Swan Conductors and Torsion in the Logarithmic de Rham Complex

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ABSTRACT. We prove, for an arithmetic scheme X/S over a discrete valuation ring whose special fiber is a strict normal crossings divisor in X, that the Swan conductor of X/S is equal to the Euler characteristic of the torsion in the logarithmic de Rham complex of X/S. This is a precise logarithmic analog of a theorem by Bloch [1].

1. INTRODUCTION

Let A be a discrete valuation ring with maximal ideal $\mathfrak{m}$, perfect residue field k, and field of fractions K, $S = \operatorname{Spec} A$, with closed point s, and X/S an arithmetic surface over S, i.e. an integral, regular scheme which is proper, flat and of relative dimension one over S. We also assume that the reduced special fiber $X_{s,\mathrm{red}}$ is a strict normal crossings divisor in X, by this we mean that $X_{s,\mathrm{red}}$ is a normal crossings divisor in X, and that the irreducible components of $X_{s,\mathrm{red}}$ are regular schemes.

There are two numerical invariants for X/S. One of the invariants is based on etale cohomology: the Swan conductor of the Galois representation on the cohomology of the generic fiber; and the other one on de Rham cohomology: the Euler characteristic of the torsion in the logarithmic de Rham complex. Both invariants measure the bad reduction of the special fiber of the arithmetic surface. Below we will prove that these two numerical invariants are in fact the same.

We use [5] as a general reference on logarithmic geometry. Let us endow S with the log structure corresponding to the natural inclusion

$$\mathcal{O}_S \backslash \{0\} \to \mathcal{O}_S.$$

Similarly, endow X with the log structure corresponding to the natural map

$$\mathcal{O}_X \cap j_* \mathcal{O}_{X_K}^* \to \mathcal{O}_X,$$

where $j : X_K \to X$ is the inclusion. Then the structure map from X to S becomes a map of fine log schemes. Let $\Omega_{X/S,\log}$ denote its logarithmic de Rham complex,

$$\mathcal{O}_X \to \Omega^1_{X/S,\log} \to \Omega^2_{X/S,\log}.$$

Taking A-torsion in $\Omega_{X/S,\log}$, we obtain the complex $\Omega_{X/S,\log,\mathrm{tors}}$, which is a complex supported on the special fiber of X. We will be mainly interested in the Euler characteristic $\chi(\Omega_{X/S,\log,\mathrm{tors}})$ of $\Omega_{X/S,\log,\mathrm{tors}}$, which will be defined as follows. If K is a bounded complex of coherent sheaves on X that is exact on the generic fiber of X, the hypercohomology groups $\mathrm{H}(X, K)$ are modules of finite length over A, and we

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put

$$\chi(K^{\cdot}) = \sum_i (-1)^i \mathrm{length}_A(\mathrm{H}^i(X, K^{\cdot})).$$

On the other hand, X_K/K has a Swan conductor $\mathrm{Sw}(X_K/K)$, defined as follows (cf. [1], [6]). Let K' be the strict henselization of the completion of K (with respect to its discrete valuation), and ℓ be a prime different from the characteristic p of k. The action of the wild inertia group of $\mathrm{Gal}(\overline{K}'/K')$ on $\mathrm{H}^i_{\text{ét}}(X_{\overline{K}'}, \mathbb{Q}_\ell)$ factors through the wild inertia group of $\mathrm{Gal}(L/K')$, for a finite Galois extension L of K', being the continuous action of a pro-p group on a finite dimensional ℓ-adic vector space. Let $\mathrm{SW}_{L/K'}$ be the Swan module over $\mathbb{Z}_\ell$, which is the projective $\mathbb{Z}_\ell[\mathrm{Gal}(L/K')]$ module having as character the Swan character of $\mathrm{Gal}(L/K')$. Then $\mathrm{Sw}_{\overline{K}/K}(\mathrm{H}^i_{\text{ét}}(X_K, \mathbb{Q}_\ell))$ is defined to be

$$\dim_{\mathbb{Q}_\ell} \mathrm{Hom}_{\mathbb{Z}_\ell[\mathrm{Gal}(\overline{K}'/K')]}(\mathrm{SW}_{L/K'}, \mathrm{H}^i_{\text{ét}}(X_{\overline{K}'}, \mathbb{Q}_\ell)).$$

This is independent of the choice of L since for a finite Galois extension L' of K' containing L; we have

$$\mathrm{SW}_{L/K'} = \mathrm{SW}_{L'/K'} \otimes_{\mathbb{Z}_\ell[\mathrm{Gal}(L'/K')]} \mathbb{Z}_\ell[\mathrm{Gal}(L/K')].$$

Finally, we define

$$\mathrm{Sw}(X_K/K) = \sum_{i=0}^{2} (-1)^i \mathrm{Sw}_{\overline{K}/K}(\mathrm{H}^i_{\text{ét}}(X_K, \mathbb{Q}_\ell)).$$

Then we have the following theorem.

Theorem 1. *With the notation above, we have*

$$\chi(\Omega_{X/S,\log,\mathrm{tors}}^{\cdot}) = -\mathrm{Sw}(X_K/K).$$

This is a logarithmic version of a theorem by Bloch ([1], Theorem 1). In fact we follow the method in [1] closely. However we are able to reduce some of the steps by explicit computations since our hypotheses on the special fiber of X/S makes this case easier to handle than the general situation in [1]. In some sense, Bloch's Theorem 1 in [1] says that $\chi(\Omega_{X/S,\mathrm{tors}}^{\cdot})$ counts the total number of vanishing cycles (including the wildly vanishing ones). Putting the above log structure on the scheme X/S can be considered as, according to the philosophy of Kato and Illusie, obtaining some kind of space that contains the generic fiber X_K/K but is less than the total space X/S and, at the same time, remembers some of the information on the special fiber. Then the above formula in the logarithmic case may be viewed as saying that $\chi(\Omega_{X/S,\log,\mathrm{tors}}^{\cdot})$ counts the total number of vanishing cycles of X/S that, in some sense, vanish before they reach the special fiber. From the philosophy of logarithmic geometry, then the Theorem 1 above is completely as expected.

