Drinfel'd-Ihara relations for p-adic multi-zeta values

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Abstract. We prove that the p-adic multi-zeta values satisfy the Drinfel'd-Ihara relations in Grothendieck-Teichmüller theory ([10], [22]). This requires a detailed study of the crystalline theory of tangential basepoints in the higher dimensional case and Coleman integrals ([5]) as they relate to the frobenius invariant path of Vologodsky ([32]). The main result (Theorem 1.8.1) is used in [14, pp. 1133-1135]

1. Introduction

1.1. For $s_1, \cdots, s_{k-1} \geq 1$ and $s_k > 1$, the multi-zeta values are defined as

$$\zeta(s_k, \cdots, s_2, s_1) := \sum_{n_k > \cdots > n_2 > n_1 > 0} \frac{1}{n_k^{s_k} \cdots n_2^{s_2} n_1^{s_1}}.$$

They were first defined and studied by Euler. More recently, they appeared in many different branches of mathematics, including deformations of Hopf algebras ([10]), the geometry of modular varieties ([19]), renormalization, knot theory etc. Their main importance stems from their being periods for mixed Tate motives over $\mathbb{Z}$ ([9], [18]).

There are two important classes of relations that are known for the multi-zeta values. One class of these, called the shuffle relations, is an immediate consequence of the Euler-Kontsevich integral representation of multi-zeta values ([18], [29]) and is of geometric nature. The other class, called the (regularized) harmonic shuffle relations, can be proven via the series representation above in a non-geometric way ([18], [29]). These relations together are called the double shuffle relations and conjecturally are the only algebraic relations with rational coefficients between the multi-zeta values.

1.2. If $\omega_1, \cdots, \omega_k$ are one-forms on a manifold M, and $\alpha : [0,1] \to M$ is a smooth path then the iterated integral is defined as:

$$\int_\alpha \omega_k \circ \cdots \circ \omega_1 := \int_{1 > t_k > \cdots > t_1 > 0} \pi_k^* \alpha^* \omega_k \wedge \cdots \wedge \pi_1^* \alpha^* \omega_1,$$

where $\pi_i : [0,1]^k \to [0,1]$ is the i-th projection. With this notation, the Euler-Kontsevich formula for the multi-zeta values:

$$\zeta(s_k, \cdots, s_1) = \int_{[0,1]} \big(\frac{dz}{z}\big)^{\circ(s_k-1)} \circ \frac{dz}{1-z} \circ \cdots \circ \big(\frac{dz}{z}\big)^{\circ(s_1-1)} \circ \frac{dz}{1-z},$$

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expresses these numbers as periods. This expression realizes $\zeta(s_k,\cdots,s_1)$ as $(-1)^k$-times the coefficient of $e_0^{s_k-1}e_1\cdots e_0^{s_1-1}e_1$ in a formal non-commutative power series $\Phi_{KZ}(e_0,e_1)\in\mathbb{C}\langle\langle e_0,e_1\rangle\rangle$. Conversely, the multi-zeta values determine Φ_{KZ} uniquely.

Letting $\omega:=e_0\frac{dz}{z}+e_1\frac{dz}{z-1}$, $\Phi_{KZ}(e_0,e_1)$ is defined as the limit:

$$\lim_{t\to 0}\exp(-e_1\log(t))\cdot(1+\sum_{1\le k}\int_{[t,1-t]}\omega^{\circ k})\cdot\exp(e_0\log(t)).$$

1.3. The Drinfel'd associator $\Phi_{KZ}(e_0,e_1)$ first appeared in ([10]), where it was shown that it gives an element in $\mathrm{M}_{2\pi i}(\mathbb{C})$. The pro-variety $\mathrm{M}/\mathbb{Q}$ has a canonical morphism to $\mathbb{A}^1$, and its fiber M_λ over $\lambda\in\mathbb{A}^1(K)$ is a torsor on the left under the unipotent part of the Grothendieck-Teichmüller group GT_1, and a torsor on the right under its graded version $\mathrm{GRT}_1(=\mathrm{M}_0)$ ([10], [27]).

The assertion that $\Phi_{KZ}(e_0,e_1)\in\mathrm{M}_{2\pi i}(\mathbb{C})$ is saying that Φ_{KZ} satisfies the Grothendieck-Teichmüller relations with $\mu=2\pi i$ ((5.3), (2.12), (2.13) in [10]). Since multi-zeta appear as coefficients, any relation on Φ_{KZ} implies a similar relation on the multi-zeta values. By a result of Furusho (Theorem 0.2, [15]) and independently of Deligne and Terasoma ([29]), these relations imply the double shuffle relations for the multi-zeta values.

1.4. The definition of $\Phi_{KZ}(e_0,e_1)$ can be rephrased in terms of the Betti-de Rham comparison theorem for the unipotent fundamental group of $X:=\mathbb{P}^1\setminus\{0,1,\infty\}$. Namely, let t_{01} and t_{10} be the standard tangential basepoints at 0 and 1 (§5.4) and $\gamma\in{}_{t_{10}}P_{B,t_{01}}$ be the standard real path from t_{01} to t_{10} in the Betti fundamental groupoid of $X_{\mathbb{C}}$. By the Betti-de Rham comparison theorem (Proposition 10.32, [7]), this gives a path

$$ {}_{dR}\mathrm{comp}_B(\gamma)\in{}_{t_{10}}P_{dR,t_{01}}(\mathbb{C}) \tag{1.4.1}$$

in the de Rham fundamental groupoid of X. Since $H^1(\overline{X},\mathcal{O})=0$, the de Rham fiber functor ω_{dR} (§2.2.5) gives an identification

$$ {}_{t_{10}}P_{dR,t_{01}}\xrightarrow{\sim}\pi_{1,dR}(X,\omega(dR)). \tag{1.4.2}$$

Viewing $\pi_{1,dR}(X,\omega(dR))(\mathbb{C})$ as group-like elements of $\mathbb{C}\langle\langle e_0,e_1\rangle\rangle$ (§6.8) and using (1.4.1) and (1.4.2) gives and element in $\mathbb{C}\langle\langle e_0,e_1\rangle\rangle$ which is nothing other than $\Phi_{KZ}(e_0,e_1)$.

1.5. In ([9]), Deligne and Goncharov constructed an abelian category $\mathrm{MTM}_{\mathcal{O}_K}$ of mixed Tate motives over the ring of integers $\mathcal{O}_K$, of a a number field K, using Voevodsky's triangulated category $\mathrm{DM}_{\mathrm{Nis}}^{\mathrm{eff},-}(K)$. Moreover, they show that the unipotent fundamental group $\pi_1(X,t_{01})$ of X naturally defines an object of $\mathrm{MTM}_{\mathbb{Z}}$. Then the motivic philosophy and the construction of Φ_{KZ} suggest that for the other comparison theorems, namely the Betti-étale and crystalline-de Rham, one would have elements in M similar to Φ_{KZ}.

1.6. In the Betti-étale case, such an element was constructed by Drinfel'd and Ihara ([10], [21]). Let

$$\chi:\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\to\widehat{\mathbb{Z}}^\times$$

denote the cyclotomic character and $\widehat{F}_2$ denote the pro-finite completion of the free group generated by x and y. Then for every $\sigma\in\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the action of σ on the fundamental groupoid of paths from t_{01} to t_{10} determines an element $f_\sigma \in \widehat{F}_2$ such that (Theorem, §1.7, [21]):

$$f_\sigma(x,y)f_\sigma(y,x) = 1;$$

$$f_\sigma(z,x)z^m f_\sigma(y,z)y^m f_\sigma(x,y)x^m = 1,$$

for $xyz = 1$ and $m = (\chi(\sigma) - 1)/2$; and

$$f_\sigma(x_{12},x_{23})f_\sigma(x_{34},x_{45})f_\sigma(x_{51},x_{12})f_\sigma(x_{23},x_{34})f_\sigma(x_{45},x_{51}) = 1,$$

with x_{ij} as in (§3.1, [22]).

1.7. The crystalline-de Rham case is the main topic of this article. For X/k a smooth variety over a perfect field k of characteristic p, one has a category $\mathrm{Isoc}^\dagger_{uni}(X/W)$ of unipotent overconvergent isocrystals on X/W (§2.3.1), where W is the ring of Witt vectors of k. This is a tannakian category whose fundamental group at a fiber functor is the unipotent crystalline fundamental group of X. Suppose that X has a compactification $\overline{X}$ which is smooth, projective and such that $D := \overline{X} \setminus X$ is a simple normal crossings divisor in $\overline{X}$. If $\overline{X}_{log}$ denotes the canonical log structure on $\overline{X}$ associated to the divisor D (§2.1.1) and $\mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W)$ denote the category of unipotent log convergent isocrystals on $\overline{X}_{log}$ then a theorem of Shiho ([30]) implies that the restriction functor

$$\mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}^\dagger_{uni}(X/W)$$

is an equivalence of categories (Lemma 2, [31]). These categories are endowed with the frobenius functor induced by the frobenius morphism on $\overline{X}_{log}$.

Assume that $\overline{\mathfrak{X}}/W$ is a smooth, projective scheme with geometrically connected fibers and $\mathfrak{D} \subseteq \overline{\mathfrak{X}}$ is a relative simple normal crossings divisor. Let $\mathfrak{X} := \overline{\mathfrak{X}} \setminus \mathfrak{D}$, and let the subscripts η and s denote the generic and special fibers respectively.

Let K be the fraction field of W. Deligne's theory of canonical extensions gives an equivalence of categories (II.5.2, [8]; §2.2.4):

$$\mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{log,\eta}/K) \to \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/K)$$

from the the category of unipotent vector bundles with connection on $\overline{\mathfrak{X}}_{log,\eta}$ to that on $\mathfrak{X}_\eta$.

The crystalline-de Rham comparison theorem is an equivalence of categories (§11, [7]; §2.4):

$$\mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{log,\eta}) \to \mathrm{Isoc}^c_{uni}(\overline{\mathfrak{X}}_{log,s}/W). \tag{1.7.1}$$

Combining with the above and choosing a (tangential) basepoint $\mathfrak{x}$ this gives an isomorphism of the crystalline and de Rham fundamental groups:

$$\pi^\dagger_{1,crys}(\mathfrak{X}_s,\mathfrak{x}_s) \xrightarrow{\sim} \pi_{1,dR}(\mathfrak{X}_\eta,\mathfrak{x}_\eta).$$

This induces a frobenius map F_* on $\pi_{1,dR}(\mathfrak{X}_\eta,\mathfrak{x}_\eta)$. When $H^1(\overline{\mathfrak{X}}_\eta,\mathcal{O}) = 0$, there is a canonical isomorphism $\pi_{1,dR}(\mathfrak{X}_\eta,\mathfrak{x}_\eta) \simeq \pi_{1,dR}(\mathfrak{X}_\eta,\omega_{dR})$, with ω_{dR} the de Rham fiber functor of Deligne (§5.9, §12.4, [7]; 2.2.5).

Applying this to $\overline{\mathfrak{X}} = \mathbb{P}^1_{\mathbb{Z}_p}$, $\mathfrak{D} = \{0,1,\infty\}$, and the tangential basepoint t_{01} and noting that the universal enveloping algebra of $\pi_{1,dR}(\mathfrak{X}_\eta,\omega(dR))$ is the set of formal associative, and non-commutative power series ring $\mathbb{Q}_p\langle\langle e_0,e_1\rangle\rangle$, we have a frobenius map:

$$F_* : \mathbb{Q}_p\langle\langle e_0,e_1\rangle\rangle \to \mathbb{Q}_p\langle\langle e_0,e_1\rangle\rangle.$$

Note that $\mathbb{Q}_p\langle\langle e_0, e_1\rangle\rangle$ is naturally a Hopf algebra under the co-product induced by $\Delta(e_i) = 1 \otimes e_i + e_i \otimes 1$, $i = 1, 2$. The frobenius map is determined by $F_*(e_0) = pe_0$ and $F_*(e_1) = pg^{-1}e_1 g$, for a unique group-like power series g whose e_0 and e_1 coefficients are 0. In fact, g is the image of the de Rham path from t_{01} to t_{10} under F_* and so is the exact analog of Φ_{KZ} in the crystalline case.

Deligne (unpublished) defines the p-adic multi-zeta value $\zeta_p(s_k, \cdots, s_1)$ as $p^{-\sum s_i}$ times the coefficient of $e_0^{s_k-1}e_1 \cdots e_0^{s_1-1}e_1$ in g (§4.3, [31]). A series representation for p-adic muti-zeta values is given for $k \leq 2$ in ([31]). An essentially equivalent definition is given and studied in ([13]); the double shuffle relations were proved by Furusho and Jafari ([16]).

1.8. Let K be a field of characteristic 0. The set $\mathrm{M}_0(K)$ of Drinfel'd associators with $\mu = 0$ consists of $\varphi(X, Y) \in K\langle\langle X, Y\rangle\rangle$ such that $\varphi(X, Y)$ is group-like for the usual co-product on $K\langle\langle X, Y\rangle\rangle$, i. e. that

$$\varphi(0, 0) = 1 \quad \text{and} \quad \Delta(\varphi(X, Y)) = \varphi(X, Y) \otimes \varphi(X, Y);$$

and that φ satisfies the following 2-cycle, 3-cycle and 5-cycle relations:

$$\varphi(X, Y)\varphi(Y, X) = 1$$

in $K\langle\langle X, Y\rangle\rangle$,

$$\varphi(Z, X)\varphi(Y, Z)\varphi(X, Y) = 1$$

in $K\langle\langle X, Y, Z\rangle\rangle/(X + Y + Z)$ and

$$\varphi(X_{23}, X_{34})\varphi(X_{40}, X_{01})\varphi(X_{12}, X_{23})\varphi(X_{34}, X_{40})\varphi(X_{01}, X_{12}) = 1$$

in $K\langle\langle X_{ij}\rangle\rangle_{0\leq i,j\leq 4}/R$, where R is the ideal in $K\langle\langle X_{ij}\rangle\rangle_{0\leq i,j\leq 4}$ generated by the following elements:

$$X_{ii}, \quad \text{for} \quad 0 \leq i \leq 4;$$

$$\sum_{0\leq j\leq 4} X_{ij}, \quad \text{for} \quad 0 \leq i \leq 4;$$

$$X_{ij} - X_{ji}, \quad \text{for} \quad 0 \leq i, j \leq 4;$$

and

$$[X_{ij}, X_{kl}],$$

for $0 \leq i, j, k, l \leq 4$ such that $\{i, j\} \cap \{k, l\} = \emptyset$.

Our main theorem in this paper is the following, which is the exact crystalline version of the corresponding results in the Betti (§1.3) and étale (§1.6) cases:

Theorem 1.8.1. *With notation as above, we have*

$$g \in \mathrm{M}_0(\mathbb{Q}_p).$$

In order to prove this, we need to check that g satisfies the 2-cycle, 3-cycle and 5-cycle relations above. Note that these immediately imply relations on the p-adic multi-zeta values. In particular, by Furusho's result mentioned above ([15]), these give another proof of the p-adic double shuffle relations ([16]).

1.9. The proof of Theorem 1.8.1 relies on the construction and study of crystalline tangential basepoints in higher dimensions and its relation to limits of the frobenius invariant path. In defining tangential basepoints, we use the language of log geometry which appears naturally in this context. This gives an equivalent definition to the one given in (§3, [31]) in the one dimensional case.

Let $\overline{X}/k$ and D be as in (§1.7), and let $x \in D(k)$. The natural equivalence of categories $\mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}^\dagger_{uni}(X/W)$ together with the pull-back via the inclusion $x_{log} \to \overline{X}_{log}$ gives a functor

$$\mathrm{Isoc}^\dagger_{uni}(X/W) \to \mathrm{Isoc}^c_{uni}(x_{log}/W).$$

Choosing a tangent vector $v \in N^\times_{D_x/\overline{X}}(x)$ at x transversal to the the divisor D, gives a splitting of the log structure on x_{log}, i. e. an isomorphism $x_{log} \simeq k_{x,log}$. Since $k_{x,log}$ has a canonical lifting, namely $W_{x,log}$, we get a realization functor for $\mathrm{Isoc}^c_{uni}(k_{x,log}/W)$. Combining this with the above gives us the fiber functor

$$\omega(v) : \mathrm{Isoc}^\dagger_{uni}(X/W) \to \mathrm{Vec}_K.$$

Suppose that $\overline{\mathfrak{X}}$, $\mathfrak{D}$ be also as in (§1.7), and $\mathfrak{x} \in \mathfrak{D}(W)$. If $\mathfrak{v} \in N^\times_{\mathfrak{D}_\mathfrak{x}/\overline{\mathfrak{X}}}(\mathfrak{x})$, then the fiber functor $\omega(\mathfrak{v}_\eta)$ on $\mathrm{Mic}_{uni}(\mathfrak{X}_\eta/K)$ is a realization of the fiber functor $\omega(\mathfrak{v}_s)$ through the equivalence (1.7.1). Choosing a similar $\mathfrak{w} \in N^\times_{\mathfrak{D}_\mathfrak{y}/\overline{\mathfrak{X}}}(\mathfrak{y})$, and assuming $W = \mathbb{Z}_p$, we obtain a frobenius action on ${}_{\mathfrak{w}_\eta}\overline{P}_{dR}(\mathfrak{X}_\eta)_{\mathfrak{v}_\eta}$, the fundamental groupoid from $\mathfrak{x}_\eta$ to $\mathfrak{y}_\eta$ defined over $\mathbb{Q}_p$. Since the frobenius depends on the choice of a tangent vector we integrated it into the notation above.

