k-i) - \underline{\mathcal{X}}(1;k-j)))$, if $l = 1$.

Claim 6.1.3. $\lim_{N\to\infty} lq^N g_{\mathcal{F}}[e_ie_je_k][lq^N] = 0$.

Proof. The proof is exactly as that of Claim 4.2.1. We only need to add that (i)–(vi) imply the existence of the limit. □

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A direct consequence of this is the following proposition and its corollary.

Proposition 6.1.4. *For distinct i, j, k, $(1-\zeta^{\underline{i-k}})g_i[e_je_k]$ is equal to the following sum of series:*

$$\begin{aligned}
&\mathcal{X}(1,(1);k-j,i-k;f_2)-\zeta^{\underline{i-k}}\mathcal{X}(1,(1);j-i,k-j)\\
&+\zeta^{\underline{i-k}}\mathcal{X}(1,1;i-j,k-i;d_2)-\frac{\zeta^{\underline{i-k}}\mathcal{X}(2;j-k)}{1-\zeta^{\underline{j-k}}}+\frac{\zeta^{\underline{i-k}}\mathcal{X}(2;i-j)}{1-\zeta^{\underline{i-j}}}\\
&-\frac{\zeta^{\underline{i-j}}\mathcal{X}(1;j-i)\mathcal{X}(1;k-j)}{1-\zeta^{\underline{k-j}}}+\frac{\zeta^{\underline{i-k}}\mathcal{X}(1;j-k)\mathcal{X}(1;k-i)}{1-\zeta^{\underline{j-k}}}\\
&+\frac{(\zeta^{\underline{i-j}}-\zeta^{\underline{i-k}})\mathcal{X}(1;i-j)\mathcal{X}(1;k-j)}{(1-\zeta^{\underline{i-j}})(1-\zeta^{\underline{k-j}})}\\
&-\zeta^{\underline{i-k}}\frac{\mathcal{X}(1;j-i)}{1-\zeta^{\underline{j-i}}}(\mathcal{X}(1;k-i)-\mathcal{X}(1;k-j)).
\end{aligned}$$

Proof. This follows from combining Claim 6.1.3 and (6.1.2), which expresses $g_k^{-1}[e_ie_j]$ in terms of $g_i[e_je_k]$, and rearranging the terms. □

Recall the notation $S(s,(t);i,j)$, where the parenthesis stands for the regularized version of the series $S(s,t;i,j)$ as explained in the beginning of § 6.

Corollary 6.1.5. *We have the following expression for $g_{\mathcal{F}}[e_ie_0^{s-1}e_j]$, using the notation above:*

$$-S(s,1;j,i;d_2)+(-1)^{s-1}S(s,(1);i,j)+g_i[e_0^{s-1}e_j](S(1;i)-S(1;j)).$$

Proof. We saw in Sect. 4.1 that $g_{\mathcal{F}}[e_ie_0^{s-1}e_j]$ is equal to

$$-S(s,1;j,i;d_2)+(-1)^{s-1}S(s,1;i,j)+g_i[e_0^{s-1}e_j](T(1;i)-T(1;j)).$$

The fact that $\lim_{N\to\infty} lq^N g_{\mathcal{F}}[e_ie_0^{s-1}e_j][lq^N]=0$, which we have shown in Sect. 4.2, shows that we obtain the above answer after regularization. □

Similarly we have the following corollary.

Corollary 6.1.6. *The coefficients of $g_{\mathcal{F}}[e_ie_je_k]$ define an M-power series. In fact,*

$$\begin{aligned}
g_{\mathcal{F}}[e_ie_je_k]=\ &S(1,1,1;k,j,i;d_2,d_3)-S(1,(1),(1);j,k,i;d_3)\\
&-S(1,1,(1);j,i,k;d_2)+S(1,(1),(1);i,j,k)\\
&-g_j[e_k](S(1,1;j,i;d_2)-S(1,1;k,i;d_2))+g_i[e_j](S(1,(1);i,k)\\
&-S(1,(1);j,k))-g_k[e_j]S(1,(1);i,k)+g_k^{-1}[e_ie_j]S(1;k)\\
&+g_j[e_k]S(1,(1);i,j)-g_j[e_i]g_j[e_k]S(1;j)+g_i[e_je_k]S(1;i).
\end{aligned}$$

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6.2. An explicit formula for $F(t;i)$

Recall that $F(t;i)(n) = p^t \sum_{\substack{0<k<n \\ p\nmid k}} \frac{\zeta^{ik}}{k^t}$, is an M-power series function. In Corollary 6.2.3 below we will give a series expression for this function when $p|n$. The coefficients of this expansion are in terms of cyclotomic p-adic multi-zeta values. The proof will be based on the following observation which we will continue to use throughout the paper.