Here is a brief outline of the paper. We begin with a lemma on the local structure of maps of the form X/S, which will be important in explicit computations later on. Then, applying the Riemann-Roch theorem to the torsion part of the logarithmic de Rham complex, we express the Euler characterstic of this complex in terms of the localized Chern character of this complex. Next, we show in Lemma 2 that

the localized character of this complex is the same as that of the more manageable complex $C^{\cdot}$. Next in Lemmas 3 and 4, we show that the localized Chern character of this complex is the same as the localized second Chern class of $\Omega^1_{X/S,log}$. Combining this with the conductor formula of Bloch [1] and a Theorem in [6] that computes the difference between the difference between the second localized Chern classes of $\Omega^1_{X/S}$ and $\Omega^1_{X/S,log}$ finishes the proof. In the last part of the paper we give some applications of the main result.

2. Proof

We need a lemma on the local structure of X/S, c.f. Proposition 4.4.4 and Corollary 4.4.7 of [6].

Lemma 1. *For each point $x \in X_s$ there is a Zariski open neighborhood U of x in X, a scheme P, etale over $\mathbb{A}^2_S = \mathrm{Spec}A[t,s]$, and a closed immersion $i : U \to P$, such that U is defined in P by an equation of the form $\pi - ut^a s^b$, where π is a uniformizer of A, u is a unit in P, t and s form a system of parameters in $\mathcal{O}_{X,x}$ when restricted to X, and a, b are nonnegative integers.*

Proof. Since $X_{s,red}$ is a strict normal crossings divisor in X, there are at most two components of $X_{s,red}$ passing through x. If there is only one component we choose t to be a local defining function for that component, and choose s so that $\{t, s\}$ is a system of parameters at x. If there are two components we choose t and s to be the local defining functions for these two components. Now, by considering the multiplicity of X_s along the components of $X_{s,red}$ passing through x, we see that X_s is defined by $t^a s^b = 0$ in a neighborhood of x in X for some nonnegative integers a and b. Using t and s, we get a map from an open neighborhood U of x to $\mathbb{A}^2$. Since $\{s, t\}$ is a system of parameters at t, and the residue field k is perfect, by restricting U if necessary, we may assume that this map is unramified. By restricting U if necessary, we can factor this map as a closed immersion into a scheme P followed by an etale map from P to $\mathbb{A}^2_S$ ([4], Corollaire 18.4.7). Now since $\pi/(t^a s^b)$ is a unit in U, by restricting U and P if necessary, we can find a unit u in P such that $\pi - ut^a s^b$ vanishes on U. Since U is a divisor in P, and $\pi - ut^a s^b$ vanishes on U to see that U is defined by $\pi - ut^a s^b$, it suffices to note that $\pi - ut^a s^b$ is not in $\mathfrak{m}^2_{P,x}$ (note that, since P is smooth over S, $\{\pi, t, s\}$ is a system of parameters for $\mathcal{O}_{P,x}$). $\qquad\square$

Continuing with the notation of the lemma and denoting the conormal sheaves with N, we get an exact sequence:

$$0 \to N_{U/P} \to \Omega^1_{P/S}|_U \to \Omega^1_{U/S} \to 0.$$

In fact, this sequence is exact for any closed imbedding of U into a scheme P smooth over S. To see the injectivity of the map $N_{U/P} \to \Omega^1_{P/S}|_U$ we proceed as follows. If we denote the kernel of this map by M, we see that since U_K/K is smooth, the map is injective over U_K, and hence $M|_{U_K} = 0$. On the other hand, since the imbedding $U \to P$ is regular $N_{U/P}$ is locally free. And therefore M, being a subsheaf of a locally free coherent sheaf on an integral scheme U that is supported on a proper closed subscheme of U, is zero. In the following we endow P with the log structure

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associated with the inclusion

$$\mathcal{O}_V^* \to \mathcal{O}_P,$$

where $V = P - \{t^a s^b = 0\}$. We denote by $\Omega^1_{P/S,\log}$ the sheaf of log differentials where we endow S with the trivial log structure. Then for the logarithmic differentials we get the similar exact sequence

$$0 \to N_{U/P} \otimes_A \mathfrak{m}^{-1} \to \Omega^1_{P/S,\log}|_U \to \Omega^1_{U/S,\log} \to 0,$$

where the map

$$\delta : N_{U/P} \otimes_A \mathfrak{m}^{-1} \to \Omega^1_{P/S,\log}|_U$$

is the map sending

$$(\pi - u t^a s^b) \otimes \pi^{-1} \text{ to } u^{-1}\mathrm{d}u + a \cdot \mathrm{d}\log(t) + b \cdot \mathrm{d}\log(s).$$