Let $\mathbb{Q}_{p,st} := \mathbb{Q}_p[l(p)]$, where $l(p)$ is a symbol standing for a possible branch of the p-adic logarithm. For any $a, b \in \mathfrak{X}_\eta(\mathbb{Q}_p)$, not necessarily of finite reduction, Vologodsky (based on the work of Coleman, Besser, etc.) defines a canonical path ${}_bc_a \in {}_bP_{dR,a}(\mathbb{Q}_{p,st})$ that is fixed under frobenius. We study the limit of this path along a tangent vector at a point in $\mathfrak{D}_\eta$.

The universal enveloping algebra of the de Rham fundamental group of $M_{0,5}/\mathbb{Q}_p$ is $\mathbb{Q}_p\langle\langle X_{ij}\rangle\rangle_{0\leq i,j\leq 4}/R$, with the notation above. We apply the results on tangential basepoints and the limit of the frobenius invariant path to the five "infinitesimal imbeddings" of $M_{0,4}$ into $M_{0,5}$. Namely, we look at a parameter of imbeddings of $M_{0,4}$ into $M_{0,5}$ and let the parameter go to zero. We, then, express the crystalline invariant paths in terms of Coleman integrals and explicitly compute the limits of these Coleman integrals which finishes the proof of the 5-cycle relation. The proofs of the 2-cycle and 3-cycle relations are straightforward.

1.10. Outline. In §2, we start with the basics of logarithmic geometry and the de Rham and crystalline fundamental groups. We describe the trivializations of a log point in Lemma 2.1.2 in §2.1.3. This will be important when we are defining the tangential basepoints. Logarithmic differentials and its relation to the log product over a frame is reviewed in §2.1.4. In Lemma 2.1.3 in §2.1.5, we remark a canonical isomorphism between two log points which will be used in comparing tangential basepoints to ordinary basepoints in the tangent space. In §2.2, we review the de Rham fundamental group: in particular, canonical extensions of unipotent connections and the de Rham fiber functor for a variety X that satisfies $H^1(\overline{X}, \mathcal{O}) = 0$. In §2.3, we review the crystalline fundamental group, where we review the comparison theorem between unipotent overconvergent isocrystals and unipotent log convergent isocrystals. We describe the comparison between the de Rham and crystalline fundamental groups in §2.4.

In §3, we define the de Rham and crystalline versions of tangential basepoints and the comparison between them. In §4, we show that evaluating at a tangential basepoint is equivalent to pulling back to the tangent space and then evaluating at the ordinary basepoint in the tangent space. This will be used when we compute the limit of the frobenius invariant path. In §5, we describe the de Rham fundamental group of $M_{0,5}$ and define a canonical set of tangential basepoints on it.

In §6, we describe the limit of the frobenius invariant path of Besser ([3]) and Vologodsky ([32]). We give an alternative description of this path in §6.5 and §6.6, which will be used later on, and relate it to the tangential basepoints. In §6.7, we show that changing a tangential basepoint by multiplication by a root of unity does not change the frobenius invariant path. Note that the analog of this is not true in the Betti-de Rham case. This will be important while we are proving the Drinfel'd-Ihara relations. In §6.8, we describe the main object of our study, p-adic multi-zeta values.

In §7, we prove the Drinfel'd-Ihara relations. The proofs of the 2-cycle and 3-cycle relations are fairly straightforward. In §7.3, we prove the 5-cycle relation, by first expressing the frobenius invariant path in terms of Coleman integrals and then taking a limit which enables us to interpret certain frobenius invariant paths on $M_{0,5}$ in terms of those on $M_{0,4}$.

We assume that the reader is familiar with the basic notions of logarithmic geometry, as in (§1-2, [23]) or (§1, [24]); and the basic notions of rigid geometry as in ([28]).

1.11. Notations and conventions. If $\mathcal{C}$ is a category and A is an object of $\mathcal{C}$, we will abuse notation, for the sake of brevity, and write $A \in \mathcal{C}$.

We denote the category of vector spaces over a field K by Vec_K.

By a variety X over a field k, we mean a separated and geometrically integral k-scheme X, which is of finite type over k. If X/k is smooth, we say that $D \subseteq X$ is a simple normal crossings divisor, if $D := \sum_{j \in J} D_j$ is the sum of smooth divisors D_j meeting transversally. Since we will always be dealing with simple normal crossings divisors we will not distinguish between a divisor and its support.

If A is a finite dimensional k-space, we denote the corresponding affine variety, namely $\mathrm{Spec}\, Sym^\bullet A\check{}$, by $\mathbb{V}(A)$.

If X and Y are S-schemes, we remove the subscript S in the fiber product $X \times_S Y$ of X and Y over S, and denote it by $X \times Y$, if it is likely to cause no confusion. If A is a ring, sometimes we will denote the associated scheme by A, rather than $\mathrm{Spec}\, A$.

If $\mathfrak{X}/W$ is a scheme over a discrete valuation ring W, we let $\mathfrak{X}_s$ and $\mathfrak{X}_\eta$ denote the special and generic fibers respectively.

If $X/\mathbb{Q}_p$ is a variety, we will denote the associated rigid analytic variety with the same notation in order to go easy on the subscripts, since this is likely to no confusion.

If $a \in R\langle\langle x_i \rangle\rangle_{i \in I}$, the ring of associative formal power series ring in x_i, then we let $a[x^J]$ denote the coefficient of x^J in a, for a multi-index J.

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2. On log geometry and the crystalline fundamental group

2.1. Conventions on log geometry. Our main references for logarithmic geometry are [23], [24], and (§4, [25]). We will pass between a set of tangent vectors transversal to a divisor D at x and trivializations of a log point x_{log} using Lemma 2.1.2 below; we will use (2.1.5) to describe connections with logarithmic singularities; and (2.1.6) to give a description of tangential basepoints.

2.1.1. *Canonical log structure.* Let $\overline{X}/k$ be a smooth variety over a field k, $D \subseteq \overline{X}$, a simple normal crossings divisor, $X := \overline{X} \setminus D$ and $x \in D(k)$. Let $\{D_j | j \in J_x\}$, be the set of irreducible components of D passing through x and $D_x := \cup_{j \in J_x} D_j$.

Denote by $\overline{X}_{log}$ the canonical log structure on $\overline{X}$ defined by D (§1.3, [23]): the underlying scheme of $\overline{X}_{log}$ is $\overline{X}$ and the log structure is defined by the inclusion

$$M_{\overline{X}} := \mathcal{O}_{\overline{X}} \cap j_*(\mathcal{O}_X^\times) \to \mathcal{O}_{\overline{X}},$$

where $j : X \hookrightarrow \overline{X}$ is the open imbedding. $\overline{X}_{log}$ is a fine saturated log scheme (§1.3, [24]) and there is a canonical isomorphism of monoids

$$\overline{M}_{\overline{X},x} := M_{\overline{X},x}/\mathcal{O}^\times_{\overline{X},x} \xrightarrow{\sim} \mathrm{Cart}^-(\overline{X}, D_x), \tag{2.1.1}$$

where $\mathrm{Cart}^-(\overline{X}, D_x)$ denotes the monoid of anti-effective Cartier divisors on $\overline{X}$ supported on D_x. If the D_j are defined locally by $t_j \in \mathcal{O}_{\overline{X},x}$, and $t := \Pi_{j \in J_x} t_j$ then $M_{\overline{X},x} = \mathcal{O}_{\overline{X},x} \cap ((\mathcal{O}_{\overline{X},x})_t)^\times$.

2.1.2. *Log point at x.* Let x_{log} denote the log scheme obtained by pulling back the log structure on $\overline{X}_{log}$ via the map $\mathrm{Spec}\, k \to \overline{X}$ corresponding to x. Note that the monoid M_x on x_{log} is $M_{\overline{X},x} \otimes_{\mathcal{O}^\times_{\overline{X},x}} k^\times$, and (2.1.1) gives

$$\overline{M}_x := M_x/k^\times \xrightarrow{\sim} \mathrm{Cart}^-(\overline{X}, D_x). \tag{2.1.2}$$

2.1.3. *Splittings of the log structure on x_{log}.* With the notation in (2.1.2), let

$$S(D_x, \overline{X}) := \{\varphi : \varphi \text{ is a splitting of } M_x \to \overline{M}_x\}.$$

Let $I_j \subseteq \mathfrak{m}_x$ be the ideal defining D_j in $\mathcal{O}_{X,x}$ and $d : \mathfrak{m}_x \to \mathfrak{m}_x/\mathfrak{m}_x^2 = \Omega^1_{\overline{X},x}$, the canonical projection. Then

$$S(D_x, \overline{X}) \simeq \{(\cdots, \bar{t}_j, \cdots)_{j \in J_x} : \bar{t}_j \in (dI_j \setminus \{0\}), \text{ for } j \in J_x\}. \tag{2.1.3}$$

Let

$$N_{D_x/\overline{X}}(x) := \prod_{j \in J_x} N_{D_j/\overline{X}}(x),$$

be the fiber at x of the product of the normal bundles of D_j in $\overline{X}$.

Let $N^\times_{D_j/\overline{X}}(x) := N_{D_j/\overline{X}}(x) \setminus \{0\}$, and $N^\times_{D_x/\overline{X}}(x) := \prod_{j \in J_x} N^\times_{D_j/\overline{X}}(x)$. Note that by (2.1.3), there is a one-to-one correspondence between $S(D_x, \overline{X})$ and the set $\{(\cdots, \alpha_j, \cdots) | \alpha_j : N_{D_j/\overline{X}}(x) \simeq k, \text{ a linear isomorphism, for } j \in J_x\}$ and hence with $N^\times_{D_x/\overline{X}}(x)$.

Definition 2.1.1. Let $\mathrm{Spec}\, k_{x,log}$ denote the log scheme with underlying scheme $\mathrm{Spec}\, k$, and log structure associated to the pre-log structure $\mathrm{Cart}^-(\overline{X}, D_x) \to k$ that maps all the nonzero elements of $\mathrm{Cart}^-(\overline{X}, D_x)$ to 0.

Then by (2.1.2) and (2.1.3) we have:

Lemma 2.1.2. *There are natural bijections between* $S(D_x, \overline{X})$, $N^{\times}_{D_x/\overline{X}}(x)$, *and* $\{\alpha | \alpha : x_{log} \xrightarrow{\sim} \operatorname{Spec} k_{x,log}\}$, *the set of isomorphisms between the log schemes* x_{log} *and* $\operatorname{Spec} k_{x,log}$ *over* k.

2.1.4. *Logarithmic differentials.* For any map $f : Y_{log} \to Z_{log}$ of log schemes, let $\Omega^1_{Y_{log}/Z_{log}}$ denote the associated sheaf of Kähler differentials (§1.5, [23]) on Y.

We will be interested in this only when $Z_{log} = \operatorname{Spec} k$, endowed with the trivial log structure, and when Y_{log} is either $\overline{X}_{log}$, or x_{log} as in (§2.1.1 and §2.1.2).

Let $P := \Gamma(Y, \overline{M}_Y)$. The identity map $P \to \Gamma(Y, \overline{M}_Y)$ gives a frame $Y_{log} \to [P]$, in the sense of (Definition 4.1.3, [25]). Let $Y_{log} \times_{[P]} Y_{log}$ denote the log product of Y_{log} with itself over k and $[P]$ (Definition 4.2.4, [25]). There is an exact closed immersion (Corollary 4.2.8, [25])

$$\tilde{\Delta} : Y_{log} \to Y_{log} \times_{[P]} Y_{log}, \tag{2.1.4}$$

whose conormal bundle is canonically isomorphic to $\Omega^1_{Y_{log}/k}$.

Let $\tilde{\Delta}(Y)^{(1)}$ denote the first infinitesimal neighborhood of $Y_{log} \times_{[P]} Y_{log}$. Then $\mathcal{O}_{\tilde{\Delta}(Y)^{(1)}}$ can be thought of as a sheaf of $\mathcal{O}_Y$-algebras through the first projection $Y_{log} \times_{[P]} Y_{log} \to Y$. If we endow $\mathcal{O}_Y \oplus \Omega^1_{Y_{log}/k}$ with the ring structure such that $(f_1, \omega_1) \cdot (f_2, \omega_2) := (f_1 f_2, f_1 \omega_2 + f_2 \omega_1)$ then we have a natural isomorphism of sheaves of $\mathcal{O}_Y$-algebras (Corollary 4.2.8, [25]):

$$\mathcal{O}_{\tilde{\Delta}(Y)^{(1)}} \simeq \mathcal{O}_Y \oplus \Omega^1_{Y_{log}/k}. \tag{2.1.5}$$

Case (a): $Y_{log} = \overline{X}_{log}$.

In this case, $\Omega^1_{\overline{X}_{log}/k} = \Omega^1_{\overline{X}/k}(\log(D))$, where, if locally around a point $x \in \overline{X}$, D is defined by $x_1 \cdots x_r = 0$ for a system of parameters $(x_1, \cdots, x_n)$ then $\Omega^1_{\overline{X}/k}(\log(D))$ is locally generated by

$$\frac{dx_1}{x_1}, \cdots, \frac{dx_r}{x_r}, dx_{r+1}, \cdots, dx_n.$$

Let $(\overline{X} \times \overline{X})^{\sim}$ denote the blow-up of $\overline{X} \times_k \overline{X}$ along $\cup_{i \in J} D_i \times D_i$, where $D = \cup_{i \in J} D_i$. The fiber of $(\overline{X} \times \overline{X})^{\sim}$ over the point $(x, x) \in \overline{X} \times \overline{X}$ is

$$\prod_{i \in J_x} \mathbb{P}(N_{D_i \times D_i/\overline{X} \times \overline{X}}(x, x)) = \prod_{i \in J_x} \mathbb{P}(N_{D_i/\overline{X}}(x)^{\oplus 2}) = \prod_{i \in J_x} \mathbb{P}^1_k, \tag{2.1.6}$$

where J_x is the set of $i \in J$ such that $x \in D_i$, and $\mathbb{P}$ denotes projectivization.

Endow $(\overline{X} \times \overline{X})^{\sim}$ with the canonical log structure associated to the exceptional divisor of the blow-up. This gives the log scheme $(\overline{X} \times \overline{X})^{\sim}_{log}$. The diagonal map from $\Delta : X \to X \times X$ extends to a map

$$\tilde{\Delta} : \overline{X}_{log} \to (\overline{X} \times \overline{X})^{\sim}_{log}.$$

In this case, $P = \operatorname{Cart}^-(\overline{X}, D)$ and $\overline{X}_{log} \times_{[P]} \overline{X}_{log}$ is an open subscheme of $(\overline{X} \times \overline{X})^{\sim}$ which contains the image of $\tilde{\Delta}(\overline{X})$ (Example in §4.2, Corollary 4.2.8, [25]).

Case (b): $Y_{log} = x_{log}$.

In this case, if for $j \in J_x$, the component D_j of D_x is locally defined by $x_j = 0$, then

$$\Omega^1_{x_{log}/k} = \oplus_{j \in J_x} k \frac{dx_j}{x_j}. \tag{2.1.7}$$

Let $\{x_j \frac{\partial}{\partial x_j}\}_{j\in J_x}$ denote the dual basis.

We have $P = \mathrm{Cart}^-(\overline{X}, D_x) \simeq \oplus_{j\in J_x}\mathbb{N}$ and

$$x_{log} \times_{[P]} x_{log} = \prod_{j\in J_x} \mathbb{G}_m, \tag{2.1.8}$$

(Corollary 4.2.6; Example §4.2, [25]), which embeds canonically into the fiber of (x, x) in $(\overline{X} \times \overline{X})^{\sim}$ using (2.1.6).

2.1.5. *Comparison of two log points.* This section will be useful in studying tangential basepoints. Namely, it will help us localize near a point in a variety by passing from the variety to its tangent space.

Let $D_{(x)} \subseteq \mathbb{V}(N_{D_x/\overline{X}}(x))$ denote the simple normal crossings divisor that is the union of the coordinate axes through the origin. In other words, if $t_1, \cdots, t_n$ is a regular system of parameters on $\overline{X}$ at x, such that D is defined locally by $t_1 \cdots t_r = 0$ then $D_{(x)}$ is defined by $dt_1 \cdots dt_r = 0$. Let $\mathbb{V}(N)_{log} := \mathbb{V}(N_{D_x/\overline{X}}(x))_{log}$ be the scheme $\mathbb{V}(N_{D_x/\overline{X}}(x))$ endowed with the log structure associated to the divisor $D_{(x)}$, and 0_{log} the log scheme induced by the inclusion $0 \to \mathbb{V}(N_{D_x/\overline{X}}(x))$.

Lemma 2.1.3. *The log schemes x_{log} and 0_{log} are canonically isomorphic.*

Proof. Note that both log schemes have the same underlying scheme $\mathrm{Spec}\, k$. Let the component D_j of D be defined locally at x by the ideal I_j, for $1 \le j \le r$.

If $\{t_j\}_{1\le j\le n}$ is as above then $I_j = (t_j)$, for $1 \le j \le r$. We have a map

$$\varphi : \otimes_{1\le j\le r} Sym^{\bullet}(dI_j/dI_j^2) \to \mathcal{O}_{\overline{X},x},$$

with the property $\varphi(dt_j) = t_j$. Since $M_{\mathbb{V}(N)} = (\mathcal{O}_{\mathbb{V}(N),0})^{\times}_{dt_1\cdots dt_r}$, and $M_{\overline{X},x} = (\mathcal{O}_{\overline{X},x})^{\times}_{t_1\cdots t_r}$, this gives a map $M_{\mathbb{V}(N)} \to M_{\overline{X},x}$ and hence a map

$$M_{\mathbb{V}(N)} \otimes_{\mathcal{O}^{\times}_{\mathbb{V}(N),0}} k^{\times} \to M_{\overline{X},x} \otimes_{\mathcal{O}^{\times}_{\overline{X},x}} k^{\times}$$

which we continue to denote by φ.