Proposition 6.2.1. *Let f be a rigid analytic function on $\mathcal{U}_M$ such that*

$$df = dg + h\omega_0 + \sum_{1\le i\le M} \alpha_i \omega_i,$$

with $g(z) = \sum_{0<n} a_n z^n$, $h(z) = \sum_{0<n} b_n z^n$, $\alpha_i \in \mathbb{C}_p$. Suppose that for $0 < l \le pM$, $\lim_{N\to\infty}(lq^N a_{lq^N} + b_{lq^N})$ exists. Then

$$\lim_{N\to\infty}(lq^N a_{lq^N} + b_{lq^N}) = \sum_{1\le i\le M} \zeta^{-il}\alpha_i.$$

Proof. Let $f(z) := \sum_{0<n} \gamma_n z^n$. Since by assumption f is a rigid analytic function on $\mathcal{U}_M$, Corollary 3.0.4 states that if $\lim_{N\to\infty} lq^N \gamma_{lq^N}$ exists then it is equal to 0. The differential equation in the statement of the proposition gives

$$\gamma_n = a_n + \frac{b_n}{n} - \sum_{1\le i\le M} \frac{\zeta^{-in}\alpha_i}{n}.$$

Therefore $lq^N\gamma_{lq^N} = lq^N a_{lq^N} + b_{lq^N} - \sum_{1\le i\le M} \zeta^{-il}\alpha_i$ and the statement follows. □

Suppose that $t \ge 1$ and $s \ge 2$. Comparing the coefficients of $e_j e_0^{t-1} e_k e_0^{s-1}$ in (2.2.10) gives the following identity:

$$dg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] = \mathcal{F}^*\omega_j g_{\mathcal{F}}[e_0^{t-1} e_k e_0^{s-1}] - pg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-2}]\omega_0 - pg_j[e_0^{t-1} e_k e_0^{s-1}]\omega_{\underline{j}}.$$

Since $g_{\mathcal{F}}[e_0^{t-1} e_k e_0^{s-1}] = (-1)^{s-1}\binom{s+t-2}{t-1} S(s+t-1;k)$ by (4.1.8), letting $(x)_k := x(x+1)\cdots(x+(k-1))$, we obtain

$$\frac{(-1)^s}{(t-1)!}(s)_{t-1} dS(s+t-1, 1; k, j; d_2) = \mathcal{F}^*\omega_j g_{\mathcal{F}}[e_0^{t-1} e_k e_0^{s-1}].$$

This gives $dg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] = \frac{(-1)^s}{(t-1)!}(s)_{t-1} dS(s+t-1, 1; k, j; d_2)$

$$- pg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-2}]\omega_0 - pg_j[e_0^{t-1} e_k e_0^{s-1}]\omega_{\underline{j}}.$$

Proposition 6.2.2. *If we let $g_{\mathcal{F}}(z)[e_j e_0^{t-1} e_k e_0^{s-1}] = \sum_{0<n} c_n z^n$, then*

$$\lim_{N\to\infty} c_{lq^N} = \zeta^{-\underline{jl}} g_j[e_0^{t-1} e_k e_0^s],$$

for any $0 < l \le pM$.

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Proof. We will proceed by induction on s. We have $g_{\mathcal{F}}[e_j e_0^{t-1} e_k] =$

$$-S(t,1;k,j;d_2) + (-1)^{t-1} S(t,1;j,k) + g_j[e_0^{t-1}e_k](T(1;j) - T(1;k)),$$

by Sect. 4.1. The coefficient of z^n in $\underline{g}_{\mathcal{F}}[e_j e_0^{t-1} e_k]$ is $\frac{K(n)}{n}$ where $K(n) :=$

$$-\zeta^{-n\underline{j}}\underline{F}(t;j-k;d_2)(n) + (-1)^{t-1}\zeta^{-\underline{k}n}\underline{F}(t;k-j)(n)$$
$$+\underline{g}_j[e_0^{t-1}e_k](\zeta^{-\underline{j}n} - \zeta^{-\underline{k}n}).$$

By Lemma 6.1.2, we know that $\lim_{N\to\infty} F(t;k-j)(lq^N)$ is equal to

$$(-1)^{t-1} g_j[e_0^{t-1}e_k](1 - \zeta^{(\underline{k}-\underline{j})l}).$$

This implies by Example 5.0.4 (iv) that $\frac{K(n)}{n} =$

$$K^{(1)}(n) = -\zeta^{-n\underline{j}}\underline{F}(t;j-k;d_2)(n) + \zeta^{-\underline{k}n}\underline{F}^{(1)}(t;k-j)(n),$$

is an M-power series function and hence by Remark 5.0.3 (ii),

$$\lim_{N\to\infty} K^{(1)}(lq^N) = \lim_{N\to\infty} \zeta^{-\underline{k}}\underline{F}^{(1)}(t;k-j)(lq^N)$$

exists, for any $0 < l \leq pM$.