This resolution of $\Omega^1_{U/S,\log}$ gives a map

$$\alpha_U : \Omega^1_{U/S,\log} \to \det\Omega^1_{U/S,\log} \cong \mathrm{Hom}(N_{U/P} \otimes_A \mathfrak{m}^{-1}, \Lambda^2\Omega^1_{P/S,\log}|_U),$$

by the formula

$$\alpha_U(a)(b) = \widetilde{a} \wedge \delta(b),$$

where $\widetilde{a}$ is a section of $\Omega^1_{P/S,\log}$ that maps to a. Since any two resolutions of $\Omega^1_{U/S,\log}$ are homotopic and the maps α_U for different resolutions are compatible with the isomorphisms induced by the homotopies on $\det\Omega^1_{U/S,\log}$, we get a map

$$\alpha : \Omega^1_{X/S,\log} \to \det\Omega^1_{X/S,\log}.$$

Let Z_U be the closed subscheme of U defined by the section of the locally free sheaf $\Omega^1_{P/S,\log}$ corresponding to δ. As above, this does not depend on the imbedding, and hence defines a closed subscheme Z of X. Note that $\Omega^1_{X/S,\log}$ is an invertible sheaf over $X - Z$, and hence $\alpha|_{X-Z}$ is an isomorphism. Let C denote the complex

$$\alpha : \Omega^1_{X/S,\log} \to \det\Omega^1_{X/S,\log},$$

with $\Omega^1_{X/S,\log}$ in degree 1.

For any bounded complex K of locally free coherent sheaves on X, which is exact outside a proper closed subscheme Y, we have the bivariant class

$$\mathrm{ch}_Y^X(K) \text{ in } \mathrm{A}(Y \to X)_{\mathbb{Q}}$$

([3], Chapter 18), which is the same for any other bounded complex of locally free coherent sheaves that is quasi-isomorphic to K and exact outside Y. Therefore we can define $\mathrm{ch}_Y^X(F)$ for any coherent sheaf F supported on Y. If Y is X_s we will use the notation ch_s for $\mathrm{ch}_{X_s}^X$. Now by the Riemann-Roch theorem ([3], Chapter 18; [7], Lemma 2.4), we have

$$\chi(\Omega_{X/S,\log,\mathrm{tors}}) = \deg(\mathrm{ch}_s(\Omega_{X/S,\log,\mathrm{tors}}) \cap \mathrm{Td}(X/S)),$$

where $\mathrm{Td}(X/S)$ is the Todd class of X/S. Since C is exact outside Z, after choosing any resolution of $\Omega^1_{X/S,\log}$ we can define $\mathrm{ch}_s(C)$. Using the following lemma, we will work with C instead of $\Omega_{X/S,\log,\mathrm{tors}}$.

Lemma 2. *With the notation above, we have*

$$\mathrm{ch}_s(C) = \mathrm{ch}_s(\Omega_{X/S,\log,\mathrm{tors}}).$$

Proof. Note that $\ker(\alpha) = \Omega^1_{X/S,\log,\mathrm{tors}}$, since $\det\Omega^1_{X/S,\log}$ is an invertible sheaf and α is an isomorphism on the generic fiber. Therefore to finish the proof of the lemma we need to show that $\mathrm{ch}_s(\mathrm{coker}(\alpha)) = \mathrm{ch}_s(\Omega^2_{X/S,\log,\mathrm{tors}})$. First note that

$$\mathrm{coker}(\alpha) \cong \det\Omega^1_{X/S,\log} \otimes \mathcal{O}_Z.$$

To see this we can work locally and choose an imbedding of U as in Lemma 1. Let

$$u^{-1}\mathrm{d}u + a \cdot \mathrm{d}\log(t) + b \cdot \mathrm{d}\log(s) = (a + t \cdot x)\mathrm{d}\log(t) + (b + s \cdot y)\mathrm{d}\log(s),$$

for some x, y in $\mathcal{O}_U$. Then Z is defined in U by the ideal

$$(a + t \cdot x, b + s \cdot y), \text{ if } a \neq 0 \text{ and } b \neq 0, \text{ or by}$$
$$(a + t \cdot x, y), \text{ if } a \neq 0 \text{ and } b = 0.$$

$\Omega^1_{X/S,\log}$ is generated by $\mathrm{d}t$, $\mathrm{d}s$, $\mathrm{d}\log(t)$, (if $a \neq 0$), and $\mathrm{d}\log(s)$ (if $b \neq 0$) subject to the relations

$$(a + t \cdot x)\mathrm{d}\log(t) + (b + s \cdot y)\mathrm{d}\log(s) = 0,$$
$$t \cdot \mathrm{d}\log(t) = \mathrm{d}t, \text{ and } s \cdot \mathrm{d}\log(s) = \mathrm{d}s.$$

Note that we may view $\det\Omega^1_{X/S,\log}$ as a subsheaf of $\Omega^1_{X_K/K,\log}$. If $a \neq 0$ and $b \neq 0$ $\det\Omega^1_{X/S,\log}$ is generated by

$$\frac{1}{b + s \cdot y}\mathrm{d}\log(t), \text{ if } b + s \cdot y \neq 0, \text{ or by}$$

$$\frac{1}{a + t \cdot x}\mathrm{d}\log(s), \text{ if } a + t \cdot x \neq 0.$$

Assume without loss of generality that $b + s \cdot y$ is nonzero. Then the image of $\Omega^1_{X/S,\log}$ in $\det\Omega^1_{X/S,\log}$ is generated by

$$\mathrm{d}\log(t) = (b + s \cdot y)\frac{1}{b + s \cdot y}\mathrm{d}\log(t), \text{ and } \mathrm{d}\log(s) = (a + t \cdot x)\frac{1}{b + s \cdot y}\mathrm{d}\log(t).$$

Therefore the cokernel of α is $\det\Omega^1_{X/S,\log} \otimes \mathcal{O}_Z$. If $a \neq 0$ and $b = 0$ then $\det\Omega^1_{X/S,\log}$ is generated by

$$\frac{1}{y}\mathrm{d}\log(t), \text{ if } y \neq 0, \text{ or by}$$

$$\frac{1}{a + t \cdot x}\mathrm{d}\log(s), \text{ if } a + t \cdot x \neq 0,$$

in $\Omega^1_{X_K/K,\log}$, and we similarly arrive at the conclusion.