This map is independent of the choice of $\{t_j\}_{1\le j\le n}$. If $\{s_j\}_{1\le j\le n}$ is another set then $s_j/t_j \in \mathcal{O}^{\times}_{\overline{X},x}$, for $1 \le j \le r$. Let ψ be the map corresponding to $\{s_j\}_{1\le j\le n}$. To show the independence it is enough to show that ψ and φ agree on $\{ds_j\}_{1\le j\le r}$. Note that

$$\varphi(ds_j) = \varphi(d(\frac{s_j}{t_j} \cdot t_j)) = \varphi(d(\frac{s_j}{t_j}) \cdot t_j + \frac{s_j}{t_j}(x) \cdot dt_j) = \frac{s_j}{t_j}(x) \cdot t_j$$

and $\psi(ds_j) = s_j$. Therefore

$$\frac{\varphi(ds_j)}{\psi(ds_j)} = \frac{s_j}{t_j}(x) \cdot \frac{t_j}{s_j}$$

and $\varphi(ds_j)$ and $\psi(ds_j)$ are equal in $M_{\overline{X},x} \otimes_{\mathcal{O}^{\times}_{\overline{X},x}} k^{\times}$, for $1 \le j \le r$.

This map makes the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k^{\times} & \longrightarrow & M_{0_{log}} & \longrightarrow & \overline{M}_{0_{log}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & k^{\times} & \longrightarrow & M_{x_{log}} & \longrightarrow & \overline{M}_{x_{log}} & \longrightarrow & 0 \end{array}$$

commute, where α is the natural identification. This implies that the middle map is also an isomorphism. □

2.2. The de Rham fundamental group. We review the theory of the de Rham fundamental group (10.24-10.53, [7]).

2.2.1. *The categories* $\mathrm{Mic}_{reg}(X/K)$ *and* $\mathrm{Mic}_{uni}(X/K)$. Let K be a field of characteristic 0 and X/K, a smooth variety over K. Let $\mathrm{Mic}_{reg}(X/K)$ denote the category whose objects consist of vector bundles with integrable connection (E,∇) that has regular singularities at infinity (I.5, [1]); and whose morphisms are vector bundle morphisms which are compatible with the connections. Since we assume that X/K is geometrically integral, $\mathrm{End}((\mathcal{O},d)) = K$, and $\mathrm{Mic}_{reg}(X/K)$ naturally forms a tensor category over K (§5, [7]).

We will be mostly interested in a full-subcategory $\mathrm{Mic}_{uni}(X/K)$ of $\mathrm{Mic}_{reg}(X/K)$. A vector bundle with integrable connection (E,∇) is *unipotent*, if it has an increasing filtration by sub-vector bundles with connection $\mathrm{Fil}_i(E,\nabla)\subseteq(E,\nabla)$, $0\le i\le n$, such that $\mathrm{Fil}_0(E,\nabla)=0$, $\mathrm{Fil}_n(E,\nabla)=(E,\nabla)$ and $\mathrm{gr}_i(E,\nabla)\simeq(\mathcal{O},d)$ or 0. Since regularity is stable under extensions (I, Proposition 5.2, [1]), unipotent connections are regular. Unipotent vector bundles with connection then form a tensor full-subcategory $\mathrm{Mic}_{uni}(X/K)$ of $\mathrm{Mic}_{reg}(X/K)$.

Definition 2.2.1. If $\omega:\mathrm{Mic}_{uni}(X/K)\to\mathrm{Vec}_K$ is a fiber functor, with values in K-spaces, we let $\pi_{1,dR}(X/K,\omega)$ denote the fundamental group of this tensor category at ω (§6, [7]) and call it the *de Rham fundamental group* of X/K at ω.

2.2.2. *Connections with logarithmic poles.* We follow (I.4, [1]; II.3.8, [8]; §11.1, [7]).

Let Y_{log}/K be either $\overline{X}_{log}$ or x_{log} as in §2.1.4. A vector bundle with connection (E,∇) on Y_{log} is a vector bundle E on Y and a K-linear map

$$\nabla: E\to E\otimes\Omega^1_{Y_{log}/K}$$

satisfying the Leibniz property and it is *integrable*, if $\nabla^2: E\to E\otimes\Omega^2_{Y_{log}/K}$ is 0. We denote the corresponding category by $\mathrm{Mic}(Y_{log}/K)$.

Continuing with the notation in §2.1.4, let $p_i: Y_{log}\times_{[P]}Y_{log}\to Y_{log}$, for $i=1,2$, denote the two projections.

A vector bundle with connection (E,∇) on Y_{log} is equivalent to giving an isomorphism

$$p_2^*E|_{\tilde{\Delta}(Y)^{(1)}}\tilde{\to}p_1^*E|_{\tilde{\Delta}(Y)^{(1)}},$$

that induces the identity on $\tilde{\Delta}(Y)$. If the connection is integrable, then this isomorphism extends to an isomorphism,

$$p_2^*E|_{\tilde{\Delta}(Y)^{\widehat{}}}\tilde{\to}p_1^*E|_{\tilde{\Delta}(Y)^{\widehat{}}},$$

to the formal completion $\tilde{\Delta}(Y)^{\widehat{}}$ of $\tilde{\Delta}(Y)$ in $Y_{log}\times_{[P]}Y_{log}$, since we assume that K is of characteristic 0.

2.2.3. *Residues of connections with logarithmic poles.* We follow (I.4, [1]; II.3.8, [8]).

Let $\{t_j|j\in J_x\}$ be part of a local set of parameters in a neighborhood U of $x\in\overline{X}$, as in §2.1.1. Then letting $i_{j,U}: D_j\cap U\to\overline{X}$ be the locally closed immersion, we have a linear map:

$$\nabla_{j,U}: i_{j,U}^*\overline{E}\to i_{j,U}^*\overline{E}$$

induced by pairing ∇ with the derivation $t_j \frac{\partial}{\partial t_j}$. Even though $t_j \frac{\partial}{\partial t_j}$ depends on the choice of a system of parameters extending $\{t_j | j \in J_x\}$, $\nabla_{j,U}$ is independent of any choice of system of parameters and hence patches to give a map $\nabla_j : i_j^* \overline{E} \to i_j^* \overline{E}$, where $i_j : D_j \to \overline{X}$ is the closed immersion, which is called the *residue map along* D_j. It turns out that the characteristic polynomial $P_j(z) \in \mathcal{O}_{D_j}[z]$ of ∇_j is, in fact, in $K[z]$ (II.3.10, [8]). Therefore, the eigenvalues of the residue map, called exponents, are constant along the divisor D_j.

The construction of residues is compatible with duality and tensor products, i.e.

$$(\nabla^1 \otimes \nabla^2)_j = id \otimes \nabla_j^2 + \nabla_j^1 \otimes id, \tag{2.2.1}$$

and an exact sequence of vector bundles with connection

$$0 \to (\overline{E}_1, \nabla_1) \to (\overline{E}_2, \nabla_2) \to (\overline{E}_3, \nabla_3) \to 0$$

on $\overline{X}_{log}$ induces a corresponding exact sequence of vector bundles with endomorphisms

$$0 \to (i_j^* \overline{E}_1, \nabla_{1,j}) \to (i_j^* \overline{E}_2, \nabla_{2,j}) \to (i_j^* \overline{E}_3, \nabla_{3,j}) \to 0,$$

on D_j.

The residue map also gives the following description of $\mathrm{Mic}(x_{log}/k)$.

Definition 2.2.2. For $x \in \overline{X}(k)$, let $\mathcal{T}_x$ denote the following category. Its objects are pairs $(V, (T_j)_{j \in J_x})$ of vector spaces V over k and linear operators T_j on V, indexed by J_x, such that for every $i, j \in J_x$, $[T_i, T_j] = 0$. The morphisms are maps of vector spaces that commute with the operators.

The map that sends $(E, \nabla) \in \mathrm{Mic}(x_{log}/k)$ to $(E, (\nabla_j)_{j \in J_x})$, where

$$\nabla_j = (\mathrm{id}_E \otimes x_j \frac{\partial}{\partial x_j}) \circ \nabla$$

(c. f. §2.1.4 (b)), induces an equivalence of categories

$$\mathrm{Mic}(x_{log}/k) \to \mathcal{T}_x, \tag{2.2.2}$$

whose restriction to unipotent objects on both sides induces the equivalence

$$\mathrm{Mic}_{uni}(x_{log}/k) \to \mathcal{T}_{x,uni}. \tag{2.2.3}$$

One sees that the objects on the right hand side, in fact, consist of vector spaces V together with commuting *nilpotent* linear operators T_j on V, indexed by J_x.

2.2.4. *Canonical extensions of unipotent connections.* Let X/K and $\overline{X}/K$ be as above. Let $(E, \nabla) \in \mathrm{Mic}_{uni}(X/K)$ be a unipotent vector bundle with connection. Then there exists a vector bundle with connection (E_{can}, ∇) on $\overline{X}_{log}$ which extends (E, ∇) and has the property that all the residue maps ∇_j are nilpotent (II.5.2, [8]; I. Theorem 4.9, [1]). This extension, which is unique up to unique isomorphism, is called the *canonical extension* of (E, ∇) to $\overline{X}$. To ease the notation, we will sometimes denote E_{can} by $\overline{E}$.

The functor

$$can : \mathrm{Mic}_{uni}(X/K) \to \mathrm{Mic}(\overline{X}_{log}/K),$$

that sends (E, ∇) to (E_{can}, ∇) is exact, and is compatible with taking duals and tensor products. This follows from the corresponding properties of the residue map in §2.2.2 and the stability of nilpotence under these constructions. The functor *can*, then, induces an equivalence of categories

$$can : \mathrm{Mic}_{uni}(X/K) \to \mathrm{Mic}_{uni}(\overline{X}_{log}/K), \tag{2.2.4}$$

where $\mathrm{Mic}_{uni}(\overline{X}_{log}/K)$ is the full-subcategory of $\mathrm{Mic}(\overline{X}_{log}/K)$ whose set of objects is the essential image of *can*.

2.2.5. *The de Rham fiber functor* $\omega(dR)$. Assume further that $\overline{X}/K$ is proper and $H^1(\overline{X}, \mathcal{O}) = 0$. Then Deligne defines a fiber functor (§5.9, §12.4, [7]):

$$\omega(dR) : \mathrm{Mic}_{uni}(X/K) \to Vec_K,$$

from $\mathrm{Mic}_{uni}(X/K)$ to Vec_K, the category of vector spaces over K, as follows.

The condition $H^1(\overline{X}, \mathcal{O}) = 0$ implies that the underlying bundle E_{can} of the canonical extension of $(E, \nabla) \in \mathrm{Mic}_{uni}(X/K)$ is trivial (Proposition 12.3, [8]). This gives a canonical isomorphism $\Gamma(\overline{X}, E_{can}) \otimes_K \mathcal{O}_{\overline{X}} \xrightarrow{\sim} E_{can}$. Since the canonical extension functor is a tensor functor (§2.2.3), it follows from the above that

$$\omega(dR) : \mathrm{Mic}_{uni}(X/K) \to Vec_K$$

that sends (E, ∇) to $\Gamma(\overline{X}, E_{can})$ is a fiber functor. This called the *de Rham fiber functor*. Note that with this notation we have a canonical isomorphism:

$$\omega(dR)(E, \nabla) \otimes_K \mathcal{O}_{\overline{X}} \xrightarrow{\sim} E_{can},$$

for any $(E, \nabla) \in \mathrm{Mic}_{uni}(X/K)$.

Remark. A priori the de Rham fiber functor depends on the choice of a compactification $\overline{X}$ with $H^1(\overline{X}, \mathcal{O}) = 0$. Let $\overline{X}_1$ and $\overline{X}_2$ be two such compactifications. By resolving the singularities of the closure of the diagonal Δ_X in $\overline{X}_1 \times \overline{X}_2$ we find a compactification $\tilde{X}$ of X which maps to $\overline{X}_1$ and $\overline{X}_2$. Let $(E, \nabla) \in \mathrm{Mic}_{uni}(X/K)$ and let $\overline{E}_i$ denote the canonical extension of X to $\overline{X}_i$, for $i = 1, 2$. Let $\pi_i : \tilde{X} \to \overline{X}_i$ denote the projection. Since the exponents of the pull-backs are linear combinations of the original exponents (I, [1]), we see that $(\pi_i^* \overline{E}_i, \nabla)$ both have zero exponents and hence are both the canonical extension of (E, ∇) to $\tilde{X}$. Therefore, since $\overline{E}_i$ are trivial bundles,

$$\Gamma(\overline{X}_1, \overline{E}_1) = \Gamma(\tilde{X}, \pi_1^* \overline{E}_1) \xrightarrow{\sim} \Gamma(\tilde{X}, \pi_2^* \overline{E}_2) = \Gamma(\overline{X}_2, \overline{E}_2).$$

This shows that, up to isomorphism, $\omega(dR)$ does not depend on the compactification.

2.3. The crystalline fundamental group. In this section, we review the theory of the crystalline fundamental group (§11, [7]; §2.4, [31]).

2.3.1. *Unipotent overconvergent isocrystals.* We follow (§2.3, [2]; §2.4.1, [31]). Let k be a perfect field of characteristic p, $W := W(k)$, the ring of Witt vectors over k, and K, the field of fractions of W.

If $\mathcal{Q}/W$ is a formal scheme and $Z_0 \subseteq \mathcal{Q} \times_W k$ is a locally closed subscheme, let $]Z_0[_{\mathcal{Q}} \subseteq \mathcal{Q}_K$ denote the tube of Z_0 in $\mathcal{Q}$ ((1.1.2), [2]). Assume further that $Z \subseteq \mathcal{Q} \times_W k$ is a closed subscheme with $j : Z_0 \hookrightarrow Z$, an open imbedding. For a sheaf of abelian groups E on $]Z[_{\mathcal{Q}}$, $j^\dagger E$ denotes the sheaf on $]Z[_{\mathcal{Q}}$, which is characterized by the property that for any quasi-compact $U \subseteq]Z[_{\mathcal{Q}}$,

$$\Gamma(U, j^\dagger E) = \varinjlim_{V} \Gamma(U \cap V, E),$$

where V runs through the strict neighborhoods of $]Z_0[_{\mathcal{Q}}$ in $]Z[_{\mathcal{Q}}$ (§2.1.1, [2]).

Assume, moreover, that $\mathcal{Q}/W$ is smooth in a neighborhood of Z_0 in $\mathcal{Q}$. Let E be a $j^\dagger\mathcal{O}_{]Z[}$-module with integrable connection ∇ on $]Z[_{\mathcal{Q}}$. We imbed Z diagonally into $\mathcal{Q}^2 := \mathcal{Q}\times_W \mathcal{Q}$, and let $p_1, p_2 :]Z[_{\mathcal{Q}^2} \to]Z[_{\mathcal{Q}}$ denote the two projections induced by those from $\mathcal{Q}^2$ to $\mathcal{Q}$. We say that ∇ is *overconvergent along* $Z\setminus Z_0$, if there is an isomorphism $p_2^*E \xrightarrow{\sim} p_1^*E$ on $]Z[_{\mathcal{Q}^2}$, which induces ∇ when restricted to the formal completion of the diagonal ((2.2.5), [2]).

Let Y/k be a variety over k, the category of overconvergent isocrystals $\mathrm{Isoc}^\dagger(Y/W)$ is defined as follows. Let $\overline{Y}$ be a compactification of Y, and $\{U_i\}_{i\in I}$ an open covering of $\overline{Y}$ together with closed imbeddings of $U_i \hookrightarrow \mathcal{P}_i$ into formal schemes $\mathcal{P}_i/W$, which are smooth in a neighborhood of $U_i\cap Y$. Let $j_i : U_i\cap Y \to U_i$ denote the corresponding open imbedding. Then an overconvergent isocrystal on Y is given by a collection of $j_i^\dagger\mathcal{O}_{]U_i[}$ modules with integrable connection which are overconvergent along $U_i\setminus(U_i\cap Y)$; and a collection of isomorphisms between their restrictions to $]U_i\cap U_j[_{\mathcal{P}_i\times\mathcal{P}_j}$, which satisfy the co-cycle condition (§2.3, [2]). The morphisms are given by a collection of maps of $j_i^\dagger\mathcal{O}_{]U_i[}$-modules with connection which are compatible with the isomorphisms on $]U_i\cap U_j[_{\mathcal{P}_i\times\mathcal{P}_j}$. The category is independent, up to canonical equivalence, of all the choices and depends only on Y/W.

For an overconvergent isocrystal $(E,\nabla)\in\mathrm{Isoc}^\dagger(Y/W)$, and data $\{U_i\to\mathcal{P}_i\}_{i\in I}$ as above, we call the corresponding $j_i^\dagger\mathcal{O}_{]U_i[}$-module with connection on $]U_i[_{\mathcal{P}_i}$, the *realization* of (E,∇) on $\mathcal{P}_i$ and denote it by $(E,\nabla)_{\mathcal{P}_i}$.

An overconvergent isocrystal (E,∇) on Y/W is said to be unipotent, if it has a finite filtration by sub-overconvergent isocrystals $\mathrm{Fil}_i(E,\nabla)$, $0\le i\le n$, such that $\mathrm{Fil}_0(E,\nabla)=0$, $\mathrm{Fil}_n(E,\nabla)=(E,\nabla)$ and $\mathrm{gr}_i(E,\nabla)=(\mathcal{O},d)$ or 0. We denote by $\mathrm{Isoc}^\dagger_{uni}(Y/W)$ the full-subcategory of $\mathrm{Isoc}^\dagger(Y/W)$ consisting of unipotent overconvergent isocrystals on Y/W. $\mathrm{Isoc}^\dagger_{uni}(Y/W)$ has a natural structure of a tensor category over K.

Definition 2.3.1. If $\omega : \mathrm{Isoc}^\dagger_{uni}(Y/W)\to\mathrm{Vec}_K$ is a fiber functor over K, we let $\pi^\dagger_{1,crys}(Y/W,\omega)$ denote the corresponding fundamental group of $\mathrm{Isoc}^\dagger_{uni}(Y/W)$ at ω (§6, [7]), and call it the *crystalline fundamental group* of Y/W at ω.