Since $dg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0] = tdS(t+1,1;k,j;d_2) - pg_{\mathcal{F}}[e_j e_0^{t-1} e_k]\omega_0 - pg_j[e_0^{t-1}e_k e_0]\omega_{\underline{j}}$, the existence of the above limit and Proposition 6.2.1 implies that

$$\lim_{N\to\infty} K^{(1)}(lq^N) = \lim_{N\to\infty} \zeta^{-\underline{k}}\underline{F}^{(1)}(t;k-j)(lq^N) = \zeta^{-\underline{j}l}\underline{g}_j[e_0^{t-1}e_k e_0].$$

This implies that $g_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0]$ is equal to

$$\frac{1}{(t-1)!}\sum_{0\leq r\leq 1}(r+1)_{t-1}S(t+r,2-r;k,j;d_2) + (-1)^t S(t,(2);j,k),$$

and the coefficient of z^n in this expression is

$$\frac{\zeta^{-n\underline{j}}}{(t-1)!}\sum_{0\leq r\leq 1}(r+1)_{t-1}\frac{F(t+r;j-k;d_2)(n)}{n^{2-r}}$$
$$+(-1)^t\zeta^{-n\underline{k}}F^{(2)}(t;k-j;f_1)(n).$$

Since $\frac{F(t;i;d_2)}{n^k} = F^{(k)}(t;i;d_2)$, we inductively arrive at $g_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] =$

$$\frac{(-1)^s}{(t-1)!}\sum_{0\leq r\leq s-1}(r+1)_{t-1}S(t+r,s-r;k,j;d_2) + (-1)^{s+t}S(t,(s);j,k).$$

The coefficient of z^n in this expression is

$$\frac{(-1)^s\zeta^{-n\underline{j}}}{(t-1)!}\sum_{0\leq r\leq s-1}F^{(s-r)}(t+r;j-k;d_2)(n) + (-1)^{s+t}\zeta^{-n\underline{k}}F^{(s)}(t;k-j)(n).$$

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Now let $g_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] = \sum c_n z^n$. Since c_n is expressed in terms of the values of an M-power series function by the above expression, the limit $\lim_{N\to\infty} c_{lq^N}$ exists. In order to find this limit, we employ Proposition 6.2.1 in the differential equation for $dg_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^s]$ and find that

$$\lim_{N\to\infty} c_{lq^N} = \zeta^{-\underline{jl}} g_j[e_0^{t-1} e_k e_0^s].$$

Using the expression above this gives $\lim_{N\to\infty} F^{(s)}(t; k-j)(lq^N) =$

$$(-1)^{s+t}\zeta^{(\underline{k}-\underline{j})l} g_j[e_0^{t-1} e_k e_0^s] = \frac{(-1)^t}{(t-1)!}\zeta^{(\underline{k}-\underline{j})l}(s+1)_{t-1} g_j[e_0^{s+t-1} e_k].$$

These limits determine the M-power series function $F(t; i)$ completely as

$$F(t; i)(n) = (-1)^{t-1}(g[e_0^{t-1} e_i](1-\zeta^{\underline{in}}) - \frac{1}{(t-1)!}\sum_{1\le r} \zeta^{\underline{in}}(r+1)_{t-1} g[e_0^{r+t-1} e_i] n^r),$$

for $p|n$. □

The proof of the above proposition has the following corollaries.

Corollary 6.2.3. *For $t \ge 1$, $p|n$ and $M \not| i$, we have*

$$p^t \sum_{\substack{0<k<n \\ p\not| k}} \frac{\zeta^{\underline{ik}}}{k^t} = (-1)^{t-1}(g[e_0^{t-1} e_i](1-\zeta^{\underline{in}}) - \frac{1}{(t-1)!}\sum_{1\le r} \zeta^{\underline{in}}(r+1)_{t-1} g[e_0^{r+t-1} e_i] n^r).$$

Corollary 6.2.4. *For, $t, s \ge 1$ and $j \ne k$, we have* $g_{\mathcal{F}}[e_j e_0^{t-1} e_k e_0^{s-1}] =$

$$\frac{(-1)^s}{(t-1)!}\sum_{0\le r\le s-1} (r+1)_{t-1} S(t+r, s-r; k, j; d_2) + (-1)^{s+t} S(t, (s); j, k).$$

6.3. Computation of $g_i[e_j e_k e_0^s]$

We already made this computation for $s = 0$ in Sect. 6.1. The computation will be based on induction on s. From now on we assume that $s > 0$. The pattern of the proof is the same as that in Sect. 6.1, some parts simplified using Proposition 6.2.1.