Next, if $a \neq 0$ and $b \neq 0$, $\Omega^2_{X/S,\log}$ is generated by $\mathrm{d}\log(t) \wedge \mathrm{d}\log(s)$ with the relations

$$(a + t \cdot x)\mathrm{d}\log(t) \wedge \mathrm{d}\log(s) = 0, \text{ and } (b + s \cdot y)\mathrm{d}\log(t) \wedge \mathrm{d}\log(s) = 0.$$

And if $a \neq 0$ and $b = 0$, $\Omega^2_{X/S,\log}$ is generated by $\mathrm{d}\log(t) \wedge \mathrm{d}s$ with the relations

$$(a + t \cdot x)\mathrm{d}\log(t) \wedge \mathrm{d}s = 0, \text{ and } y\mathrm{d}\log(t) \wedge \mathrm{d}s = 0.$$

This shows that $\Omega^2_{X/S,\log}$ is an invertible sheaf on Z. And using again the local desription we see that $\Omega^1_{X/S,\log}|Z$ is locally free of rank 2, and we have

$$\Omega^2_{X/S,\log,\mathrm{tors}} = \Omega^2_{X/S,\log} = \Lambda^2\Omega^1_{X/S,\log}|_Z.$$

Restricting the resolution of $\Omega^1_{X/S,\log}$ over U to Z_U, we obtain

$$0 \rightarrow L^1i^*\Omega^1_{X/S,\log}|_{Z_U} \rightarrow N_{U/P}|_{Z_U} \otimes_A \mathfrak{m}^{-1} \rightarrow \Omega^1_{P/S,\log}|_{Z_U} \rightarrow \Omega^1_{U/S,\log}|_{Z_U} \rightarrow 0,$$

where $i : Z \rightarrow X$ is the inclusion. Here, the second and the fourth arrows are isomorphisms. In particular, we have

$$L^1i^*\Omega^1_{X/S,\log}|_{Z_U} \cong N_{U/P}|_{Z_U} \otimes_A \mathfrak{m}^{-1}.$$

The exact sequence also shows that

$$\Lambda^2\Omega^1_{X/S,\log}|_Z \cong \det\Omega^1_{X/S,\log}|_Z \otimes L^1i^*\Omega^1_{X/S,\log}.$$

Using a filtration of $\mathcal{O}_Z$ with graded pieces supported on integral subschemes of Z, we see that to prove the lemma it is enough to show that $L^1i^*\Omega^1_{X/S,\log}|_T \cong \mathcal{O}_T$, for any integral curve T in Z. For the rest of the proof we use the method of the proof of Proposition 3.1 in [7], in this very explicit (and easier) case. First note that if $k : U \rightarrow Q$ is a closed immersion with Q smooth over S, then k is a regular immersion. Since the inclusion $T_U \rightarrow U$ is also a regular immersion, we have an exact sequence of locally free sheaves on T_U :

$$0 \rightarrow N_{U/Q}|_{T_U} \rightarrow N_{T_U/Q} \rightarrow N_{T_U/U} \rightarrow 0.$$

And similarly, we have an exact sequence

$$0 \rightarrow N_{Q_s/Q}|_{T_U} \rightarrow N_{T_U/Q} \rightarrow N_{T_U/Q_s} \rightarrow 0;$$

in particular, $N_{Q_s/Q}|_{T_U} \rightarrow N_{T_U/Q}$ is injective. Furthermore, for an immersion $U \rightarrow P$ as in Lemma 1, we claim that

$$0 \rightarrow N_{P_s/P}|_{T_U} \rightarrow N_{T_U/P} \rightarrow N_{T_U/U} \rightarrow 0$$

is exact. To see this we only need to check that

$$N_{P_s/P}|_{T_U} = \ker(N_{T_U/P} \rightarrow N_{T_U/U}).$$

Since this is a local question on X, by restricting U and P we will assume that T_U is defined by t on U. Denoting $\pi - ut^as^b$ by g, we need to show that

$$\pi/(\pi^2, \pi t) = \ker((t,g)/(t,g)^2 \rightarrow (t,g)/(t^2,g)).$$

Since by assumption T is contained in Z, X/S is not smooth along T, and so $a \geq 2$. Therefore $ut^as^b \in (t^2)$, and $\pi \in (t^2, g)$. This shows that $\pi/(\pi^2, \pi t)$ is in the kernel. To see the converse we only need to note that

$$g - \pi = ut^as^b = 0 \text{ in } (t,g)/(t,g)^2,$$

since $ut^as^b \in (t^2)$. This proves the claim. Tensoring the exact sequence with $\mathfrak{m}^{-1}$ and observing that

$$N_{P_s/P}|_{T_U} \otimes_A \mathfrak{m}^{-1} \cong \mathcal{O}_{T_U},$$

we obtain the exact sequence

$$0 \to \mathcal{O}_{T_U} \to N_{T_U/P} \otimes_A \mathfrak{m}^{-1} \to N_{T_U/U} \otimes_A \mathfrak{m}^{-1} \to 0.$$