2.3.2. *Unipotent log convergent isocrystals.* Shiho defines the log convergent site on a fine saturated log scheme Y_{log} over k (§2, [30]). As usual we will be interested in the cases when Y_{log} is $\overline{X}_{log}$ or x_{log}.

Case (a): $Y_{log}=\overline{X}_{log}$.

If we assume that $\overline{X}_{log}$ is the canonical log scheme associated to a simple normal crossings divisor $D\subseteq\overline{X}$ (§2.1.1), the category of log convergent isocrystals on $\overline{X}_{log}$ can be described as follows.

First suppose that there is a formal scheme $\mathcal{Q}/W$ and a *relative simple normal crossings divisor* $\mathcal{D}\subseteq\mathcal{Q}$, i. e. that $\mathcal{D}$ is a simple normal crossings divisor lying in the smooth locus of $\mathcal{Q}/W$ and is flat over W, satisfying the following properties: there is a closed immersion $i:\overline{X}\hookrightarrow\mathcal{Q}\times_W k$, such that $\mathcal{Q}/W$ is smooth in a neighborhood of $\overline{X}$ and $D=i^*(\mathcal{D})$. Let $(\mathcal{Q}\times\mathcal{Q})^\sim$ denote the blow-up as in §2.1.4, and $\overline{p}_1$ and $\overline{p}_2$ denote the two maps from $]\tilde{\Delta}(\overline{X})[_{(\mathcal{Q}\times\mathcal{Q})^\sim}$ to $]\overline{X}[_{\mathcal{Q}}$ that are induced by the two projections.

Then a *realization* of a log convergent isocrystal on $\overline{X}_{log}$ corresponding to the data $(i,\mathcal{Q},\mathcal{D})$ is given by a locally free $\mathcal{O}_{]\overline{X}[}$-module of finite rank E on $]\overline{X}[_{\mathcal{Q}}$ and an integrable connection ∇ on E with logarithmic singularities along $]\overline{X}[_{\mathcal{Q}} \cap \mathcal{D}_K$ such that there is an isomorphism

$$\overline{p}_2^* E \xrightarrow{\sim} \overline{p}_1^* E$$

on $]\tilde{\Delta}(\overline{X})[_{(\mathcal{Q}\times\mathcal{Q})^\sim}$ that induces the map associated to ∇, when restricted to the formal completion of $\tilde{\Delta}(\mathcal{Q}_K) \cap]\tilde{\Delta}(\overline{X})[_{(\mathcal{Q}\times\mathcal{Q})^\sim}$ in $]\tilde{\Delta}(\overline{X})[_{(\mathcal{Q}\times\mathcal{Q})^\sim}$.

If there is no such global imbedding, choose an open cover $\{U_j\}_{j\in J}$ of $\overline{X}$, such that there are formal schemes $\mathcal{P}_j/W$, relative normal crossings divisors $\mathcal{D}^{(j)} \subseteq \mathcal{P}_j$ and imbeddings $U_j \hookrightarrow \mathcal{P}_j \times_W k$ as above. For $j_1, j_2 \in J$, let $(\mathcal{P}_{j_1} \times \mathcal{P}_{j_2})^\sim$ denote the blow-up of $\mathcal{P}_{j_1} \times \mathcal{P}_{j_2}$ along $\cup_{i\in I}(\mathcal{D}_i^{(j_1)} \times \mathcal{D}_i^{(j_2)})$ and

$$\overline{q}_k :]\tilde{\Delta}(U_{j_1} \cap U_{j_2})[_{(\mathcal{P}_{j_1}\times\mathcal{P}_{j_2})^\sim} \to]U_{j_k}[_{\mathcal{P}_{j_k}},$$

for $k = 1, 2$, denote the two maps induced by the projections.

Then the *realization* of a log convergent isocrystal on $\overline{X}/W$ corresponding to this data is described as follows. For every $j \in J$, there is an $\mathcal{O}_{]U_j[}$-module E_j on $]U_j[_{\mathcal{P}_j}$ and an isomorphism

$$\alpha_j : \overline{p}_2^* E_j \xrightarrow{\sim} \overline{p}_1^* E_j \tag{2.3.1}$$

on $]\tilde{\Delta}(U_j)[_{(\mathcal{P}_j\times\mathcal{P}_j)^\sim}$ as above, and for every distinct $j_1, j_2 \in J$, there is an isomorphism

$$f_{j_1 j_2} : \overline{q}_2^* E_{j_2} \xrightarrow{\sim} \overline{q}_1^* E_{j_1}, \tag{2.3.2}$$

which satisfy the co-cycle condition, and which are compatible with the maps (2.3.1) in the obvious sense. A morphism from an isocrystal $\{E_j, \alpha_j, f_{j_1 j_2}\}$ to another isocrystal $\{F_j, \beta_j, g_{j_1 j_2}\}$ is given by a collection of morhpisms $E_j \to F_j$ on $]U_j[_{\mathcal{P}_j}$ that are compatible with the α_j and β_j's and with the $f_{j_1 j_2}$ and $g_{j_1 j_2}$'s. The category of log convergent isocrystals is independent of the data of the local liftings $\{\mathcal{P}_j, \mathcal{D}^{(j)}, U_j \hookrightarrow \mathcal{P}_j \times_W k\}$, up to canonical isomorphism, and is denoted by $\mathrm{Isoc}^c(\overline{X}_{log}/W)$.

A log convergent isocrystal (E, ∇) on $\overline{X}$ is *unipotent* if it has a filtration $\mathrm{Fil}_i(E, \nabla)$, $0 \le i \le n$, such that $\mathrm{Fil}_0(E, \nabla) = 0$, $\mathrm{Fil}_n(E, \nabla) = (E, \nabla)$, and the graded pieces are $(\mathcal{O}, d)$ or 0. Let $\mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W)$ denote the full-subcategory of $\mathrm{Isoc}^c(\overline{X}_{log}/W)$ consisting of unipotent log convergent isocrystals.

Assume from now on that $\overline{X}/k$ is proper and $X := \overline{X} \setminus D$. By restriction we obtain a natural functor:

$$\mathrm{Isoc}^c(\overline{X}_{log}/W) \to \mathrm{Isoc}^\dagger(X/W),$$

which, when restricted to unipotent objects, induces an equivalence of categories (Lemma 2, [31]):

$$\mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}^\dagger_{uni}(X/W). \tag{2.3.3}$$

Definition 2.3.2. If $\omega : \mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W) \to \mathrm{Vec}_K$ is a fiber functor, we let $\pi_{1,crys}(\overline{X}_{log}/W, \omega)$ denote the fundamental group of $\mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W)$ at ω.

A fiber functor ω on $\mathrm{Isoc}^\dagger_{uni}(X/W)$ then induces an isomorphism

$$\pi^\dagger_{1,crys}(X/W, \omega) \xrightarrow{\sim} \pi_{1,crys}(\overline{X}_{log}/W, \omega),$$

by (2.3.3).

Case (b): $Y_{log} = x_{log}$.

Let $\mathfrak{x}_{log}$ be a log scheme structure on $\operatorname{Spec} W$ such that the pull-back log structures on $\operatorname{Spec} k$ and $\operatorname{Spec} K$ via the canonical maps to $\operatorname{Spec} W$ induce isomorphisms $\overline{M}_W \xrightarrow{\sim} \overline{M}_k = P$ and $\overline{M}_W \xrightarrow{\sim} \overline{M}_K$. The *realization* of an object of $\operatorname{Isoc}^c(\mathfrak{x}_{s,log}/W)$ with respect to this data is then given by a finite dimensional vector space E over K, endowed with an integrable connection

$$\nabla : E \to E \otimes \Omega^1_{\mathfrak{x}_{\eta,log}/K}$$

on $\mathfrak{x}_{\eta,log}$ such that there is an isomorphism

$$p_2^* E \xrightarrow{\sim} p_1^* E \tag{2.3.4}$$

on $\tilde{\Delta}(\mathfrak{x}_s)_{]\mathfrak{x}\times_{[P]}\mathfrak{x}[}$ (c. f. §2.1.4), which induces the map associated to ∇ when restricted to the formal completion of $\tilde{\Delta}(\mathfrak{x}_\eta)$ in $\mathfrak{x}_{\eta,log} \times_{[P]} \mathfrak{x}_{\eta,log}$.

Note that, using the isomorphism (2.1.8), we have that

$$]\tilde{\Delta}(\mathfrak{x}_s)[_{\mathfrak{x}\times_{[P]}\mathfrak{x}} = \prod_{j\in J_x} D(1, 1^-), \tag{2.3.5}$$

where $D(a, r^-)$ is the open p-adic disk of radius r and center a.

Choosing a fiber functor $\omega : \operatorname{Isoc}^c_{uni}(\mathfrak{x}_{s,log}/W) \to \operatorname{Vec}_K$, we define the log crystalline fundamental group $\pi_{1,crys}(\mathfrak{x}_{s,log}/W, \omega)$, as in Definition 2.3.2.

2.3.3. *Frobenius.*

Case (a): $Y_{log} = \overline{X}_{log}$.

The relative p-power frobenius map $F : \overline{X}_{log} \to \overline{X}^{(p)}_{log}$ induces the following functors

$$F^* : \operatorname{Isoc}^c_{uni}(\overline{X}^{(p)}_{log}/W) \to \operatorname{Isoc}^c_{uni}(\overline{X}_{log}/W)$$

and

$$F^* : \operatorname{Isoc}^\dagger_{uni}(X^{(p)}/W) \to \operatorname{Isoc}^\dagger_{uni}(X/W).$$

Let us describe the first of these functors.

Let $\sigma : W \to W$ denote the frobenius map on W, and for a scheme Z/W let $Z^{(\sigma)}/W$ denote the base change of Z/W via σ.

Let $\{U_j\}_{j\in J}$ be a cover of $\overline{X}$ such that there is data $(\mathcal{P}_j, \mathcal{D}^{(j)}, U_j \hookrightarrow \mathcal{P}_j, \mathcal{F}_j)$ with $(\mathcal{P}_j, \mathcal{D}^{(j)}, U_j \hookrightarrow \mathcal{P}_j)$ is as in §2.3.2 above and $\mathcal{F}_j : \mathcal{P}_{j,log} \to \mathcal{P}^{(\sigma)}_{j,log}$ is a lifting of the frobenius F, where $\mathcal{P}_j$ is endowed with the log structure defined by $\mathcal{D}^{(j)}$. Let $(E_j, \alpha_j, f_{j_1 j_2})$ be the realization of a log convergent crystal (E, ∇) on $\overline{X}^{(p)}_{log}$ corresponding to the data $(\mathcal{P}^{(\sigma)}_j, \mathcal{D}^{(j,\sigma)}, U^{(p)}_j \hookrightarrow \mathcal{P}^{(\sigma)}_j)$. Then $F^*(E, \nabla)$ is the log convergent crystal on $\overline{X}_{log}$ whose realization is given by

$$(\mathcal{F}^*_{j,K}(E_j), (\mathcal{F}_{j,K} \times \mathcal{F}_{j,K})^{\sim *}\alpha_j, (\mathcal{F}_{j_1,K} \times \mathcal{F}_{j_2,K})^{\sim *} f_{j_1 j_2}).$$

This induces maps

$$F_* : \pi_{1,crys}(\overline{X}_{log}/W, \omega) \to \pi_{1,crys}(\overline{X}^{(p)}_{log}/W, \omega \circ F^*) \tag{2.3.6}$$

and

$$F_* : \pi^\dagger_{1,crys}(X/W, \omega) \to \pi^\dagger_{1,crys}(X^{(p)}/W, \omega \circ F^*). \tag{2.3.7}$$

Case (b): $Y_{log} = x_{log}$.

First note that, if x_{log} is given by the map $M_x = M_k \xrightarrow{\gamma} k$, then $x_{log}^{(p)}$ is the log scheme structure on $\operatorname{Spec} k$ associated to the map

$$\gamma^{(p)} : M_k \otimes_{k^\times} k^\times \to k,$$

where $k^\times \to M_k$ is the canonical inclusion, $k^\times \to k^\times$ is the frobenius, and $M_k \to k$ is γ. Then the relative frobenius $F : x_{log} \to x_{log}^{(p)}$ is the map given by the identity on $\operatorname{Spec} k$, and the map

$$M_k \otimes_{k^\times} k^\times \to M_k,$$

which is induced by the multiplication by p on M_k and the canonical inclusion $k^\times \to M_k$.

Let $\mathfrak{x}_{log}$ be a lifting of x_{log} as in §2.3.2 and $\mathcal{F} : \mathfrak{x}_{log} \to \mathfrak{x}_{log}^{(\sigma)}$ be a lifting of F. Let (E, α), where α is as in (2.3.4), be the realization of an object of $\mathrm{Isoc}^c_{uni}(x_{log}^{(p)}/W)$. We let $\mathcal{F}_\eta \times_{[P]} \mathcal{F}_\eta$ denote the induced map

$$\mathcal{F}_\eta \times_{[P]} \mathcal{F}_\eta : (\mathfrak{x}_{log} \times_{[P]} \mathfrak{x}_{log})_\eta \to (\mathfrak{x}_{log}^{(\sigma)} \times_{[P]} \mathfrak{x}_{log}^{(\sigma)})_\eta.$$

Then the realization of the pull-back of (E, α) with respect to F is given by $(\mathcal{F}_\eta^* E, (\mathcal{F}_\eta \times_{[P]} \mathcal{F}_\eta)^* \alpha)$.

Let $[p] : \prod_{j \in J_x} D(1, 1^-) \to \prod_{j \in J_x} D(1, 1^-)$, be defined by

$$[p]((z_j)_{j \in J_x}) = (z_j^p)_{j \in J_x}.$$

Using the identification (2.3.5) the pull-back above is given by $(E, [p]^*(\alpha))$.

Let $(\mathfrak{x}^1_{log}, \mathcal{F}^1)$ and $(\mathfrak{x}^2_{log}, \mathcal{F}^2)$ be two liftings of (x_{log}, F); and (E_i, α_i) denote the corresponding realizations of a log isocrystal on $x_{log}^{(p)}$, together with the isomorphism

$$f_{12} : \overline{p}_2^* E_2 \to \overline{p}_1^* E_1$$

on $]\tilde{\Delta}(x^{(p)})[_{\mathfrak{x}^{1(\sigma)} \times_{[P]} \mathfrak{x}^{2(\sigma)}}$. Then the isomorphism between the pull-backs of $\mathcal{F}_\eta^{1*} E_1$ and $\mathcal{F}_\eta^{2*} E_2$ to $]\tilde{\Delta}(x)[_{\mathfrak{x}^1 \times_{[P]} \mathfrak{x}^2}$ is given by $(\mathcal{F}^1_\eta \times_{[P]} \mathcal{F}^2_\eta)^* f_{12}$.

Let $\mathcal{F}^1$ and $\mathcal{F}^2$ be two liftings of frobenius on $\mathfrak{x}_{log}$, and (E, α) a realization of an object in $\mathrm{Isoc}^c_{uni}(x_{log}^{(p)}/W)$ as above. The map $(\mathcal{F}^1_\eta \times_{[P]} \mathcal{F}^2_\eta)^* \alpha$ gives the isomorphism between the pull-backs of $\mathcal{F}^{i*}_\eta E$, $i = 1, 2$, to $]\tilde{\Delta}(x)[_{\mathfrak{x} \times_{[P]} \mathfrak{x}}$. When evaluated at $\tilde{\Delta}(\mathfrak{x}_K)$, this isomorphism gives an automorphism of E, which can be described as follows.

Suppose that $\{m_i\}_{i \in J_x} \subseteq M_W$, is a subset of M_W such that it induces *the* basis for $\overline{M}_W = P = \oplus_{i \in J_x} \mathbb{N}$. Then the maps induced by $\mathcal{F}^i$ from $M_W \otimes_{W^\times} W^\times$ to M_W are given by

$$\mathcal{F}^{i*}(m_j) = a_j^{(i)} m_j^p,$$

for some $a_j^{(i)} \in 1 + pW$, $j \in J_x$ and $i = 1, 2$.

Then with the identification above, $\mathcal{F}^1_\eta \times_{[P]} \mathcal{F}^2_\eta$ on $\prod_{j \in J_x} D(1, 1^-)$ is given by

$$(\mathcal{F}^1_\eta \times_{[P]} \mathcal{F}^2_\eta)^*(z_j) = \frac{a_j^{(2)}}{a_j^{(1)}} z_j^p,$$

and hence, with this identification, the point $(a_j^{(2)}/a_j^{(1)})_{j \in J_x}$ corresponds to $(\mathcal{F}^1_\eta \times_{[P]} \mathcal{F}^2_\eta)(\tilde{\Delta}(\mathfrak{x}_\eta))$.

Let N_j denote the residue map of (E,α) along m_j, i. e. $N_j = (\mathrm{id}_E \otimes m_j \frac{\partial}{\partial m_j}) \circ \nabla$ (§2.2.3). Then the above automorphism of E is given by (§2.4 (b)):

$$\prod_{j \in J_x} \exp(\log(a_j^{(2)}/a_j^{(1)})N_j).$$

2.4. Comparison of the de Rham and crystalline fundamental groups. Let $\overline{\mathfrak{X}}/W$ be a smooth, proper, and integral scheme, $\mathfrak{D} \subseteq \overline{\mathfrak{X}}$ a relative simple normal crossings divisor and $\mathfrak{X} := \overline{\mathfrak{X}} \setminus \mathfrak{D}$. Let $\hat{\overline{\mathfrak{X}}}$ be the completion of $\overline{\mathfrak{X}}$ along $\overline{\mathfrak{X}}_s$.