The differential equation gives $dg_{\mathcal{F}}[e_i e_j e_k e_0^s] = g_{\mathcal{F}}[e_j e_k e_0^s]\mathcal{F}^*\omega_i$

$$-pg_{\mathcal{F}}[e_i e_j e_k e_0^{s-1}]\omega_0 - pg_i[e_j e_k e_0^s]\omega_{\underline{i}}$$
$$-pg_j^{-1}[e_i] g_j[e_k e_0^s]\omega_{\underline{j}} - pg_{\mathcal{F}}[e_i] g_j[e_k e_0^s]\omega_{\underline{j}}.$$

In order to be able to use Proposition 6.2.1 we note that, using Corollary 6.2.4, $g_{\mathcal{F}}[e_j e_k e_0]\mathcal{F}^*\omega_i = d(-\sum_{0\le r\le 1} S(1+r, 2-r, 1; k, j, i; d_2, d_3) + S(1, (2), 1; j, k, i; d_3))$ and $-g_{\mathcal{F}}[e_i]\omega_{\underline{j}} = dS(1, 1; i, j)$.

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As above first we will show that the individual limits exist. To start with note that $lq^N S(a,b,1;k,j,i;d_2,d_3)[lq^N]=0$,

$$lq^N S(a,(b),1;j,k,i;d_3))[lq^N]=\zeta^{-\underline{il}}F(a,(b);k-j,i-k;f_2)(lq^N)$$

and

$$lq^N S(a,1;i,j)[lq^N]=\zeta^{-\underline{jl}}F(a;j-i)(lq^N).$$

On the other hand using Corollary 6.1.6 we obtain an expression for $g_{\mathcal{F}}[e_ie_je_k]$. We have:

(i) $S(a,(b),(1);j,k,i;d_3)[lq^n]=\zeta^{-\underline{il}}F^{(1)}(a,(b);k-j,i-k;f_2)(lq^N)$
(ii) $S(a,b,(1);j,i,k;d_2)[lq^N]=\zeta^{-\underline{kl}}F^{(1)}(a,b;i-j,k-i;d_2)(lq^N)$
(iii) $S(a,(b),(1);i,j,k)[lq^N]=\zeta^{-\underline{kl}}F^{(1)}(a,(b);j-i,k-j)(lq^N)$
(iv) $S(a,(1);i,k)[lq^N]=\zeta^{-\underline{kl}}F^{(1)}(a;k-i)(lq^N)$.

Since the above limits exist as $N\to\infty$ we can use Proposition 6.2.1 and obtain that the limit of the following as $N\to\infty$ is equal to $-g_i[e_je_ke_0]\zeta^{-\underline{il}}-g_j^{-1}[e_i]g_j[e_ke_0]\zeta^{-\underline{jl}}$:

$$\begin{aligned}
&\zeta^{-\underline{il}}F(1,(2);k-j,i-k;f_2)(lq^N)+g_j[e_ke_0]\zeta^{-\underline{jl}}F(1;j-i)(lq^N)\\
&+\zeta^{-\underline{il}}F^{(1)}(1,(1);k-j,i-k;f_2)(lq^N)+\zeta^{-\underline{kl}}F^{(1)}(1,1;i-j,k-i;d_2)(lq^N)\\
&-\zeta^{-\underline{kl}}F^{(1)}(1,(1);j-i,k-j)(lq^N)+g_k[e_j]\zeta^{-\underline{kl}}F^{(1)}(1;k-i)(lq^N)\\
&-g_j[e_k]\zeta^{-\underline{jl}}F^{(1)}(1;j-i)(lq^N)-g_i[e_j]\zeta^{-\underline{kl}}F^{(1)}(1;k-i)(lq^N)\\
&+g_i[e_j]\zeta^{-\underline{kl}}F^{(1)}(1;k-j)(lq^N).
\end{aligned}$$

This gives the following formula, with i, j, and k pairwise distinct, for $g_i[e_je_ke_0]$:

$$\begin{aligned}
&-\mathcal{X}(1,(2);k-j,i-k;f_2)-\frac{\zeta^{\underline{i-j}}}{1-\zeta^{\underline{k-j}}}\mathcal{X}(2;k-j)\mathcal{X}(1;j-i)\\
&-\mathcal{X}^{(1)}(1,(1);k-j,i-k;f_2)-\zeta^{\underline{i-k}}\mathcal{X}^{(1)}(1,1;i-j,k-i;d_2)\\
&+\zeta^{\underline{i-k}}\mathcal{X}^{(1)}(1,(1);j-i,k-j)+\frac{\zeta^{\underline{i-j}}\mathcal{X}(1;i-j)\mathcal{X}(2;k-j)}{(1-\zeta^{\underline{i-j}})(1-\zeta^{\underline{k-j}})}\\
&-\frac{\zeta^{\underline{i-k}}\mathcal{X}(1;j-k)\mathcal{X}^{(1)}(1;k-i)}{1-\zeta^{\underline{j-k}}}+\frac{\zeta^{\underline{i-j}}\mathcal{X}(1;k-j)\mathcal{X}^{(1)}(1;j-i)}{1-\zeta^{\underline{k-j}}}\\
&+\frac{\zeta^{\underline{i-k}}\mathcal{X}(1;j-i)\mathcal{X}^{(1)}(1;k-i)}{1-\zeta^{\underline{j-i}}}-\frac{\zeta^{\underline{i-k}}\mathcal{X}(1;j-i)\mathcal{X}^{(1)}(1;k-j)}{1-\zeta^{\underline{j-i}}}.
\end{aligned}$$