On the other hand, using the isomorphism

$$L^1 i^* \Omega^1_{X/S,\log}|_{T_U} \cong N_{U/P}|_{T_U} \otimes_A \mathfrak{m}^{-1}$$

and the exact sequence

$$0 \to N_{U/P}|_{T_U} \otimes_A \mathfrak{m}^{-1} \to N_{T_U/P} \otimes_A \mathfrak{m}^{-1} \to N_{T_U/U} \otimes_A \mathfrak{m}^{-1} \to 0,$$

we get an exact sequence

$$0 \to L^1 i^* \Omega^1_{X/S,\log}|_{T_U} \to N_{T_U/P} \otimes_A \mathfrak{m}^{-1} \to N_{T_U/U} \otimes_A \mathfrak{m}^{-1} \to 0.$$

Therefore we see that

$$L^1 i^* \Omega^1_{X/S,\log}|_{T_U} \cong \mathcal{O}_{T_U}$$

by viewing them both as the kernel of

$$N_{T_U/P} \otimes_A \mathfrak{m}^{-1} \to N_{T_U/U} \otimes_A \mathfrak{m}^{-1}.$$

If we take imbeddings of U into P and P' as above, taking $Q = P \times P'$, we get the inclusions

$$L^1 i^* \Omega^1_{X/S,\log}|_{T_U} \to N_{U/Q}|_{T_U} \otimes_A \mathfrak{m}^{-1} \to N_{T_U/Q} \otimes_A \mathfrak{m}^{-1},$$
$$\text{and } \mathcal{O}_{T_U} \cong N_{Q_s/Q}|_{T_U} \otimes_A \mathfrak{m}^{-1} \to N_{T_U/Q} \otimes_A \mathfrak{m}^{-1}.$$

Using this we see that the isomorphism $L^1 i^* \Omega^1_{X/S,\log}|_{T_U} \cong \mathcal{O}_{T_U}$ does not depend on the choice of the local imbedding satisfying Lemma 1. Therefore we obtain

$$L^1 i^* \Omega^1_{X/S,\log}|_T \cong \mathcal{O}_T.$$

This finishes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Using Lemma 2 we see that

$$\chi(\Omega_{X/S,\log,\text{tors}}) = \deg(\text{ch}_s(\Omega_{X/S,\log,\text{tors}}) \cap \text{Td}(X/S)) = \deg(\text{ch}_s(C) \cap \text{Td}(X/S)).$$

Let

$$F : 0 \to F_m \to \cdots \to F_0 \to 0$$

be a complex of locally free coherent sheaves on X, which is exact outside a proper closed set Y. This exact sequence on $X - Y$ gives a canonical trivialization over $X - Y$, of the line bundle

$$\det F = \otimes_{0 \le i \le m} (\det F_i)^{\otimes (-1)^i}.$$

This gives a rational section s of $\det F$. Denote the image of the divisor of s in the Chow group $A_* Y$ of Y, by γ. We will need the following lemma.

Lemma 3. *We have the equality*

$$\text{ch}^X_{Y,1}(F) \cap [X] = \gamma, \ \ in \ A_* Y.$$

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Proof. Let $f_i = \text{rank} F_i$, $F_{-1} = 0$, and

$$G_i = \text{Grass}_{f_i}(F_i \oplus F_{i-1}), \text{ the Grasmannian of } f_i \text{ planes in } F_i \oplus F_{i-1},$$

for $0 \le i \le m$. Let ξ_i be the tautological subbundle, of rank f_i, of $F_i \oplus F_{i-1}$ on G_i. Let

$$G = G_m \times \cdots \times G_0, \text{ with the projections } p_i : G \to G_i, \text{ and } \pi : G \to X.$$

Let

$$\xi = \sum_{i=0}^{m} (-1)^i p_i^* \xi_i, \text{ and } \det \xi = \otimes_{0 \le i \le m} (\det p_i^* \xi_i)^{\otimes (-1)^i}.$$

Furthermore, we denote the kernel of $d_i : F_i \to F_{i-1}$ by K_i, of rank k_i, and $H_i = \text{Grass}_{k_i}(F_i)$. Finally, let W be the closure of $\varphi(X \times \mathbb{A}^1)$ in $G \times \mathbb{P}^1$, where $\varphi : X \times \mathbb{A}^1 \to G \times \mathbb{A}^1$ is the map sending (x, λ) to

$$(\text{Graph}(\lambda \cdot d_m(x)), \cdots, \text{Graph}(\lambda \cdot d_0(x)), \lambda).$$

Over $\varphi((X - Y) \times \mathbb{A}^1)$, $\det \xi$ has a natural trivialization since

$$\det p_i^* \xi_i \cong \det \pi^* K_i \otimes \det \pi^* K_{i-1} \text{ over } \varphi((X - Y) \times \mathbb{A}^1).$$

This trivialization gives a divisor, say D, on $\varphi(X \times \mathbb{A}^1)$. D is supported on $G_Y \times \mathbb{A}^1$.

$$\pi_*([D_0]) = \gamma \text{ in } A_* Y$$

Let $t \in \mathbb{P}^1 - \{0, \infty\}$. As

$$\pi_*([D_0]) = \pi_*([D_t]),$$

we will be done if we can show that $\pi_*([D_t])$ is equal to $\text{ch}_{Y,1}^X(F) \cap [X]$. For this, by definition, we need to show that

$$\pi_*([D_t]) = \pi_*(\text{ch}_1(\xi) \cap [T]),$$

where $[T] = [W_\infty] - [\widetilde{X}]$, and $\widetilde{X}$ is the irreducible component of W_∞ which projects birationally onto X. Note that over $X - Y$, φ can be extended to a function

$$\varphi : (X - Y) \times \mathbb{P}^1 \to G \times \mathbb{P}^1$$

as follows. For $(x, [\lambda_0, \lambda_1]) \in (X - Y) \times \mathbb{P}^1$, and $0 \le i \le m$, let $\varphi_i'(x, [\lambda_0, \lambda_1])$ denote the point of G_i over x that corresponds to

$$\{(v_i, v_{i-1}) \in F_i(x) \oplus K_{i-1}(x) : \lambda_0 v_{i-1} = \lambda_1 d_i v_i\} \subseteq F_i(x) \oplus F_{i-1}(x).$$