Case (a): $Y_{log} = \overline{X}_{log}$.

There is a natural equivalence of categories

$$\mathrm{Mic}_{uni}(\mathfrak{X}_\eta/K) \to \mathrm{Isoc}^c_{uni}(\overline{\mathfrak{X}}_{s,log}/W), \tag{2.4.1}$$

(Proposition 2.4.1, [4]) which is a composition of the following functors. The canonical extension functor in (2.2.4) gives the equivalence of categories

$$\mathrm{Mic}_{uni}(\mathfrak{X}_\eta/K) \to \mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{\eta,log}/K).$$

Choosing the lifting $(\hat{\overline{\mathfrak{X}}}, \hat{\mathfrak{D}})$ as a lifting of $(\overline{\mathfrak{X}}_s, \mathfrak{D}_s)$, and noting that the underlying rigid analytic space $\overline{\mathfrak{X}}_{\eta,an}$ of $\overline{\mathfrak{X}}_\eta$ is $\hat{\overline{\mathfrak{X}}}_\eta$, we obtain a natural equivalence of categories (§11, [7]):

$$\mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{\eta,log}/K) \to \mathrm{Isoc}^c_{uni}(\overline{\mathfrak{X}}_{s,log}/W),$$

whose inverse associates to a unipotent log convergent isocrystal on $\overline{\mathfrak{X}}_{s,log}$ its realization corresponding to the lifting $\mathfrak{X}_{log}/W$. The composition of these functors is (2.4.1).

If we use the equivalence (2.3.3), we get another equivalence of categories:

$$\gamma : \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/K) \to \mathrm{Isoc}^\dagger_{uni}(\mathfrak{X}_s/W). \tag{2.4.2}$$

A fiber functor ω from $\mathrm{Isoc}^\dagger_{uni}(\mathfrak{X}_s/W)$ to Vec_K, then induces an isomorphism

$$\pi_{1,crys}(\mathfrak{X}_s/W, \omega) \xrightarrow{\sim} \pi_{1,dR}(\mathfrak{X}_\eta/K, \omega \circ \gamma).$$

This implies that the frobenius map (2.3.7) on the crystalline fundamental group induces a corresponding map

$$F_* : \pi_{1,dR}(\mathfrak{X}_\eta/K, \omega\circ\gamma) \to \pi_{1,dR}(\mathfrak{X}_\eta^{(\sigma)}/K, \omega \circ F^* \circ \gamma) \tag{2.4.3}$$

on the de Rham fundamental group.

Case (b): $Y_{log} = x_{log}$.

Let $\mathfrak{x}_{log}/W$ be a lifting of x_{log} as above. Then we have a natural functor:

$$\mathrm{Isoc}^c_{uni}(\mathfrak{x}_{s,log}/W) \to \mathrm{Mic}_{uni}(\mathfrak{x}_{\eta,log}/K), \tag{2.4.4}$$

which sends an isocrystal on $\mathfrak{x}_{s,log}/W$ to its realization corresponding to the lifting $\mathfrak{x}_{log}/W$. Using the equivalence of categories (2.2.3), we also get a functor

$$\alpha_{\mathfrak{x}} : \mathrm{Isoc}^c_{uni}(\mathfrak{x}_{s,log}/W) \to \mathcal{T}_{\mathfrak{x}_\eta,uni}. \tag{2.4.5}$$

In order to prove that (2.4.4) and (2.4.5) are equivalences of categories, it suffices to prove that (2.4.5) is an equivalence of categories in the case when $\mathfrak{x}_{log}$ is the log scheme $\mathrm{Spec}\, W$ endowed with the log structure associated to the pre-log structure $\mathbb{N}^{\oplus r} \to W$, which sends all non-zero elements to 0.

Suppose that we are given a finite dimensional K-space and a collection $\{N_i\}_{1\leq i\leq r}$ of commuting nilpotent operators on V. We can define a log convergent isocrystal on $\mathfrak{x}_{s,log}$ whose realization associated to the lifting $\mathfrak{x}_{log}$ is given by the vector space V, together with the collection of automorphisms of V whose value at $(z_i)_{1\leq i\leq r} \in \prod_{1\leq i\leq r} D(1,1^-)$ (c.f. (2.3.5)) is given by

$$\prod_{1\leq i\leq r} \exp(\log(z_i)N_i). \tag{2.4.6}$$

This isocrystal has image $(V, \{N_i\}_{1\leq i\leq r})$ under $\alpha_{\mathfrak{x}}$.

Conversely, any unipotent log isocrystal on $\mathfrak{x}_{s,log}$ has to be of this form. This can be seen as follows. Suppose that we are given an isocrystal (E, ∇) with image $(E, \{N_i\}_{1\leq i\leq r})$ under $\alpha_{\mathfrak{x}}$. Then the connection defines an isomorphism from $\overline{p}_2^* E|_{\tilde{\Delta}^{(1)}}$ to $\overline{p}_1^* E|_{\tilde{\Delta}^{(1)}}$, where $\tilde{\Delta}$ denotes $\tilde{\Delta}(\mathfrak{x}_\eta)$ and the superscript (1) denotes the first infinitesimal neighborhood. The integrability of the connection implies that this isomorphism canonically extends to the formal completion $\tilde{\Delta}^{\hat{}}$ of $\tilde{\Delta}$ in $\mathfrak{x}_{\eta,log} \times_{[P]} \mathfrak{x}_{\eta,log}$. Using the identification (2.3.5), this isomorphism is given by (2.4.6). Since the natural map from analytic functions on $\prod_{1\leq i\leq r} D(1,1^-)$ to formal power series around $(1,\cdots,1) \in \prod_{1\leq i\leq r} D(1,1^-)$ is injective, it follows that the above isomorphism is given by the formula (2.4.6) over all of $\prod_{1\leq i\leq r} D(1,1^-)$.

3. Tangential basepoints

Let $\overline{X}/k$ be as above, where k is of characteristic 0 (resp. p). If $x \in X(k)$, then we have a natural fiber functor on $\mathrm{Mic}_{uni}(X/k)$ (resp. $\mathrm{Isoc}^{\dagger}_{uni}(X/W)$). In this section, we will define a similar fiber functor for $v \in N^{\times}_{D_x/\overline{X}}(x)$ (c. f. §2.1.3), where $x \in (\overline{X} \setminus X)(k)$.

3.1. de Rham case. Let k be a field of characteristic zero.

Then for $x \in X(k)$, the map

$$\omega(x) : \mathrm{Mic}_{uni}(X/k) \to \mathrm{Vec}_k,$$

which sends (E, ∇) to $E(x)$ is a fiber functor.

Now assume that $x \in (\overline{X} \setminus X)(k)$. Using the equivalence of categories (2.2.4)

$$can : \mathrm{Mic}_{uni}(X/k) \xrightarrow{\sim} \mathrm{Mic}_{uni}(\overline{X}_{log}/k),$$

the pull-back $\mathrm{Mic}_{uni}(\overline{X}_{log}/k) \to \mathrm{Mic}_{uni}(x_{log}/k)$ via the inclusion $x_{log} \to \overline{X}_{log}$, and $\mathrm{Mic}_{uni}(x_{log}/k) \xrightarrow{\sim} \mathcal{T}_{x,uni}$ (2.2.3), we get a functor

$$\mathrm{Mic}_{uni}(X/k) \to \mathcal{T}_{x,uni}.$$

Composing this with the functor $\mathcal{T}_{x,uni} \to \mathrm{Vec}_k$ which forgets the operators, we get the fiber functor

$$\omega(x) : \mathrm{Mic}_{uni}(X/k) \to \mathrm{Vec}_k, \tag{3.1.1}$$

which sends (E, ∇) to $E_{can}(x)$.

We would like to emphasize that, in this de Rham case, the construction of the fiber functor does not, in fact, depend on the choice of a set of tangent vectors at x. Namely, given $v \in N^{\times}_{D_x/\overline{X}}(x)$, $\omega(x)$, which only depends on x, could also be defined as composing the map $\mathrm{Mic}_{uni}(X/k) \to \mathrm{Mic}_{uni}(x_{log}/k)$ above with the equivalence

$\mathrm{Mic}_{uni}(x_{log}/k) \xrightarrow{\sim} \mathrm{Mic}_{uni}(k_{x,log}/k)$ induced by v (Lemma 2.1.2) and the natural forgetful fiber functor on $\mathrm{Mic}_{uni}(k_{x,log}/k)$ (c. f. §3.2).

3.2. Crystalline case. Let k be a perfect field of characteristic $p > 0$.

Assume first that $x \in X(k)$. Then we have the pull-back morphism

$$\mathrm{Isoc}^{\dagger}_{uni}(X/W) \to \mathrm{Isoc}^{\dagger}_{uni}(x/W).$$

A lifting $\mathfrak{x}$ of x induces a map $\mathrm{Isoc}^{\dagger}_{uni}(x/W) \to \mathrm{Mic}_{uni}(\mathfrak{x}_K/K) = \mathrm{Vec}_K$ as in (2.4.4). This fiber functor

$$\omega(x) : \mathrm{Isoc}^{\dagger}_{uni}(X/W) \to \mathrm{Vec}_K$$

is defined up to canonical isomorphism.

Now assume that $x \in (\overline{X} \setminus X)(k)$ and let $v \in N^{\times}_{D_x/\overline{X}}(x)$. The closed immersion $x_{\log} \to \overline{X}_{\log}$ induces a pullback functor

$$\mathrm{Isoc}^{c}_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}^{c}_{uni}(x_{log}/W),$$

and v induces the equivalence (2.1.2):

$$\mathrm{Isoc}^{c}_{uni}(x_{log}/W) \xrightarrow{\sim} \mathrm{Isoc}^{c}_{uni}(k_{x,log}/W).$$

Applying (2.4.5) to $k_{x,log}$ and $W_{x,log}$ we get a fiber functor on $\mathrm{Isoc}^{c}_{uni}(\overline{X}_{log}/W)$. The equivalence of categories in (2.3.3) then provides the fiber functor:

(3.2.1) $$\omega(v) : \mathrm{Isoc}^{\dagger}_{uni}(X/W) \to \mathrm{Vec}_K$$

we were looking for.

3.3. Comparison of the de Rham and crystalline basepoints. Let k be a perfect field of characteristic p, and $\overline{\mathfrak{X}}, \mathfrak{D}$, etc. as in §2.4. Let $\mathfrak{x} \in \mathfrak{D}$ and $\mathfrak{v} \in N^{\times}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}(\mathfrak{x})$ with reduction $\mathfrak{v}_s$ in $N^{\times}_{\mathfrak{D}_{s,\mathfrak{x}}/\overline{\mathfrak{X}}_s}(\mathfrak{x}_s)$. This data will give us a comparison between the fiber functors $\omega(\mathfrak{v}_s)$ on $\mathrm{Isoc}^{\dagger}_{uni}(\mathfrak{X}_s/W)$ and $\omega(\mathfrak{x}_\eta)$ on $\mathrm{Mic}_{uni}(\mathfrak{X}_\eta/K)$.

If id_v and $id_{\mathfrak{v}}$ denote the trivializations as in Lemma 2.1.2, we have a commutative diagram

$$\begin{array}{ccc} k_{\mathfrak{x}_s,log} & \longrightarrow & W_{\mathfrak{x},log} \\ \downarrow{\scriptstyle id_v} & & \downarrow{\scriptstyle id_{\mathfrak{v}}} \\ \mathfrak{x}_{s,log} & \longrightarrow & \mathfrak{x}_{log} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{X}}_{s,log} & \longrightarrow & \overline{\mathfrak{X}}_{log}. \end{array}$$

For $(E, \nabla) \in \mathrm{Isoc}^{c}_{uni}(\overline{\mathfrak{X}}_{s,log}/W)$, let $(E_{\overline{\mathfrak{X}}}, \nabla) \in \mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{\eta,log}/K)$ denote its realization on the log rigid analytic space $\overline{\mathfrak{X}}_{\eta,log}$. Then, associated to the data given above, we have

$$\omega(\mathfrak{v}_s)(E, \nabla) \xrightarrow{\sim} E_{\overline{\mathfrak{X}}}(\mathfrak{x}_\eta) = \omega(\mathfrak{x}_\eta)(E_{\overline{\mathfrak{X}}}, \nabla).$$

This gives an identification of the functors $\omega(\mathfrak{v}_s)$ and $\omega(\mathfrak{x}_\eta)$, if we take in to account the equivalence of categories (2.4.2). We would like to emphasize that this identification *depends* on the model, i. e. on $\overline{\mathfrak{X}}, \mathfrak{x}$, and $\mathfrak{v}$.

Let $(\overline{\mathfrak{Y}}, \mathfrak{E}, \mathfrak{y}, \mathfrak{u})$ be another data of a lifting. Associated to this lifting we have an isomorphism

$$\omega(\mathfrak{v}_s)(E, \nabla) \xrightarrow{\sim} E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_\eta) = \omega(\mathfrak{y}_\eta)(E_{\overline{\mathfrak{Y}}}, \nabla).$$

Corresponding to these two liftings the isomorphism from $E_{\overline{\mathfrak{X}}}(\mathfrak{x}_\eta)$ to $E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_\eta)$ is described as follows.

Let $(\overline{\mathfrak{X}}\times\overline{\mathfrak{Y}})^{\sim}_{log}$ denote the log scheme obtained by blowing up $\overline{\mathfrak{X}}\times\overline{\mathfrak{Y}}$ as in §2.3.2. The fiber of the blow-up over $(\mathfrak{x},\mathfrak{y})$ is isomorphic to

$$\prod_i \mathbb{P}(N_{\mathfrak{F}_i/\overline{\mathfrak{X}}\times\overline{\mathfrak{Y}}}),$$

with $\mathfrak{F}_i = \mathfrak{D}_i \times \mathfrak{E}_i$. In particular, if $\mathfrak{v} := (\cdots, \mathfrak{v}_i, \cdots)$, $\mathfrak{u} := (\cdots, \mathfrak{u}_i, \cdots)$ and

$$[\mathfrak{v},\mathfrak{u}] := (\cdots, [\mathfrak{v}_i,\mathfrak{u}_i], \cdots),$$

then $[\mathfrak{v},\mathfrak{u}]$ defines a point in the fiber of $(\mathfrak{x},\mathfrak{y})$.

If $(E,\nabla) \in \mathrm{Isoc}^c_{uni}(\overline{\mathfrak{X}}_{s,log}/W)$ then we have a canonical isomorphism between the pull-backs of $(E_{\overline{\mathfrak{X}}},\nabla)$ and $(E_{\overline{\mathfrak{Y}}},\nabla)$ to the tube of $\overline{\mathfrak{X}}_{s,log}$ in $(\overline{\mathfrak{X}}\times\overline{\mathfrak{Y}})^{\sim}_{\eta}$, and hence evaluating this isomorphism at $[\mathfrak{v},\mathfrak{u}]_\eta$ gives the isomorphism

$$E_{\overline{\mathfrak{X}}}(\mathfrak{x}_\eta) \xrightarrow{\sim} E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_\eta)$$

we were looking for.

4. Comparison of tangential basepoints and ordinary basepoints

In this section, we describe the relation between the tangential and ordinary basepoints. We will see that taking the fiber functor at a tangential basepoint v at x, is equivalent to pulling the object back from X to $\mathbb{V}^\times(N_{D_x/\overline{X}}(x))$ and then applying the fiber functor at v, viewed as a point in the variety $\mathbb{V}^\times(N_{D_x/\overline{X}}(x))$. Since there is, in general, no algebraic map from $\mathbb{V}^\times(N_{D_x/\overline{X}}(x))$ to X, the pull-back functor is defined by passing to the log point in both spaces and using the identification in §2.1.5. In order to achieve this, we will show that pulling back to the log point in $\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}$ gives an equivalence of categories.

4.1. de Rham case. In this subsection, we continue the standard notation with k a field of characteristic 0. We let

$$\mathbb{V}^\times(N_{D_x/\overline{X}}(x)) := \prod_{j\in J_x} \mathbb{V}(N_{D_j/\overline{X}}(x)) \setminus \{0\},$$

and

$$\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x)) := \prod_{j\in J_x} \overline{\mathbb{V}(N_{D_j/\overline{X}}(x))},$$

where the bar on the factors in the last expression denotes the projective completion (of curves). Clearly these varieties are isomorphic to $\mathbb{G}_m^{\times r}$ and $(\mathbb{P}^1)^{\times r}$, if $|J_x| = r$. Let

$$\pi_j : \overline{\mathbb{V}}(N_{D_x/\overline{X}}(x)) \to \overline{\mathbb{V}(N_{D_j/\overline{X}}(x))}$$

denote the j-th projection, for $z \in \{0,\infty\}$, $L_z := \cup_{j\in J_x}\pi_j^{-1}((z))$, and $D_{\overline{\mathbb{V}}} := L_0 \cup L_\infty$. Note that by the notation in §2.1.5, $L_0 = D_{(x)}$. Denote by $\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}$, the log scheme which is $\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))$ endowed with the log structure associated to $D_{\overline{\mathbb{V}}}$.

Lemma 4.1.1. *The natural pull-back functor*

$$\mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/k) \to \mathrm{Mic}_{uni}(0_{log}/k)$$

is an equivalence of categories.

Proof. Let $\mathcal{T}_{r,uni}$ denote the tannakian category over k, which consist of pairs of a vector space and r commuting nilpotent operators on it, as in Definition 2.2.2. Then by §2.2.3, the natural functor from $\mathrm{Mic}_{uni}(0_{log}/k)$ to $\mathcal{T}_{r,uni}$ is an equivalence of categories.

To prove the statement, after choosing coordinates, it suffices to prove it for $\mathcal{P}_{r,log} := (\mathbb{P}_1)^{\times r}_{log}$ with the log structure associated to the union of the coordinate axes passing through 0 and those passing through ∞.