This also implies the following expression of $g_{\mathcal{F}}[e_ie_je_ke_0]$ in terms of regularized series: $g_{\mathcal{F}}[e_ie_je_ke_0]=$

$$-\sum_{\substack{0\le p,q,r\le 1\\ p+q+r=1}}S(1+p,1+q,1+r;k,j,i;d_2,d_3)+S(1,(2),(1);j,k,i;d_3)$$

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$$+S(1,(1),(2);j,k,i;d_3)+S(1,1,(2);j,i,k;d_2)+g_j[e_ke_0]S(1,(1);i,j)$$
$$-S(1,(1),(2);i,j,k)+g_k[e_j]S(1,(2);i,k)-g_j[e_k]S(1,(2);i,j)$$
$$+g_i[e_j]S(1,(2);j,k)-g_i[e_j]S(1,(2);i,k).$$

We use this information in the differential equation for $dg_{\mathcal{F}}[e_ie_je_ke_0^2]$ above and this gives a formula for $g_i[e_je_ke_0^2]$. Inducting on s, we find the following formulas for $g_{\mathcal{F}}[e_ie_je_ke_0^{s-1}]$ and $g_i[e_je_ke_0^s]$. Namely, $g_{\mathcal{F}}[e_ie_je_ke_0^{s-1}]=$

$$(-1)^{s-1}\sum_{\substack{0\le p,q,r\\ p+q+r=s-1}}S(1+p,1+q,1+r;k,j,i;d_2,d_3)+(-1)^sg_k[e_j]S(1,(s);i,k)$$
$$+(-1)^s\sum_{\substack{0\le p,q\\ p+q=s-1}}S(1,(1+p),(1+q);j,k,i;d_3)+(-1)^sS(1,1,(s);j,i,k;d_2)$$
$$+\sum_{0\le r\le s-1}(-1)^rg_j[e_ke_0^{s-1-r}]S(1,(1+r);i,j)+(-1)^{s-1}S(1,(1),(s);i,j,k)$$
$$+(-1)^sg_i[e_j]S(1,(s);j,k)+(-1)^{s-1}g_i[e_j]S(1,(s);i,k).$$

Then using the differential equation for $dg_{\mathcal{F}}[e_ie_je_ke_0^s]$ and using Proposition 6.2.1 and noting that for the limit $\lim_{N\to\infty}$ the coefficient

(i) $S(a,(b);i,k)[q^N]$ contributes $\zeta^{-\underline{k}}\mathcal{X}^{(b)}(a;k-i)$
(ii) $S(a,(b),(c);j,k,i;d_3)[q^N]$ contributes $\zeta^{-\underline{i}}\mathcal{X}^{(c)}(a,(b);k-j,i-k;f_2)$
(iii) $S(a,b,(c);j,i,k;d_2)[q^n]$ contributes $\zeta^{-\underline{k}}\mathcal{X}^{(c)}(a,b;i-j,k-j;d_2)$
(iv) $S(a,(b);i,j)[q^N]$ contributes $\zeta^{-\underline{j}}\mathcal{X}^{(b)}(a;j-i)$
(v) $S(a,(b),(c);i,j,k)[q^N]$ contributes $\zeta^{-\underline{k}}\mathcal{X}^{(c)}(a,(b);j-i,k-j)$;

we obtain the following identity: $-\zeta^{-\underline{i}}g_i[e_je_ke_0^s]-\zeta^{-\underline{j}}g_j^{-1}[e_i]g_j[e_ke_0^s]=$

$$(-1)^{s+1}\zeta^{-\underline{k}}g_k[e_j]\mathcal{X}^{(s)}(1;k-i)+(-1)^{s+1}\zeta^{-\underline{k}}\mathcal{X}^{(s)}(1,1;i-j,k-j;d_2)$$
$$+(-1)^{s+1}\zeta^{-\underline{i}}\sum_{0\le r\le s}\mathcal{X}^{(r)}(1,(s+1-r);k-j,i-k;f_2)$$
$$+\zeta^{-\underline{j}}\sum_{0\le r\le s}(-1)^rg_j[e_ke_0^{s-r}]\mathcal{X}^{(r)}(1;j-i)+(-1)^s\zeta^{-\underline{k}}\mathcal{X}^{(s)}(1,(1);j-i,k-j)$$
$$+(-1)^{s+1}\zeta^{-\underline{k}}g_i[e_j]\mathcal{X}^{(s)}(1;k-j)+(-1)^s\zeta^{-\underline{k}}g_i[e_j]\mathcal{X}^{(s)}(1;k-i).$$

This can be summarized as the following proposition.