Then

$$\varphi(x, [\lambda_0, \lambda_1]) = (\varphi_m'(x, [\lambda_0, \lambda_1]), \cdots, \varphi_0'(x, [\lambda_0, \lambda_1]), [\lambda_0, \lambda_1]).$$

And $\widetilde{X}$ is the closure of $\varphi((X - Y) \times \{\infty\})$ in $G \times \{\infty\}$. Now as

$$\det p_i^* \xi_i \cong \det \pi^* K_i \otimes \det \pi^* K_{i-1} \text{ over } \varphi((X - Y) \times (\mathbb{P}^1 - \{0\})),$$

$\det \xi$ has a natural trivialization over $\varphi((X - Y) \times (\mathbb{P}^1 - \{0\}))$, which gives a divisor, say D', on $W - W_0$, supported on $G_Y \times (\mathbb{P}^1 - \{0\})$. If $t \in \mathbb{P}^1 - \{0, \infty\}$ then $[D_t] = [D_t']$ in $A_* W_t$ being divisors associated to the same line bundle $\det \xi|_{W_t}$. Noting that $[D_t'] = [D_\infty']$, we only need to show that

$$\pi_*(D_\infty') = \pi_*(D'.T) \text{ in } A_* Y,$$

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or that
$$\pi_*(D'_\infty.\widetilde{X}) = 0 \text{ in } A_*Y.$$

If

$\Psi : X - Y \to H$ is the map that sends x to $(K_m(x), \cdots, K_0(x))$, and

$\iota : H \to G$ is the map that sends $(V_m, \cdots, V_0)$ to $(V_m \oplus V_{m-1}, \cdots, V_0)$,

then
$$\widetilde{X} = \iota(\overline{\Psi(X - Y)}).$$

However, as $\iota^*(D'_\infty)$ is the divisor corresponding to a section of
$$\otimes_{0 \le i \le m} \det \iota^* \xi_i^{\otimes(-1)^i} \cong \otimes_{0 \le i \le m} (\det q_i^* \zeta_i \otimes \det q_{i-1}^* \zeta_{i-1})^{\otimes(-1)^i} \cong \mathcal{O}_H,$$

that is nonzero on H_{X-Y}, where ζ_i is the tautological subbundle of H_i, and
$$q_i : H \to H_i$$

is the projection, we see that
$$\pi_*(D'_\infty.\widetilde{X}) = 0.$$

This finishes the proof of the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $0 \to E_m \to \cdots \to E_1 \to \Omega^1_{X/S,\log} \to 0$ be a resolution of $\Omega^1_{X/S,\log}$ by locally free sheaves of finite rank. Now consider the complex
$$E : 0 \to E_m \to \cdots \to E_1 \to \det\Omega^1_{X/S,\log} \to 0,$$

where the map $E_1 \to \det\Omega^1_{X/S,\log}$ is the composition of the differential
$$d_1 : E_1 \to \Omega^1_{X/S,\log},$$

and the canonical map
$$\alpha : \Omega^1_{X/S,\log} \to \det\Omega^1_{X/S,\log}.$$

Applying Lemma 3 to E we obtain
$$
\begin{aligned}
\chi(\Omega_{X/S,\log,\mathrm{tors}}) &= \deg(\mathrm{ch}_s(C) \cap \mathrm{Td}(X/S)) \\
&= \deg(\mathrm{ch}_s(E) \cap \mathrm{Td}(X/S)) = \deg(\mathrm{ch}_{s,2}(E) \cap [X]).
\end{aligned}
$$

We will need the following lemma.

Lemma 4. *We have* $\mathrm{ch}_{s,2}(E) \cap [X] = \mathrm{c}_{s,2}(\Omega^1_{X/S,\log}) \cap [X]$ *in* $(A_*X_s)_{\mathbb{Q}}$.

Proof. First of all, note that we have $\mathrm{ch}_{s,1}(E) \cap [X] = 0$, hence
$$\mathrm{ch}_{s,2}(E) \cap [X] = \frac{\mathrm{c}_{s,1}^2(E) \cap [X]}{2} - \mathrm{c}_{s,2}(E) \cap [X] = -\mathrm{c}_{s,2}(E) \cap [X].$$

Let E' denote the complex $0 \to E_m \to \cdots \to E_1 \to 0$, where we put E_1 in degree 0. We use the notation in Lemma 3, where the objects with $'$ refer to those corresponding to E'.

We have
$$
\begin{aligned}
\mathrm{ch}_{s,2}(E) \cap [X] &= -\mathrm{c}_{s,2}(E) \cap [X] = \pi_*(-\mathrm{c}_2(\sum_{i=0}^m (-1)^i [p_i^* \xi_i]) \cap [T]), \text{ and} \\
&= \pi_*((-\mathrm{c}_2(\sum_{i=1}^m (-1)^i [p_i^* \xi_i]) - \mathrm{c}_1(\sum_{i=1}^m (-1)^i [p_i^* \xi_i]) \cdot \mathrm{c}_1(p_0^* \xi_0)
\end{aligned}
$$

$$-\mathrm{c}_2(p_0^*\xi_0)) \cap [T]).$$

Since ξ_0 is a line bundle this is equal to

$$\pi_*((-\mathrm{c}_2(\sum_{i=1}^{m}(-1)^i[p_i^*\xi_i]) \;-\; \mathrm{c}_1(\sum_{i=1}^{m}(-1)^i[p_i^*\xi_i]) \cdot \mathrm{c}_1(p_0^*\xi_0)) \cap [T])$$