Using the above equivalence, it suffices to prove that the natural functor

$$\mathrm{Mic}_{uni}(\mathcal{P}_{r,log}/k) \to \mathcal{T}_{r,uni}$$

is an equivalence.

This functor associates $(E(0), \{\nabla_j\}_{1\leq j\leq r})$ to (E, ∇), where ∇_j is the residue of ∇ along $z_j = 0$ at the point 0. If $(V, \{N_j\}_{1\leq j\leq r})$ is an object of $\mathcal{T}_{r,uni}$, then $(V \otimes_k \mathcal{O}_{\mathcal{P}_r}, d - \sum_j N_j d\log z_j)$ is an object of $\mathrm{Mic}_{uni}(\mathcal{P}_{r,log}/k)$, which has image $(V, \{N_j\})$ under this functor. This proves essential surjectivity.

Since the functor above is a tensor functor, in order to prove that this functor is fully-faithful, it suffices to show that

$$\mathrm{Hom}_{\mathrm{Mic}_{uni}(\mathcal{P}_{r,log}/k)}((\mathcal{O}_{\mathcal{P}_r}, d), (E, \nabla)) \to \mathrm{Hom}_{\mathcal{T}_{r,uni}}((k, \{0\}_j), (E(0), \{\nabla_j\}_j))$$

is an isomorphism or equivalently that

$$\mathrm{H}^0_{dR}(\mathcal{P}_{r,log}, (E, \nabla)) \to \cap_{1\leq j\leq r} \ker_{E(0)}(\nabla_j)$$

is an isomorphism.

First note that the underlying bundle E of

$$(E, \nabla) \in \mathrm{Mic}_{uni}(\mathcal{P}_{r,log}/k)$$

is trivial. This follows from the fact that $\mathrm{Ext}^1_{\mathcal{P}_r}(\mathcal{O}_{\mathcal{P}_r}, \mathcal{O}_{\mathcal{P}_r}) = \mathrm{H}^1(\mathcal{P}_r, \mathcal{O}_{\mathcal{P}_r}) = 0$ by induction on the nilpotence level (Proposition 12.3, [7]). Therefore, without loss of generality, we will assume that $(E, \nabla) = (\mathcal{O}^{\oplus n}_{\mathcal{P}_r}, d - \sum_{1\leq j\leq r} N_j \, d\log z_j)$, for some nilpotent matrices $N_j \in M_{n\times n}(k)$. If α is a global (horizontal) section of (E, ∇) then it is a constant section of $\mathcal{O}^{\oplus n}_{\mathcal{P}_r}$. This immediately implies the injectivity of the above map. In order to see that it is surjective, we note that for any $\alpha \in \cap_{1\leq j\leq r} \ker_{E(0)}(\nabla_j)$, the constant section of $\mathcal{O}^{\oplus n}_{\mathcal{P}_r}$ with fiber α at 0 is a horizontal section with respect to the connection $d - \sum_{1\leq j\leq r} N_j d\log z_j$. □

Then, the fiber functor $\omega(x) : \mathrm{Mic}_{uni}(X/k) \to \mathrm{Vec}_k$ is the composition of

$$can : \mathrm{Mic}_{uni}(X/k) \xrightarrow{\sim} \mathrm{Mic}_{uni}(\overline{X}_{log}/k),$$

$$\mathrm{Mic}_{uni}(\overline{X}_{log}/k) \to \mathrm{Mic}_{uni}(x_{log}/k) \xleftarrow{\sim} \mathrm{Mic}_{uni}(0_{log}/k) \xleftarrow{\sim} \mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/k)$$

and

$$\omega(dR) : \mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/k) \to \mathrm{Vec}_k.$$

Here the second map in the middle diagram is induced by the canonical identification in §2.1.5.

4.1.1. *Functoriality.* Let $(\overline{X}, D)$ and $(\overline{Y}, E)$ be pairs of smooth schemes over k and simple normal crosssings divisors. Assume that $x \in D$, $y \in E$ and $f : \overline{X}_{log} \to \overline{Y}_{log}$ a morphism of log schemes with $f(x) = y$. Let

$$\mathcal{P}(f) : \mathbb{V}(N_{D_x/\overline{X}}(x)) \to \mathbb{V}(N_{E_y/\overline{Y}}(y))$$

denote the unique homogeneous map that induces $f_x : x_{log} \to y_{log}$ under restriction to $0_{x,log}$ and the canonical isomorphisms $x_{log} \simeq 0_{x,log}$ and $y_{log} \simeq 0_{y,log}$ in Lemma 2.1.3. $\mathcal{P}(f)$ is called the *principal part* of f at x relative to the given divisors. $\mathcal{P}(f)$ naturally defines a map

$$\mathcal{P}(f) : \overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log} \to \overline{\mathbb{V}}(N_{E_y/\overline{Y}}(y))_{log}.$$

We, then, have a commutative diagram

$$\begin{array}{ccc} \mathrm{Mic}_{uni}(Y/k) & \xrightarrow{f^*} & \mathrm{Mic}_{uni}(X/k) \\ \downarrow & & \downarrow \\ \mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{E_y/\overline{Y}}(y))_{log}/k) & \xrightarrow{\mathcal{P}(f)^*} & \mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/k). \end{array}$$

4.2. Crystalline case. In this section, we give a similar description of the tangential basepoint functor in the crystalline case. We assume that k is perfect of characteristic p.

4.2.1. *From log point to the tangent space.* In order to do this, we need the crystalline analog of Lemma 4.1.1.

Lemma 4.2.1. *The natural pull-back functor*

$$\mathrm{Isoc}^c_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/W) \to \mathrm{Isoc}^c_{uni}(0_{log}/W)$$

is an equivalence of categories.

Proof. First, as in the proof of Lemma 4.1.1, we can reduce to the case of $\mathcal{P}_{r,log}$. In this case, the isomorphism (2.4.5) gives an equivalence

$$\mathrm{Isoc}^c_{uni}(0_{log}/W) \to \mathcal{T}_{r,uni},$$

where $\mathcal{T}_{r,uni}$ is the category defined in the proof of Lemma 4.1.1, but with k replaced with K. We would like to emphasize that there is *no such canonical isomorphism* for an arbitrary $N_{D_x/\overline{X}}(x)$, the reason being that $N_{D_x/\overline{X}}(x)$ does not have a canonical lifting to W, but such a lifting exists for $k^{\oplus r}$.

Next, applying (2.4.1) to $\mathcal{P}_{r,log}/W$, we see that the functor

$$\mathrm{Mic}_{uni}(\mathcal{P}_{r,log}/K) \to \mathrm{Isoc}^c_{uni}(\mathcal{P}_{r,log}/W)$$

is an equivalence. Then the statement we are trying to prove is equivalent to that of Lemma 4.1.1. □

Proposition 4.2.2. *The fiber functor* $\omega(v) : \mathrm{Isoc}^\dagger_{uni}(X/W) \to \mathrm{Vec}_K$ *is the composition of the following functors:*

$$\mathrm{Isoc}^\dagger_{uni}(X/W) \xleftarrow{\sim} \mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}^c_{uni}(x_{log}/W) \xleftarrow{\sim} \mathrm{Isoc}^c_{uni}(0_{log}/W),$$

$$\mathrm{Isoc}^c_{uni}(0_{log}/W) \xleftarrow{\sim} \mathrm{Isoc}^c_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/W)$$

and

$$\omega_{\mathbb{V}}(v) : \mathrm{Isoc}^c_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/W) \to \mathrm{Vec}_K,$$

where $\omega_{\mathbb{V}}(v)$ denotes the fiber functor at the basepoint $v \in \mathbb{V}(N_{D_x/\overline{X}}(x))$, and the last equivalence in the first diagram above is induced by the isomorphism in §2.1.5.

Let $\varphi^* : \mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}^c_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/W)$ denote the composition of the obvious functors above. In order to prove the proposition we first need to give an alternative description of φ^*.

4.2.2. *Description of φ^*.* First we give another description of the functor

$$\mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W) \to \mathrm{Isoc}^c_{uni}(0_{log}/W). \tag{4.2.1}$$

We use the notation of §3.3 with $\overline{\mathfrak{X}}_s = \overline{X}$, $\mathfrak{D}_s = D$, $\mathfrak{v}_s = v$ etc. Let $\mathcal{U} \subseteq \mathbb{V}(N_{\mathfrak{D}_\mathfrak{x}/\overline{\mathfrak{X}}}(\mathfrak{x}))_\eta$ be a rigid analytic polydisc around zero. Endow $\mathcal{U}$ with the associated log structure induced from that on $\overline{\mathbb{V}}(N_{\mathfrak{D}_\mathfrak{x}/\overline{\mathfrak{X}}})_\eta$.

Let $\psi : \mathcal{U}_{log} \to \overline{\mathfrak{X}}_{\eta,log}$, be a map such that:

(4.2.2) $\psi(0) = x$;

(4.2.3) ψ is a closed immersion;

(4.2.4) $d\psi_0 : (N_{\mathfrak{D}_\mathfrak{x}/\overline{\mathfrak{X}}})_\eta = T_0(\mathcal{U}) \to (T_\mathfrak{x}\overline{\mathfrak{X}})_\eta$ is the canonical inclusion;

these imply that

$$\psi_0^* := \mathrm{Cart}^-(\overline{\mathfrak{X}}_\eta, (\mathfrak{D}_\eta)_{\mathfrak{x}_\eta}) = \overline{M}_{\overline{\mathfrak{X}}_\eta,\mathfrak{x}_\eta} \to \overline{M}_{\mathcal{U},0} = \mathrm{Cart}^-(\mathcal{U}, (\mathfrak{D}_\eta)_{(\mathfrak{x}_\eta)}) \tag{4.2.5}$$

is the canonical identification.

We have a map

$$\psi^* : \mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{\eta,log}/K) \to \mathrm{Mic}_{uni}(\mathcal{U}_{log}/K).$$

Combining this with the restriction map, we obtain

$$\psi_0^* : \mathrm{Mic}_{uni}(\overline{\mathfrak{X}}_{\eta,log}/K) \to \mathrm{Mic}_{uni}(0_{\mathfrak{x}_\eta,log}/K).$$

Note that the category $\mathrm{Mic}_{uni}(0_{\mathfrak{x},log}/K) \xrightarrow{\sim} \mathcal{T}_{r,uni}/K$ does not depend, up to canonical isomorphism, on the choices of the models.

Consider another choice of a model, $\overline{\mathfrak{Y}}, \mathfrak{E}$, and $\mathfrak{y}$ and let $\mathcal{V} \subseteq \mathbb{V}(N_{\mathfrak{E}_\mathfrak{y}/\overline{\mathfrak{Y}}})_\eta$, and $\tilde{\psi} : \mathcal{V} \to \overline{\mathfrak{Y}}_\eta$ be another choice as above, with $\tilde{\psi}(0) = \mathfrak{y}_\eta$. Then we have a map

$$(\psi \times \tilde{\psi})^\sim : (\mathcal{U} \times \mathcal{V})^\sim_{log} \to (\overline{\mathfrak{X}}_\eta \times \overline{\mathfrak{Y}}_\eta)^\sim_{log}$$

induced by $\psi \times \tilde{\psi}$. The underlying map of schemes is the identity map on the exceptional divisors. Note that the exceptional divisors are respectively the products of the normal bundles of $\psi^*(\mathfrak{D}_{\eta,i}) \times \tilde{\psi}^*(\mathfrak{D}_{\eta,i})$ and $\mathfrak{D}_{\eta,i} \times \mathfrak{D}_{\eta,i}$ at $(0,0)$ and $(\mathfrak{x}_\eta, \mathfrak{y}_\eta)$. If $(E, \nabla) \in \mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W)$ then by pulling back with $(\psi \times \tilde{\psi})^\sim$ we have a canonical isomorphism between the pullbacks of $\psi_0^*(E_{\overline{\mathfrak{X}}}, \nabla)$ and $\tilde{\psi}_0^*(E_{\overline{\mathfrak{Y}}}, \nabla)$ to the tube of the diagonal in $0_{\mathfrak{x},log} \times_{[P]} 0_{\mathfrak{y},log}$. Here note that even though the diagonal is not defined, the tube of the diagonal is well-defined as the tube of the diagonal after $0_{\mathfrak{x},log}/W$ and $0_{\mathfrak{y},log}/W$ are identified by an isomorphism that induces the identity map on the special fibers. These isomorphisms on the tubes satisfy the cocycle condition.

Therefore, given $(E, \nabla) \in \mathrm{Isoc}^c_{uni}(\overline{X}_{log}/W)$ and a model as above $\psi_0^*(E_{\overline{\mathfrak{X}}}, \nabla)$ is a realization of the pull-back of (E, ∇) to 0_{log}, corresponding to the given model. This gives an explicit description of the functor in (4.2.1).

To give a description of φ^*, we need to describe

$$\mathrm{Isoc}^c_{uni}(0_{log}/W) \to \mathrm{Isoc}^c_{uni}(\overline{\mathbb{V}}(N_{D_x/\overline{X}}(x))_{log}/W).$$

Let $\mathfrak{N}/W$ be any formal vector bundle lifting $N_{D_x/\overline{X}}$. Then $\mathbb{V}(\mathfrak{N})$ is naturally endowed with a log structure and the equivalence

$$\text{(4.2.6)} \qquad \mathrm{Mic}_{uni}(\overline{\mathbb{V}}(\mathfrak{N})_{\eta,log}/K) \to \mathrm{Mic}_{uni}(0_{log}/K) \xrightarrow{\sim} \mathcal{T}_{r,uni}/K$$

realizes the equivalence in Lemma 4.2.1.

Therefore we only need to find a tensor functor which gives an inverse (up to isomorphism) of that in (4.2.6). The functor that sends $(V, \{N_j\}_{1\leq j\leq r})$ to

$$(V \otimes_K \mathcal{O}_{\overline{\mathbb{V}}(\mathfrak{N})_\eta}, d - \sum_{1\leq j\leq r} N_j d\log z_j) \in \mathrm{Mic}_{uni}(\overline{\mathbb{V}}(\mathfrak{N})_{\eta,log}/K)$$

does just that.

4.3. Proof of Proposition 4.2.2. We need to find a natural isomorphism between $\omega_{\mathbb{V}}(v) \circ \varphi^*$ and $\omega(v)$.

Choose a model $\overline{\mathfrak{X}}, \mathfrak{D}, \mathfrak{x}$ for $(\overline{X}, D, x)$ as above and let $\mathfrak{v} \in N^{\times}_{\mathfrak{D}_\mathfrak{x}/\overline{\mathfrak{X}}}(\mathfrak{x})$ be a lifting of v. Finally, let ψ be as in §4.2.2. Then corresponding to the data $(\overline{\mathfrak{X}}, \mathfrak{D}, \mathfrak{x}, \mathfrak{v})$ *and* ψ the realization of $(\omega_{\mathbb{V}}(v) \circ \varphi^*)(E, \nabla)$ is $E_{\overline{\mathfrak{X}}}(\mathfrak{x}_\eta)$.

Similarly, corresponding to the data $(\overline{\mathfrak{X}}, \mathfrak{D}, \mathfrak{x}, \mathfrak{v})$ the realization of $\omega(v)(E, \nabla)$ is also $E_{\overline{\mathfrak{X}}}(\mathfrak{x}_\eta)$.

This gives an obvious identification of the two fiber functors corresponding to each data. We need to check that these identifications are compatible with the isomorphisms on the isocrystal when we change the data.

Namely, let $(\overline{\mathfrak{Y}}, \mathfrak{E}, \mathfrak{y}, \mathfrak{u})$ be a similar lifting of $(\overline{X}, D, x, v)$ and $\tilde{\psi} : \mathcal{V} \to \overline{\mathfrak{Y}}_\eta$ be as in §4.2.2.

Then the isomorphism between $\omega(v)(E_{\overline{\mathfrak{X}}}, \nabla) = E_{\overline{\mathfrak{X}}}(\mathfrak{x}_\eta)$ and $\omega(v)(E_{\overline{\mathfrak{Y}}}, \nabla) = E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_\eta)$ is the one obtained by evaluating at $[\mathfrak{v}_\eta, \mathfrak{u}_\eta]$ the isomorphism between the pull-backs of $E_{\overline{\mathfrak{X}}}$ and $E_{\overline{\mathfrak{Y}}}$ to the tube in $(\overline{\mathfrak{X}}_\eta \times \overline{\mathfrak{Y}}_\eta)^{\sim}$.

Similarly, the isomorphism between

$$(\omega_{\mathbb{V}}(v) \circ \varphi^*)(E_{\overline{\mathfrak{X}}}, \nabla) = E_{\overline{\mathfrak{X}}}(\mathfrak{x}_\eta)$$

and

$$(\omega_{\mathbb{V}}(v) \circ \varphi^*)(E_{\overline{\mathfrak{Y}}}, \nabla) = E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_\eta)$$

is the one obtained by evaluating at $[\mathfrak{v}_\eta, \mathfrak{u}_\eta]$ the isomorphism between the pull-backs of $E_{\overline{\mathfrak{X}}}$ and $E_{\overline{\mathfrak{Y}}}$ on the tube in $(\mathbb{V}(N_{\mathfrak{D}_\mathfrak{x}/\overline{\mathfrak{X}}}) \times \mathbb{V}(N_{\mathfrak{E}_\mathfrak{y}/\overline{\mathfrak{Y}}}))^{\sim}_{\eta,log}$. Note that the isomorphism between the pull-backs to the tube in $(\mathbb{V}(N_{\mathfrak{D}_\mathfrak{x}/\overline{\mathfrak{X}}}) \times \mathbb{V}(N_{\mathfrak{E}_\mathfrak{y}/\overline{\mathfrak{Y}}}))^{\sim}_{\eta,log}$ is obtained by pulling back via $(\psi \times \tilde{\psi})^{\sim}$ the isomorphism on the tube in $(\overline{\mathfrak{X}} \times \overline{\mathfrak{Y}})^{\sim}_{\eta,log}$. However, because of condition (4.2.4) the map induced by $(\psi \times \tilde{\psi})^{\sim}$ on the exceptional divisor is the identity map.