Proposition 6.3.1. *Assume that i, j and k are pairwise distinct and that $s>0$. Then we have $g_i[e_je_ke_0^s]=$*

$$(-1)^s\zeta^{\underline{i}-\underline{k}}\frac{\mathcal{X}(1;j-k)}{1-\zeta^{\underline{j}-\underline{k}}}\mathcal{X}^{(s)}(1;k-i)+(-1)^s\zeta^{\underline{i}-\underline{k}}\mathcal{X}^{(s)}(1,1;i-j,k-j;d_2)$$
$$+(-1)^s\sum_{0\le r\le s}\mathcal{X}^{(r)}(1,(s+1-r);k-j,i-k;f_2)$$
$$-\frac{\zeta^{\underline{i}-\underline{j}}}{1-\zeta^{\underline{k}-\underline{j}}}\sum_{0\le r\le s}(-1)^r\mathcal{X}(s-r+1;k-j)\mathcal{X}^{(r)}(1;j-i)$$

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$$+(-1)^{s+1}\zeta^{\underline{i}-\underline{k}}\mathcal{X}^{(s)}(1,(1);j-i,k-j)+\zeta^{\underline{i}-\underline{j}}\frac{\mathcal{X}(1,i-j)}{1-\zeta^{\underline{i}-\underline{j}}}\frac{\mathcal{X}(s+1;k-j)}{1-\zeta^{\underline{k}-\underline{j}}}$$
$$+(-1)^{s}\zeta^{\underline{i}-\underline{k}}\frac{\mathcal{X}(1;j-i)}{1-\zeta^{\underline{j}-\underline{i}}}\mathcal{X}^{(s)}(1;k-j)+(-1)^{s+1}\zeta^{\underline{i}-\underline{k}}\frac{\mathcal{X}(1;j-i)}{1-\zeta^{\underline{j}-\underline{i}}}\mathcal{X}^{(s)}(1;k-i).$$

6.4. *Computation of* $g_i[e_je_0^{s-1}e_ke_0^{t-1}]$

We know the answer if $s=1$ from the previous section. Let us assume that $s>1$. Also first assume that $t=1$.

6.4.1. Computation of $g_i[e_je_0^{s-1}e_k]$ The differential equation gives $dg_{\mathcal{F}}[e_ie_je_0^{s-1}e_k]=$

$$g_{\mathcal{F}}[e_je_0^{s-1}e_k]\mathcal{F}^*\omega_i-pg_i[e_je_0^{s-1}e_k]\omega_{\underline{i}}-pg_{\mathcal{F}}[e_i]g_j[e_0^{s-1}e_k]\omega_{\underline{j}}$$
$$-pg_j^{-1}[e_i]g_j[e_0^{s-1}e_k]\omega_{\underline{j}}-pg_{\mathcal{F}}[e_ie_je_0^{s-1}]\omega_{\underline{k}}-pg_{\mathcal{F}}[e_i]g_k^{-1}[e_je_0^{s-1}]\omega_{\underline{k}}$$
$$-pg_k^{-1}[e_ie_je_0^{s-1}]\omega_{\underline{k}}.$$

Computing residues at ∞ in the above expression, we obtain that $g_i[e_je_0^{s-1}e_k]=$

$$g_\infty[e_je_0^{s-1}e_k]-g_\infty[e_ie_je_0^{s-1}]-g_j^{-1}[e_i]g_j[e_0^{s-1}e_k]-g_k^{-1}[e_ie_je_0^{s-1}].$$

Similarly, computing the residues in the differential equation for $dg_{\mathcal{F}}[e_ie_je_0^s]$ we obtain that

$$g_\infty[e_ie_je_0^{s-1}]=g_\infty[e_je_0^s]-g_i[e_je_0^s]=(-1)^{s+1}g_i[e_0^se_j].$$

Using the expression $g_\infty[e_je_0^{s-1}e_k]=(-1)^s s g_k[e_0^se_j]$ that we found in Sect. 4.1, we arrive at the following identity: $g_i[e_je_0^{s-1}e_k]=$

$$g_k[e_ie_je_0^{s-1}]+(-1)^s(sg_k[e_0^se_j]+g_i[e_0^se_j]+g_k[e_i]g_k[e_0^{s-1}e_j])$$
$$+g_j[e_i]g_j[e_0^{s-1}e_k].$$

Using the computation of multi-zeta values of depth one in Sect. 4.2, this can be rewritten as the following proposition.