$$= \pi_*((-\mathrm{c}_2(\sum_{i=1}^{m}(-1)^i[p_i^*\xi_i]) + \mathrm{c}_1^2(p_0^*\xi_0)) \cap [T])$$

$$-\pi_*((\mathrm{c}_1(\sum_{i=0}^{m}(-1)^i[p_i^*\xi_i]) \cdot \mathrm{c}_1(p_0^*\xi_0)) \cap [T])$$

$$= \pi_*((-\mathrm{c}_2(\sum_{i=1}^{m}(-1)^i[p_i^*\xi_i]) + \mathrm{c}_1^2(p_0^*\xi_0)) \cap [T])$$

$$-(\mathrm{c}_{s,1}(E) \cap [X]) \cdot \mathrm{c}_1(\xi_0) \quad (\text{since } p_0 = \pi).$$

Lemma 3 shows that this is equal to

$$\pi_*((-\mathrm{c}_2(\sum_{i=1}^{m}(-1)^i[p_i^*\xi_i]) + \mathrm{c}_1^2(p_0^*\xi_0)) \cap [T])$$

$$= \pi_*((-\mathrm{c}_1^2(\sum_{i=1}^{m}(-1)^{i+1}[p_i^*\xi_i]) + \mathrm{c}_2(\sum_{i=1}^{m}(-1)^{i+1}[p_i^*\xi_i])$$

$$+\mathrm{c}_1^2(p_0^*\xi_0)) \cap [T]).$$

If $f : G \to G'$ is the projection, note that we have $f^*p_i^{'*}\xi_i' \cong p_i^*\xi_i$, for $2 \leq i \leq m$. Therefore the last expression is equal to

$$\pi_*((-\mathrm{c}_1^2(\sum_{i=1}^{m}(-1)^{i+1}[p_i^*\xi_i]) + \mathrm{c}_2(\sum_{i=1}^{m}(-1)^{i+1}[f^*p_i^{'*}\xi_i'])$$

$$+\;\; \mathrm{c}_1(\sum_{i=1}^{m}(-1)^{i+1}[f^*p_i^{'*}\xi_i']) \cdot \mathrm{c}_1([p_1^*\xi_1] - [f^*p_1^{'*}\xi_1'])$$

$$+\mathrm{c}_2([p_1^*\xi_1] - [f^*p_1^{'*}\xi_1']) + \mathrm{c}_1^2(p_0^*\xi_0)) \cap [T]).$$

This is equal to

$$\pi_*(-\mathrm{c}_1^2(p_0^*\xi_0) \;+\; \mathrm{c}_1([p_0^*\xi_0] - [p_1^*\xi_1] + [f^*p_1^{'*}\xi_1']) \cdot \mathrm{c}_1([p_1^*\xi_1] - [f^*p_1^{'*}\xi_1'])$$

$$+\;\; \mathrm{c}_2([p_1^*\xi_1] - [f^*p_1^{'*}\xi_1']) + \mathrm{c}_1^2(p_0^*\xi_0)) \cap [T]) + \mathrm{c}_{s,2}(\Omega^1_{X/S,\log}) \cap [X]$$

$$= \pi_*((\mathrm{c}_1([\pi^*\det\Omega^1_{X/S,\log}] - [p_1^*\xi_1] + [\pi^*E_1]) \cdot \mathrm{c}_1([p_1^*\xi_1] - [\pi^*E_1])$$

$$+\mathrm{c}_2([p_1^*\xi_1] - [\pi^*E_1])) \cap [T]) + \mathrm{c}_{s,2}(\Omega^1_{X/S,\log}) \cap [X].$$

$$(\text{cf. the proof of Lemma 3})$$

Now, since $p_1^*\xi_1$ is the tautological subbundle of $\pi^*E_1 \oplus \pi^*\det\Omega^1_{X/S,\log}$ on G, we have an exact sequence $0 \to p_1^*\xi_1 \to \pi^*E_1 \oplus \pi^*\det\Omega^1_{X/S,\log} \to Q \to 0$, with Q a line bundle. Therefore we obtain

$$\mathrm{c}_2([p_1^*\xi_1] - [\pi^*E_1]) \;=\; \mathrm{c}_2([\pi^*\det\Omega^1_{X/S,\log}] - [Q])$$

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$$= \quad c_2(\pi^* \mathrm{det}\Omega^1_{X/S,\log}) - c_1([\pi^* \mathrm{det}\Omega^1_{X/S,\log}] - [Q]) \cdot c_1(Q) - c_2(Q).$$

Since ξ_0 and Q are line bundles

$$c_1(Q) = c_1([\pi^* E_1] + [\pi^* \mathrm{det}\Omega^1_{X/S,\log}] - [p_1^* \xi_1]), \text{ and}$$

$$c_2([p_1^* \xi_1] - [\pi^* E_1]) = -c_1([p_1^* \xi_1] - [\pi^* E_1]) \cdot c_1([\pi^* E_1] + [\pi^* \mathrm{det}\Omega^1_{X/S,\log}] - [p_1^* \xi_1]).$$

Combining this with the expression for $\mathrm{ch}_{s,2}(E) \cap [X]$ above, we obtain

$$\mathrm{ch}_{s,2}(E) \cap [X] = c_{s,2}(\Omega^1_{X/S,\log}) \cap [X].$$

$\square$

Using Lemma 4 we obtain

$$\chi(\Omega_{X/S,\log,\mathrm{tors}}) = \deg(\mathrm{ch}_{s,2}(E) \cap [X]) = \deg(c_{s,2}(\Omega^1_{X/S,\log}) \cap [X]).$$