This implies that the isomorphisms between $E_{\overline{\mathfrak{X}}}(\mathfrak{x}_\eta)$ and $E_{\overline{\mathfrak{Y}}}(\mathfrak{y}_\eta)$ agree in both cases. □

5. de Rham fundamental group of $M_{0,5}$

In this section, we will construct an exact sequence that describes the de Rham fundamental group of $M_{0,5}$. In order to do this we will use the corresponding exact sequence in the Betti case and the comparison theorem between the Betti and the de Rham fundamental groups.

5.1. de Rham basepoint of $M_{0,5}$. We have seen in §2.2.5 that there is a de Rham fiber functor $\omega(dR)$ if X/K has a smooth compactification $\overline{X}/K$ with the property that $H^1(\overline{X},\mathcal{O})=0$.

Let

$$M_{0,n}/\mathbb{Q} := \{(x_0,\cdots,x_{n-1})\in(\mathbb{P}^1)^n_{\mathbb{Q}}|x_i\neq x_j,\ \text{ for }\ 0\le i<j\le n-1\}/PGL_2,$$

where PGL_2 acts diagonally by linear fractional transformations and let

$$\overline{M}_{0,5} := ((\mathbb{P}^1)^5\setminus L)/PGL_2,$$

where $L := \{(x_0,\cdots,x_4)\in(\mathbb{P}^1)^5|\,\exists\ 0\le i<j<k\le 4 \text{ with } x_i=x_j=x_k\}$. We have $M_{0,5}\subseteq\overline{M}_{0,5}$ as an open subvariety with the complement a simple normal crossings divisor. Let $D_{ij}\subseteq\overline{M}_{0,5}$ denote the divisor defined by $x_i-x_j=0$. Let $\overline{M}_{0,5,log}$ denote $\overline{M}_{0,5}$ endowed with the log structure associated to the normal crossings divisor $\cup_{0\le i<j\le 4}D_{ij}=\overline{M}_{0,5}\setminus M_{0,5}$.

Since $\overline{M}_{0,5}$ is the blow-up of $(\mathbb{P}^1)^2$ at three points, $H^1_B((\overline{M}_{0,5})_{\mathbb{C}},\mathbb{Z})=0$. Then Grothendieck's comparison theorem gives

$$H^1_{dR}((\overline{M}_{0,5})_{\mathbb{C}},(\mathcal{O},d))\simeq H^1_B((\overline{M}_{0,5})_{\mathbb{C}},\mathbb{C})=0.$$

By the Hodge decomposition, we have $H^1((\overline{M}_{0,5})_{\mathbb{C}},\mathcal{O})=0$, and since

$$H^1(\overline{M}_{0,5},\mathcal{O})\otimes_{\mathbb{Q}}\mathbb{C}=H^1((\overline{M}_{0,5})_{\mathbb{C}},\mathcal{O})$$

we have $H^1(\overline{M}_{0,5},\mathcal{O})=0$.

Therefore there is a de Rham fiber functor

$$\omega(dR):\mathrm{Mic}_{uni}(M_{0,5}/\mathbb{Q})\to\mathrm{Vec}_{\mathbb{Q}}$$

5.2. Malcev completion and the de Rham-Betti comparison theorem. We will review the construction of Malcev completion of groups as it relates to the comparison theorem between the de Rham and Betti fundamental groups. General references for this section are §9 in [7]; Appendix A in [26]; §2.5 in [20]; and Appendice A in [9].

Let G be a (discrete) group. Denote by $\{G_i\}_{i\ge1}$ the lower central series of G, i.e. $G_1:=G$ and $G_{i+1}:=[G,G_i]$, the subgroup generated by $[g,h]$ with $g\in G$ and $h\in G_i$. Let $G^{[N]}$ denote the nilpotent uniquely divisible envelope of $G^{(N)}:=G/G_{N+1}$ (Corollary 3.8, p. 278, [26]). There is a canonical map $j:G^{(N)}\to G^{[N]}$ which is universal for maps of $G^{(N)}$ into nilpotent uniquely divisible groups. Moreover the map j is characterized by the properties that $G^{[N]}$ is nilpotent and uniquely divisible, $ker(j)$ is the torsion subgroup of $G^{(N)}$ and for every $g\in G^{[N]}$, there exists an $n\neq0$ such that $g^n\in im(j)$ (see *loc. cit.*). $G^{[N]}$ is the *Malcev completion* of $G^{(N)}$.

For any group H and a field of characteristic zero K let $K[H]$ denote the group algebra over K, considered with its standard augmented Hopf algebra structure, i.e. with the co-multiplication $\Delta:K[H]\to K[H]\otimes K[H]$ given by $\Delta(h)=h\otimes h$. Let $\mathcal{J}$ be its augmentation ideal. Let $\hat{K}[H]$ be the $\mathcal{J}-$adic completion of $K[H]$. Let $\mathcal{G}(\hat{K}[H]):=\{x\in 1+\hat{\mathcal{J}}|\Delta(x)=x\hat{\otimes}x\}$ the set of *group-like* elements in $\hat{K}[H]$, and $\mathcal{P}(\hat{K}[H]):=\{x\in\hat{K}[G]|\Delta(x)=1\hat{\otimes}x+x\hat{\otimes}1\}$, the set of *primitive* elements in $\hat{K}[H]$. The set of group-like elements form a group under multiplication and the set of primitive elements form a Lie algebra under commutator as bracket. Moreover the logarithm map $\log:\mathcal{G}(\hat{K}[H])\to\mathcal{P}(\hat{K}[H])$ induces a bijection ([20], Proposition 2.5.1). This is, of course, a bijection of pointed sets only.

If H is a uniquely divisible nilpotent group then the natural map $H \to \mathcal{G}(\hat{\mathbb{Q}}[H])$ is an isomorphism (Corollary 3.7, Appendix A, [26]). Therefore we obtain a bijection $H \to \mathcal{P}(\hat{\mathbb{Q}}[H])$ of H with a Lie algebra over $\mathbb{Q}$. In this case, abusing the notation, we denote $\mathcal{P}(\hat{\mathbb{Q}}[H])$ by $\mathrm{Lie}(H)$.

If $X/\mathbb{C}$ is a smooth, quasi-projective complex algebraic variety then this construction applied to $\pi_1(X^{an}, *)$, the topological fundamental group, gives finite dimensional nilpotent Lie algebras $\{\mathrm{Lie}(\pi_1(X^{an}, *)^{[N]})\}_{N\geq 1}$ (§9.8, [7]). Since the functor Lie induces an equivalence of categories between unipotent algebraic groups over $\mathbb{Q}$ and finite dimensional nilpotent Lie algebras over $\mathbb{Q}$ (§9.1, [7]) we obtain unipotent algebraic groups $\{\underline{\pi}_1(X^{an}, *)^{[N]}\}_{N\geq 1}$ with Lie algebras $\{\mathrm{Lie}(\pi_1(X^{an}, *)^{[N]})\}_{N\geq 1}$. These algebraic groups satisfy $\pi_1(X^{an}, *)^{[N]} = \underline{\pi}_1(X^{an}, *)^{[N]}(\mathbb{Q})$, for $N \geq 1$.

Proposition 5.2.1. *Let $X/\mathbb{C}$ be a smooth, quasi-projective variety. If $\underline{\pi}_1(X^{an}, *)$ denotes the pro-unipotent group that is the inverse limit of*

$$\{\underline{\pi}_1(X^{an}, *)^{[N]}\}_{N\geq 1},$$

then there is a canonical isomorphism

$$\pi_{1,\mathrm{dR}}(X/\mathbb{C}, *) \simeq \underline{\pi}_1(X^{an}, *)$$

Proof. See Proposition 10.32(b) in [7]. □

Lemma 5.2.2. *The natural map $\pi_4 : M_{0,5} \to M_{0,4}$ that sends $[z_0, \cdots, z_4]$ to $[z_0, \cdots, z_3]$ induces an exact sequence*

$$1 \to \pi_{1,dR}(F_{b,\mathbb{C}}, a) \to \pi_{1,dR}(M_{0,5,\mathbb{C}}, a) \to \pi_{1,dR}(M_{0,4,\mathbb{C}}, b) \to 1$$

where $a \in M_{0,5}(\mathbb{C})$, $b := \pi_4(a)$ and F_b is the fiber over b, of pro-unipotent algebraic groups over $\mathbb{C}$.

Proof. We base-change all the varieties to $\mathbb{C}$. Let $M := \mathbb{A}^2 \setminus \{(z_1, z_2)|z_1z_2(1-z_1)(1-z_2)(z_1 - z_2) = 0\}$. Then the map $M \to M_{0,5}$ defined by $(z_1, z_2) \to [0, 1, \infty, z_1, z_2]$ is an isomorphism. Similarly the map $X := \mathbb{A}^1 \setminus \{0, 1\} \to M_{0,4}$ defined by $z \to [0, 1, \infty, z]$ is an isomorphism. Under these isomorphisms π_4 transforms to $(z_1, z_2) \to z_1$.

The homotopy exact sequence for the locally trivial fibration $M^{an} \to X^{an}$ gives

$$\cdots \to \pi_2(X^{an}, b) \to \pi_1(F_b^{an}, a) \to \pi_1(M^{an}, a) \to \pi_1(X^{an}, b) \to \cdots .$$

Since F_b is connected, the map $\pi_1(M^{an}, a) \to \pi_1(X^{an}, b)$ is surjective. On the other hand, X^{an} has the unit disc as its universal covering space (Uniformization, [12]) and hence $\pi_2(X^{an}, b) = 0$. This gives the exactness of the sequence

$$1 \to \pi_1(F_b^{an}, a) \to \pi_1(M^{an}, a) \to \pi_1(X^{an}, b) \to 1.$$

The projection $M^{an} \to X^{an}$ has a (topological) section defined by $\sigma(z) = (z, (1 + |z|^2)^{1/2})$. Note that the statement in the lemma remains the same if we change a with another point a' such that $\pi_4(a) = \pi_4(a')$. Therefore from now on we will assume without loss of generality that $a = \sigma(b)$. Then we have a splitting of the last exact sequence defined by the map $\sigma_* : \pi_1(X^{an}, b) \to \pi_1(M^{an}, a)$. This induces an action of $\pi_1(X^{an}, b)$ on $\pi_1(F_b^{an}, a)$, by conjugation. And hence an action of $\pi_1(X^{an}, b)$ on $H_1(F_b, \mathbb{Z})$.

The injectivity of $H_1(F_b^{an},\mathbb{Z}) \to H_1(M^{an},\mathbb{Z})$ implies, by Lemma (2.3) of [11], that the action of $\pi_1(X^{an},b)$ on $H_1(F_b,\mathbb{Z})$ is trivial. Now in this situation, Theorem (3.1) of *loc. cit* implies that the induced sequences

$$1 \to \pi_1(F_b^{an},a)^{(N)} \to \pi_1(M^{an},a)^{(N)} \to \pi_1(X^{an},b)^{(N)} \to 1$$

are split exact for all $N \geq 1$.

By the description of the Malcev completion we see that the same is true for the sequences

$$1 \to \pi_1(F_b^{an},a)^{[N]} \to \pi_1(M^{an},a)^{[N]} \to \pi_1(X^{an},b)^{[N]} \to 1.$$

Using the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(F_b^{an},a)^{[N]} & \longrightarrow & \pi_1(M^{an},a)^{[N]} & \longrightarrow & \pi_1(X^{an},b)^{[N]} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Lie}(\pi_1(F_b^{an},a)^{[N]}) & \longrightarrow & \mathrm{Lie}(\pi_1(M^{an},a)^{[N]}) & \longrightarrow & \mathrm{Lie}(\pi_1(X^{an},b)^{[N]}) & \longrightarrow & 0 \end{array}$$

of pointed sets where the vertical arrows are the logarithm bijections imply that the lower horizontal line is exact. Passing to the corresponding unipotent algebraic groups we see that

$$1 \to \underline{\pi}_1(F_b^{an},a)^{[N]} \to \underline{\pi}_1(M^{an},a)^{[N]} \to \underline{\pi}_1(X^{an},b)^{[N]} \to 1.$$

is exact. To see the exactness of the inverse limit of the last sequence we only need to note that the maps $\underline{\pi}_1(F_b^{an},a)^{[N+1]} \to \underline{\pi}_1(F_b^{an},a)^{[N]}$ are surjective, which follows from the surjectivity of the $\pi_1(F_b^{an},a)^{(N+1)} \to \pi_1(F_b^{an},a)^{(N)}$. This gives the exactness of corresponding sequence of pro-unipotent algebraic groups

$$1 \to \underline{\pi}_1(F_b^{an},a) \to \underline{\pi}_1(M^{an},a) \to \underline{\pi}_1(X^{an},b) \to 1.$$

Finally we use Proposition 5.2.1 to deduce the statement in the lemma. □

Lemma 5.2.3. *Let $b \in M_{0,4}(\mathbb{Q})$. The natural map $\pi_4 : M_{0,5} \to M_{0,4}$ induces an exact sequence*

$$1 \to \pi_{1,dR}(F_b,\omega(dR)) \to \pi_{1,dR}(M_{0,5},\omega(dR)) \to \pi_{1,dR}(M_{0,4},\omega(dR)) \to 1$$

of pro-unipotent algebraic groups over $\mathbb{Q}$.

Proof. Let us first choose $a \in F_b(\mathbb{Q})$. Then the previous lemma shows that the sequence corresponding to $\pi_{4,\mathbb{C}}$ and the basepoint $a \in M_{0,5}(\mathbb{C})$ is exact. However we know that if X/k is a smooth, quasi-projective variety over a field k of characteristic zero, $x \in X(k)$ and k'/k is any field extension then the natural map $\pi_{1,dR}(X_{k'},x) \to \pi_{1,dR}(X_k,x)_{k'}$ of pro-unipotent algebraic groups over k' is an isomorphism (10.43 Corollaire, [7]). This fact applied to the extension $\mathbb{C}/\mathbb{Q}$ and the exact sequence in Lemma 5.2.2 implies the exactness of

$$1 \to \pi_{1,dR}(F_b,a) \to \pi_{1,dR}(M_{0,5},a) \to \pi_{1,dR}(M_{0,4},b) \to 1$$

of pro-unipotent algebraic groups over $\mathbb{Q}$.

In order to prove the exactness of the corresponding sequence with the de Rham basepoint, we need to show that the pull-back maps induced by the inclusion $i : F_b \to M_{0,5}$ and the projection $M_{0,5} \to M_{0,4}$ commute with the fiber functor $\omega(dR)$. Note that i and π_4 naturally extend to morphisms of log schemes: $\overline{i} : \overline{F}_{b,log} \to \overline{M}_{0,5,log}$ and $\overline{\pi}_4 : \overline{M}_{0,5,log} \to \overline{M}_{0,4,log}$. Since the map

$$\overline{\pi_4}^* : \mathrm{Mic}_{uni}(\overline{M}_{0,4,log}/\mathbb{Q}) \to \mathrm{Mic}_{uni}(\overline{M}_{0,5,log}/\mathbb{Q})$$

commutes with the de Rham fiber functors, the same statement is true for π_4^* as the de Rham fiber functor on $\mathrm{Mic}_{uni}(X/\mathbb{Q})$ is induced by composing the one on $\mathrm{Mic}_{uni}(\overline{X}_{log}/\mathbb{Q})$ with the canonical extension functor *can* (§2.2.4, §2.2.5). The same argument also shows that i^* commutes with $\omega(dR)$. □

5.3. Residues. For $(\overline{E}, \nabla) \in \mathrm{Mic}_{uni}(\overline{M}_{0,5,log}/\mathbb{Q})$, let $\nabla_{ij} : \overline{E}|_{D_{ij}} \to \overline{E}|_{D_{ij}}$ denote the residue of ∇ along D_{ij}. Since $\overline{E}$ is a trivial bundle, $\nabla_{ij} \in \mathrm{End}_{\mathbb{Q}}(\Gamma(D_{ij}, \overline{E}|_{D_{ij}})) = \mathrm{End}_{\mathbb{Q}}(\omega(dR)(\overline{E}, \nabla))$. Because of the identity (2.2.1), the map that assigns ∇_{ij} to $(\overline{E}, \nabla)$ defines an element $e_{ij} \in \mathrm{Lie}\,\pi_{1,dR}(M_{0,5}, \omega(dR))$.

If $\{i,j\} \cap \{k,l\} = \emptyset$ then $D_{ij} \cap D_{kl} \neq \emptyset$. Computing the residues at the point $D_{ij} \cap D_{kl}$ we see that $[e_{ij}, e_{kl}] = 0$. Similarly, by computing the residues along the fibers of the projections $M_{0,5} \to M_{0,4}$, we see that $\sum_i e_{ij} = 0$. Let

$$H := \mathrm{Lie}\,\langle\langle e_{ij} \rangle\rangle_{0 \le i,j \le 4} / (e_{ii}, e_{ij} - e_{ji}, \sum_i e_{ij}, [e_{ij}, e_{kl}] | \text{ for } \{i,j\} \cap \{k,l\} = \emptyset),$$

with $\mathrm{Lie}\,\langle\langle \cdot \rangle\rangle$ denoting the free pro-nilpotent Lie algebra generated by the arguments. We have an obvious map $H \to \mathrm{Lie}\,\pi_{1,dR}(M_{0,n}, \omega(dR))$, which induces isomorphisms

$$H_4 := \langle\langle e_{i4} \rangle\rangle_{0 \le i \le 3} / (\sum_i e_{i4}) \xrightarrow{\sim} \mathrm{Lie}\,\pi_{1,dR}(F_b, \omega(dR))$$

and

$$H/H_4 \xrightarrow{\sim} \mathrm{Lie}\,\pi_{1,dR}(M_{0,4}, \omega(dR)).$$

This implies that $H \to \mathrm{Lie}\,\pi_{1,dR}(M_{0,n}, \omega(dR))$ is an isomorphism.