Proposition 6.4.1. *Assume that i, j, and k are pairwise distinct and that $s>1$. Then $g_i[e_je_0^{s-1}e_k]=$*

$$g_k[e_ie_je_0^{s-1}]+\frac{s\mathcal{X}(s+1;j-k)}{1-\zeta^{\underline{j}-\underline{k}}}+\frac{\mathcal{X}(s+1;j-i)}{1-\zeta^{\underline{j}-\underline{i}}}-\frac{\mathcal{X}(1;i-k)}{1-\zeta^{\underline{i}-\underline{k}}}\frac{\mathcal{X}(s;j-k)}{1-\zeta^{\underline{j}-\underline{k}}}$$
$$+(-1)^{s-1}\frac{\mathcal{X}(1;i-j)\mathcal{X}(s;k-j)}{(1-\zeta^{\underline{i}-\underline{j}})(1-\zeta^{\underline{k}-\underline{j}})},$$

where $g_k[e_ie_je_0^{s-1}]$ is given by Proposition 6.3.1.

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Using the expression for $g_{\mathcal{F}}[e_l e_0^a e_m e_0^b]$ that we found in Corollary 6.2.4 in the differential equation above, we see that $dg_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k] =$

$$\begin{aligned}
&d(S(s,1,1;k,j,i;d_2,d_3)) + (-1)^s S(s,(1),1;j,k,i;d_3) + g_j[e_0^{s-1}e_k]S(1,1;i,j)\\
&+g_k^{-1}[e_j e_0^{s-1}]S(1,1;i,k) + (-1)^s \sum_{0\le r\le s-1} S(1+r,s-r,1;j,i,k;d_2)\\
&+(-1)^{s+1}S(1,(s),1;i,j,k)) - pg_i[e_j e_0^{s-1}e_k]\omega_{\underline{i}} - pg_j^{-1}[e_i]g_j[e_0^{s-1}e_k]\omega_{\underline{j}}\\
&-pg_k^{-1}[e_i e_j e_0^{s-1}]\omega_{\underline{k}}.
\end{aligned}$$

By the same arguments as above we see that the hypotheses of Proposition 6.2.1 are satisfied, and we have the following:

Proposition 6.4.2. *Suppose that $s > 1$, then we have $g_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k] =$*

$$\begin{aligned}
&S(s,1,1;k,j,i;d_2,d_3)) + (-1)^s S(s,(1),(1);j,k,i;d_3) + g_j[e_0^{s-1}e_k]S(1,(1);i,j)\\
&+g_k^{-1}[e_j e_0^{s-1}]S(1,(1);i,k) + (-1)^s \sum_{0\le r\le s-1} S(1+r,s-r,(1);j,i,k;d_2)\\
&+(-1)^{s+1}S(1,(s),(1);i,j,k) + g_i[e_j e_0^{s-1}e_k]S(1;i) + g_j^{-1}[e_i]g_j[e_0^{s-1}e_k]S(1;j)\\
&+g_k^{-1}[e_i e_j e_0^{s-1}]S(1;k).
\end{aligned}$$

6.4.2. Computation of $g_i[e_j e_0^{s-1} e_k e_0^{t-1}]$ Assume that $s, t > 1$. Then the differential equation gives that $dg_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k e_0^{t-1}] =$

$$\begin{aligned}
&g_{\mathcal{F}}[e_j e_0^{s-1}e_k e_0^{t-1}]\mathcal{F}^*\omega_i - pg_{\mathcal{F}}[e_i e_j e_0^{s-1}e_k e_0^{t-2}]\omega_0 - pg_i[e_j e_0^{s-1}e_k e_0^{t-1}]\omega_{\underline{i}}\\
&-pg_j^{-1}[e_i]g_j[e_0^{s-1}e_k e_0^{t-1}]\omega_{\underline{j}} - pg_{\mathcal{F}}[e_i]g_j[e_0^{s-1}e_k e_0^{t-1}]\omega_{\underline{j}}.
\end{aligned}$$

We will use Proposition 6.2.1 and do induction on t, starting with the formulas we found above for $g_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k]$ and $g_{\mathcal{F}}[e_a e_0^l e_b e_0^m]$. We find that $g_{\mathcal{F}}[e_i e_j e_0^{s-1} e_k e_0^{t-1}] =$

$$\begin{aligned}
&\frac{(-1)^{t-1}}{(s-1)!}\sum_{\substack{0\le r,q\\ r+q\le t-1}} (r+1)_{s-1}S(s+r,1-q,t-(q+r);k,j,i;d_2,d_3)\\
&+\sum_{0\le r\le t-1}(-1)^r g_j[e_0^{s-1}e_k e_0^{t-1-r}]S(1,(1+r);i,j)\\
&+(-1)^{t-1}g_k^{-1}[e_j e_0^{s-1}]S(1,(t);i,k) + (-1)^{s+t}S(1,(s),(t);i,j,k)\\
&+(-1)^{s+t+1}\sum_{0\le r\le s-1} S(1+r,s-r,(t);j,i,k;d_2)\\
&+(-1)^{s+t+1}\sum_{1\le r\le t} S(s,(t+1-r),(r);j,k,i;d_3)\\
&+(-1)^{t-1}g_k^{-1}[e_i e_j e_0^{s-1}]S(t;k) + \sum_{1\le r\le t}(-1)^{r-1}g_i[e_j e_0^{s-1}e_k e_0^{t-r}]S(r;i)
\end{aligned}$$