Bloch's conductor formula ([2], Theorem.1) gives

$$\deg(c_{s,2}(\Omega^1_{X/S}) \cap [X]) = -\mathrm{Art}(X_K/K).$$

The proof of Theorem 6.2.5 in [6] gives

$$\deg(c_{s,2}(\Omega^1_{X/S}) \cap [X]) - \deg(c_{s,2}(\Omega^1_{X/S,\log}) \cap [X]) = \mathrm{Sw}(X_K/K) - \mathrm{Art}(X_K/K)$$

Note that the proof of Theorem 6.2.5 is independent of the rest of [6]. These imply

$$\deg(c_{s,2}(\Omega^1_{X/S,\log}) \cap [X]) = -\mathrm{Sw}(X_K/K).$$

Combining this with the above we obtain

$$\chi(\Omega_{X/S,\log,\mathrm{tors}}) = -\mathrm{Sw}(X_K/K).$$

$\square$

We now give a consequence of the proof of Theorem 1. In the following, if C is a 0-dimensional subscheme of X, and $[G]$ and $[H]$ are curves in X, we denote by $\deg[C]$ the degree of C with respect to k, and by $[G] \cdot [H]$ the intersection number of the curves G and H. Let $D \subseteq X$ be a curve supported on the special fiber X_s, L a line bundle on X, and K and E divisors on X such that $\mathcal{O}(K) \cong \mathrm{det}\Omega^1_{X/S}$, the dualizing sheaf of X/S, and $\mathcal{O}(E) \cong L$. Then we have the following lemma.

Lemma 5. *We have the equality*

$$\chi(i_*(L|_D)) = [E] \cdot [D] - \frac{1}{2}\chi(\omega_{D/k}),$$

where $i : D \to X$ is the inclusion.

Proof. By the Riemann-Roch theorem we have

$$\chi(i_*(L|_D)) = \deg(\mathrm{ch}_D^X(L|_D) \cap \mathrm{Td}(X/S)).$$

By applying the Riemann-Roch theorem to the closed immersion $i : D \to X$ we obtain

$$\begin{aligned}
\mathrm{ch}_D^X(i_*(L|_D)) &= i_*(\mathrm{ch}_D^D(L|_D) \cdot \mathrm{Td}(N_{D/X})^{-1}) \\
&= (1 + c_1(L|_D)) \cdot (1 + \frac{1}{2}c_1(\mathcal{O}(-D)|_D)).
\end{aligned}$$

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Combining this with the above we obtain

$$\chi(i_*(L|_D)) = [E] \cdot [D] - \frac{([K] + [D]) \cdot [D]}{2}.$$

Finally using the adjunction formula for $D \to X$ we obtain the expression in the statement of the lemma. $\square$

Let Z_1 and Z_2 denote the closed subschemes of Z consisting of the components of Z which have codimension 1 and codimension 2 in X respectively. With this notation, we have the following corollary.

Corollary 1. *We have the following equality*

$$\text{Sw}(X_K/K) = \chi(\Omega^1_{X/S,\log,\text{tors}}) + (2[Z_1] - [Z_{1,\text{red}}]) \cdot [Z_1] - \frac{1}{2}\chi(\omega_{Z_1/k}) - \deg[Z_2].$$

Proof. Using Theorem 1 we see that we only need to prove the equality

$$\chi(\Omega^2_{X/S,\log}) = ([Z_{1,\text{red}}] - 2[Z_1]) \cdot [Z_1] + \frac{1}{2}\chi(\omega_{Z_1/k}) + \deg[Z_2].$$

The proof of Lemma 2 shows that

$$\chi(\Omega^2_{X/S,\log}) = \chi(\det\Omega^1_{X/S,\log}|_Z).$$

Then we have

$$\chi(\Omega^2_{X/S,\log}) = \chi(\det\Omega^1_{X/S,\log}|_{Z_1}) + \deg[Z_2].$$

Using the lemma above we obtain that

$$\chi(\Omega^2_{X/S,\log}) = [K_{\log}] \cdot [Z_1] - \frac{1}{2}\chi(\omega_{Z_1/k}) + \deg[Z_2],$$

where $K_{\log}$ is a divisor on X such that $\mathcal{O}(K_{\log}) \cong \det\Omega^1_{X/S,\log}$. Now we also have

$$[K_{\log}] = [K] - [Z_1] + [Z_{1,\text{red}}].$$

To see this we only need to look at the multiplicities at codimension 1 points. Using the notation in the proof of Lemma 2, viewing $\det\Omega^1_{X/S}$ as a subsheaf of

$$\Omega^1_{X_K/K} \cong \Omega^1_{X_K/K,\log},$$

it is generated by

$$\frac{1}{t^{a-1}y}\text{d}\log(t), \text{ if } y \neq 0, \text{ or by}$$

$$\frac{1}{t^{a-1}(a + t \cdot x)}\text{d}s, \text{ if } a + t \cdot x \neq 0,$$

in a Zariski neighborhood of a point with $a \neq 0$ and $b = 0$. Using the similar description of $\det\Omega^1_{X/S,\log}$ in the proof of Lemma 2, we arrive at the formula as claimed above. Using this and the adjunction formula we obtain that

$$\chi(\Omega^2_{X/S,\log}) = ([Z_{1,\text{red}}] - 2[Z_1]) \cdot [Z_1] + \frac{1}{2}\chi(\omega_{Z_1/k}) + \deg[Z_2].$$

$\square$

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