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$$+\sum_{1\le r\le t}(-1)^{r-1}g_j^{-1}[e_i]g_j[e_0^{s-1}e_ke_0^{t-r}]S(r;j),$$

for $s>1$ and $t\ge 1$.

We also obtain that $-\zeta^{-\underline{i}}g_i[e_je_0^{s-1}e_ke_0^{t-1}]-\zeta^{-\underline{j}}g_j^{-1}[e_i]g_j[e_0^{s-1}e_ke_0^{t-1}]=$

$$\begin{aligned}
&\zeta^{-\underline{j}}\sum_{0\le r\le t-2}(-1)^{r+1}g_j[e_0^{s-1}e_ke_0^{t-2-r}]\mathcal{X}^{(1+r)}(1;j-i)\\
&+\zeta^{-\underline{k}}(-1)^{t-1}g_k^{-1}[e_je_0^{s-1}]\mathcal{X}^{(t-1)}(1;k-i)\\
&+\zeta^{-\underline{k}}(-1)^{s+t}\mathcal{X}^{(t-1)}(1,(s);j-i,k-j)\\
&+\zeta^{-\underline{k}}(-1)^{s+t+1}\sum_{0\le r\le s-1}\mathcal{X}^{(t-1)}(1+r,s-r;i-j,k-i;d_2)\\
&+\zeta^{-\underline{i}}(-1)^{s+t+1}\sum_{1\le r\le t-1}\mathcal{X}^{(r)}(s,(t-r);k-j,i-k;f_2)\\
&+\zeta^{-\underline{i}}(-1)^{s+t+1}\mathcal{X}(s,(t);k-j,i-k;f_2)+\zeta^{-\underline{j}}g_j[e_0^{s-1}e_ke_0^{t-1}]\mathcal{X}(1;j-i).
\end{aligned}$$

Using the formula for the depth one multi-zeta values which we found in Sect. 4.2, we obtain the following.

Theorem 6.4.3. *Assume that $s,t\ge 2$. Then $g_i[e_je_0^{s-1}e_ke_0^{t-1}]=$*

$$\begin{aligned}
&(-1)^{s-1}\zeta^{\underline{i}-\underline{j}}\binom{s+t-2}{s-1}\frac{\mathcal{X}(1;i-j)}{1-\zeta^{\underline{i}-\underline{j}}}\frac{\mathcal{X}(s+t-1;k-j)}{1-\zeta^{\underline{k}-\underline{j}}}\\
&-\zeta^{\underline{i}-\underline{j}}\sum_{0\le r\le t-2}(-1)^{r+s}\binom{s+t-r-3}{s-1}\frac{\mathcal{X}(s+t-r-2;k-j)}{1-\zeta^{\underline{k}-\underline{j}}}\mathcal{X}^{(1+r)}(1;j-i)\\
&-\zeta^{\underline{i}-\underline{k}}((-1)^t\frac{\mathcal{X}(s;j-k)}{1-\zeta^{\underline{j}-\underline{k}}}\mathcal{X}^{(t-1)}(1;k-i)+(-1)^{s+t}\mathcal{X}^{(t-1)}(1,(s);j-i,k-j)\\
&+(-1)^{s+t+1}\sum_{0\le r\le s-1}\mathcal{X}^{(t-1)}(1+r,s-r;i-j,k-i;d_2))\\
&+(-1)^{s+t}(\sum_{1\le r\le t-1}\mathcal{X}^{(r)}(s,(t-r);k-j,i-k;f_2)+\mathcal{X}(s,(t);k-j,i-k;f_2))\\
&+(-1)^s\binom{s+t-2}{s-1}\zeta^{\underline{i}-\underline{j}}\frac{\mathcal{X}(s+t-1;k-j)}{1-\zeta^{\underline{k}-\underline{j}}}\mathcal{X}(1;j-i),
\end{aligned}$$

for distinct i,j,k.

Using the fact that the g_i's are group-like, one obtains a formula for all the cyclotomic p-adic multi-zeta values $\zeta_p(s_2,s_1;i_2,i_1)$ of depth two, by Theorem 6.4.3, Propositions 6.4.1 and 6.3.1.

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