5.4. Tangential basepoints on $M_{0,4}$ and $M_{0,5}$.

(i) Basepoints on $M_{0,4}$. First consider basepoints on $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. For $i, j \in \{0, 1, \infty\}$ let $\bar{t}_{ij}$ denote the unit tangent vector at the point i that points in the direction from i to j. For example, $\bar{t}_{01} := \frac{d}{dz}$ at 0, $\bar{t}_{10} := -\frac{d}{dz}$ at 1, $\bar{t}_{\infty 0} := z^2 \frac{d}{dz}$ at ∞ etc. The following claim will be useful in defining tangential basepoints on the configuration spaces.

Claim 5.4.1. *For any $\sigma \in Aut(X)$ and any $i, j \in \{0, 1, \infty\}$, $d(\sigma)(\bar{t}_{ij}) = \bar{t}_{\sigma(i)\sigma(j)}$.*

Proof. Explicit computation. □

This defines tangential basepoints t_{ij} on $M_{0,4}$, well-defined up to multiplication by ± 1, for $i, j \in \{0, 1, 2, 3\}$, with $i \neq j$, in the following manner. Take $k \in \{i, j\}$. Fix any isomorphism $\gamma : M_{0,4} \to X$, and any bijection $\alpha : \{0, 1, 2, 3\} \setminus \{k\} \to \{0, 1, \infty\}$. Let $k' \in \{0, 1, 2, 3\}$ such that $\{k, k'\} = \{i, j\}$, and $\{a, b\} = \alpha(\{0, 1, 2, 3\} \setminus \{i, j\})$ Then define $\{\pm t_{ij}\} := \{((d\gamma^{-1})_{\alpha(k')})(\bar{t}_{\alpha(k')a}), ((d\gamma^{-1})_{\alpha(k')})(\bar{t}_{\alpha(k')b})\}$. The claim above shows the well-definedness of the t_{ij} up to sign. Note, for example, that $\{\pm t_{01}\} = \{\pm t_{23}\}$. We will never fix the choice of signs since this is unnatural and will not be necessary.

(ii) Basepoints on $M_{0,5}$. We will define tangential basepoints on $\overline{M}_{0,5}$ at the points $x_{01,23}, x_{01,34}, x_{12,34}, x_{04,12}$ and $x_{04,23}$ where by $x_{i_1 i_2, j_1 j_2}$ we denote the point in $\overline{M}_{0,5}$ defined by $x_{i_1} = x_{i_2}$ and $x_{j_1} = x_{j_2}$. Let $\pi_i : \overline{M}_{0,5} \to \overline{M}_{0,4}$ denote the map that sends $[x_0, \cdots, x_i, \cdots, x_4]$ to $[x_0, \cdots, \hat{x}_i, \cdots, x_4]$.

Claim 5.4.2. *Let $0 \le i_1 < i_2 \le 4$ and $0 \le j_1 < j_2 \le 4$ such that $\{i_1, i_2\} \cap \{j_1, j_2\} = \emptyset$. Then the subset $\{t | \forall k \in \{i_1, i_2, j_1, j_2\}, d\pi_k(t) \in \{\pm t_{ab} | 0 \le a, b \le 3\}\}$ of the tangent space $T_{x_{i_1 i_2, j_1 j_2}} \overline{M}_{0,5}$ has exactly four elements. The elements of this set,* *considered as a subset of* $T_{x_{i_1 i_2, j_1 j_2}}\overline{M}_{0,5} \simeq N_{D_{i_1 i_2/\overline{M}_{0,5}}} \oplus N_{D_{j_1 j_2/\overline{M}_{0,5}}}$ *are the pairs* (v, w) *such that* $(d(x_{i_2}/x_{i_1}), v) = \pm 1$ *and* $(d(x_{j_2}/x_{j_1}), w) = \pm 1$.

Proof. Elementary computation. □

Let $t_{i_1 i_2, j_1 j_2}$ be one of the tangent vectors at $x_{i_1 i_2, j_1 j_2}$ as in the statement of the claim. There are four different choices. However, in the crystalline setting, the choice between these four points will not be important since in the normal bundle decomposition the different choices differ only by multiplication by ± 1. Again we are not fixing these choices.

6. p-adic integration

Our main references for p-adic integration are [5], [32], and [3]. From now on we will restrict to the case $k = \mathbb{F}_p$ and hence also to $W = \mathbb{Z}_p$. Let $\overline{\mathfrak{X}}$, $\mathfrak{D}$ etc. be as in §2.4. Let $P_{dR} := P_{dR}(\mathfrak{X}_\eta/\mathbb{Q}_p)$ denote the de Rham fundamental groupoid of $\mathfrak{X}_\eta/\mathbb{Q}_p$ (Définitions 10.27, [7]). Through the use of the canonical extension (§2.2.4), the fundamental groupoid naturally extends to a scheme $\overline{P}_{dR}$ affine over $\overline{\mathfrak{X}}_\eta \times \overline{\mathfrak{X}}_\eta$, whose fiber ${}_y\overline{P}_{dR,x}$ over a point $(y, x) \in \overline{\mathfrak{X}}_\eta \times \overline{\mathfrak{X}}_\eta$ represents the functor $\underline{\mathrm{Isom}}_{\mathbb{Q}_p}(\omega(x), \omega(y))$ of isomorphisms from the fiber functor $\omega(x)$ to $\omega(y)$.

The equivalence $\mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p) \simeq \mathrm{Isoc}^\dagger_{uni}(\mathfrak{X}_s/\mathbb{Z}_p)$ and the frobenius functor on the second category induce a morphism

$$F_* : {}_{\mathfrak{y}_\eta}P_{dR,\mathfrak{x}_\eta} \to {}_{\mathfrak{y}_\eta}P_{dR,\mathfrak{x}_\eta}, \tag{6.0.1}$$

for $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}(\mathbb{Z}_p)$. Similarly, if $\mathfrak{x}, \mathfrak{y} \in \mathfrak{D}(\mathbb{Z}_p)$ and $\mathfrak{v} \in N^\times_{\mathfrak{D}_\mathfrak{x}/\overline{\mathfrak{X}}}(\mathfrak{x})$, $\mathfrak{u} \in N^\times_{\mathfrak{D}_\mathfrak{y}/\overline{\mathfrak{X}}}(\mathfrak{y})$ then *depending* on these tangential basepoints (§3.3), we have a morphism

$$F_* : {}_{\mathfrak{y}_\eta}\overline{P}_{dR,\mathfrak{x}_\eta} \to {}_{\mathfrak{y}_\eta}\overline{P}_{dR,\mathfrak{x}_\eta}. \tag{6.0.2}$$

6.1. The frobenius invariant path. Let $\mathbb{Q}_{p,st}$ denote the ring of polynomials $\mathbb{Q}_p[l(p)]$, where $l(p)$ is a formal variable. In this context, one should think of $l(p)$ as (a multi-valued) $\log p$. If $D(1, 1^-)$ denotes the open p-adic disc of radius 1 centered at 1 then $\log z : D(1, 1^-) \cap \mathbb{Q}_p \to \mathbb{Q}_p$ extends uniquely to a homomorphism $\log z : \mathbb{Q}_p^* \to \mathbb{Q}_{p,st}$ such that $\log p = l(p)$ (cf. 1.14, [32]).

There is a unique path that is left invariant by (6.0.1). Namely, Vologodsky [32], extending the work of Coleman [5], Colmez [6], and Besser [3], shows (§4.3 and §4.4 in [32]) that:

Theorem 6.1.1. *For* $x, y \in \mathfrak{X}_\eta(\mathbb{Q}_p)$, *there is a unique path* ${}_yc_x \in {}_yP_{dR,x}(\mathbb{Q}_{p,st})$ *such that*

$$F_*({}_yc_x) = {}_yc_x. \tag{6.1.1}$$

Remark. For $\mathfrak{x}, \mathfrak{y} \in \mathfrak{X}(\mathbb{Z}_p)$, and $x := \mathfrak{x}_\eta, y := \mathfrak{y}_\eta$, ${}_yc_x \in {}_yP_{dR,x}(\mathbb{Q}_p)$. In this case, F_* is defined over $\mathbb{Q}_p$. Projecting ${}_yc_x$ via the morphism $\mathbb{Q}_{p,st} \to \mathbb{Q}_p$ that sends $l(p)$ to 0 and using the uniqueness in the statement of the above theorem, we see that ${}_yc_x$ is, in fact, defined over $\mathbb{Q}_p$.

Example. *(i)* We will describe the frobenius invariant path on $\mathbb{G}_m/\mathbb{Q}_p$. By Lemma 4.1.1, (2.2.4) and (2.2.3), $\mathrm{Mic}_{uni}(\mathbb{G}_m/\mathbb{Q}_p) \xrightarrow{\sim} \mathcal{T}_{uni}$. With this equivalence ${}_yc_x$ assigns an automorphism of $V_{\mathbb{Q}_{p,st}} := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p,st}$ to every pair (V, N) in $\mathcal{T}_{uni}$.

If $x, y \in \mathbb{G}_m(\mathbb{Q}_p)$ are two points with the same finite reduction let ${}_y\mathrm{par}_x$ denote the path that corresponds to the parallel transport along the connection. Then

$${}_y\mathrm{par}_x(V, N) = \exp(\log(y/x)N) : V \to V.$$

If the points do not have finite reduction then we need the construction in §4 of [32] to construct the parallel transport. Suppose that x and y have reduction $x_0 \in \{0, \infty\}$. For $(E, \nabla) \in \mathrm{Mic}_{uni}(\mathbb{G}_m/\mathbb{Q}_p)$, let E_{crys} denote the corresponding unipotent log isocrystal on $\mathbb{P}^1_{log}$ and $\Psi^{un}_{x_0}(E_{crys})$ denote the unipotent nearby cycles of E_{crys} (3.4, [32]) at x_0.

Then the parallel transport on (E, ∇) from x to y is given by the isomorphisms (4.4 in [32]):

$$E(x)_{\mathbb{Q}_{p,st}} \simeq \Psi^{un}_{x_0}(E_{crys})_{\mathbb{Q}_{p,st}} \simeq E(y)_{\mathbb{Q}_{p,st}}.$$

Following these isomorphisms gives:

$${}_y\mathrm{par}_x(V, N) = \exp(\log(y/x)N) : V_{\mathbb{Q}_{p,st}} \to V_{\mathbb{Q}_{p,st}}.$$

Let ${}_yc_x$, for any $x, y \in \mathbb{Q}_p$ be given by the above formula. We would like to show that ${}_yc_x$ is fixed by frobenius. If $\mathcal{F}$ is a lifting of the frobenius to $\mathbb{P}^1_{\mathbb{Z}_p}$ with the property that $\mathcal{F}^*((0)) = p(0)$ and $\mathcal{F}^*((\infty)) = p(\infty)$. In order to fix the choices let $\mathcal{F}(z) = z^p$. The residue of $\mathcal{F}^*(\overline{E}, \nabla)$ at 0 is p times the residue of $(\overline{E}, \nabla)$ at 0. Therefore $\mathcal{F}^*$ sends (V, N) to (V, pN).

For any path ${}_y\gamma_x$ from x to y,

$$F_*({}_y\gamma_x) = {}_y\mathrm{par}_{\mathcal{F}(y)} \cdot \mathcal{F}_*({}_y\gamma_x) \cdot {}_{\mathcal{F}(x)}\mathrm{par}_x.$$

Then we have

$$\begin{aligned} F_*({}_yc_x)(V, N) &= ({}_y\mathrm{par}_{\mathcal{F}(y)} \cdot \mathcal{F}_*({}_yc_x) \cdot {}_{\mathcal{F}(x)}\mathrm{par}_x)(V, N) \\ &= \exp(\log(y/y^p)N)\exp(\log(y/x)pN)\exp(\log(x^p/x)N) \\ &= \exp(\log(y/x)N) \\ &= {}_yc_x(V, N). \end{aligned}$$

Hence

$${}_yc_x(V, N) = \exp(\log(y/x)N) : V_{\mathbb{Q}_{p,st}} \to V_{\mathbb{Q}_{p,st}}.$$

(ii) The same kind of reasoning gives the frobenius invariant path on $(\mathbb{G}_m)^{r_1} \times (\mathbb{A}^1)^{r_2}$. First note that $\mathrm{Mic}_{uni}((\mathbb{G}_m)^{r_1} \times (\mathbb{A}^1)^{r_2}) \xrightarrow{\sim} \mathcal{T}_{r_1,uni}$. With this equivalence the frobenius invariant path ${}_yc_x$ from

$$x := (x_1, x_2, \cdots, x_{r_1+r_2}) \in ((\mathbb{G}_m)^{r_1} \times (\mathbb{A}^1)^{r_2})(\mathbb{Q}_p)$$

to

$$y := (y_1, y_2, \cdots, y_{r_1+r_2}) \in ((\mathbb{G}_m)^{r_1} \times (\mathbb{A}^1)^{r_2})(\mathbb{Q}_p)$$

is given by

$${}_yc_x(V, N_1, \cdots, N_{r_1}) = \exp(\log(\frac{y_1}{x_1})N_1) \cdot \exp(\log(\frac{y_2}{x_2})N_2) \cdots \exp(\log(\frac{y_{r_1}}{x_{r_1}})N_{r_1})$$

as an automorphism of $V_{\mathbb{Q}_{p,st}}$. □

There is a unique path ${}_vc_u$ satisfying (6.1.1) even when u and v are *tangential* basepoints. The proof of Theorem 19 in [32] extends to this case to show the existence and uniqueness of ${}_vc_u$. We will give an explicit description of this path below, which implicitly shows its existence and uniqueness in the case of tangential basepoints.

6.2. Description of the limit of the frobenius invariant path. In order to describe the path ${}_u c_v$ when u or v is a tangential basepoint, we can choose an ordinary basepoint y and note by the uniqueness that ${}_u c_v = {}_u c_y \cdot ({}_v c_y)^{-1}$. Therefore it suffices to describe ${}_u c_y$ when y is an ordinary basepoint and u a tangential basepoint.

Let $\overline{\mathfrak{X}}, \mathfrak{D}$ etc. be as in the beginning of this section; $\mathfrak{x} \in \mathfrak{D}(\mathbb{Z}_p)$. Assume that $u \in N^{\times}_{\mathfrak{D}_\eta/\overline{\mathfrak{X}}_\eta}(\mathfrak{x}_\eta)$, which is not necessarily of finite reduction with respect to the given model.

Let $\mathcal{U} \subseteq \mathbb{V}(N_{\mathfrak{D}/\overline{\mathfrak{X}}}(\mathfrak{x}))_\eta$ be a rigid analytic polydisc around zero; $\psi : \mathcal{U}_{log} \to \overline{\mathfrak{X}}_{\eta,log}$ a rigid analytic map as in §4.2.2. Recall the pull-back to the tangent space functor

$$\varphi^* : \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p) \to \mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{\mathfrak{D}/\overline{\mathfrak{X}}}(\mathfrak{x}))_{log,\eta}/\mathbb{Q}_p)$$

in §4.2.1. The rigid analytic description of φ^* in §4.2.2 implies that if $(E, \nabla) \in \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p)$ and ψ is as above then the restriction of

$$\psi^*(E, \nabla) \in \mathrm{Mic}_{uni}(\mathcal{U}_{log}/\mathbb{Q}_p)$$

to $\mathrm{Mic}_{uni}(0_{log}/\mathbb{Q}_p)$ is canonically isomorphic to the restriction of

$$\varphi^*(E, \nabla) \in \mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{\mathfrak{D}/\overline{\mathfrak{X}}}(\mathfrak{x}))_{log,\eta}/\mathbb{Q}_p)$$

to $\mathrm{Mic}_{uni}(0_{log}/\mathbb{Q}_p)$.

Lemma 6.2.1. *There is a polydisc $\mathcal{U}^\circ \subseteq \mathcal{U}$ such that the restrictions of $\varphi^*(E, \nabla)$ and $\psi^*(E, \nabla)$ to $\mathcal{U}^\circ_{log}$ are canonically isomorphic.*

Proof. Because of the above isomorphism on the restrictions

$$\varphi^*(E, \nabla)|_{0_{log}} \to \psi^*(E, \nabla)|_{0_{log}}$$

to 0_{log}; and the equivalence of categories

$$\mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{\mathfrak{D}/\overline{\mathfrak{X}}}(\mathfrak{x}))_{log,\eta}/\mathbb{Q}_p) \to \mathrm{Mic}_{uni}(0_{log}/\mathbb{Q}_p)$$

by Lemma 4.1.1, to finish the proof of the lemma it suffices to show that there is a polydisc $\mathcal{U}^\circ \subseteq \mathcal{U}$ such that $(E, \nabla)|_{\mathcal{U}^\circ}$ is in the essential image of the restriction functor

$$\mathrm{Mic}_{uni}(\overline{\mathbb{V}}(N_{\mathfrak{D}/\overline{\mathfrak{X}}}(\mathfrak{x}))_{log,\eta}/\mathbb{Q}_p) \to \mathrm{Mic}_{uni}(\mathcal{U}^\circ_{log}/\mathbb{Q}_p).$$

This follows immediately from Lemma 1 in §3.2 of [32]. □

Therefore

$$(\varphi^*(E, \nabla))(\varepsilon) \quad \text{is canonically isomorphic to } (\psi^*(E, \nabla))(\varepsilon) = E(\psi(\varepsilon))$$

for all $\varepsilon \in \mathcal{U}^\circ \setminus (\mathfrak{D}_{(\mathfrak{x})})_\eta$.

Notation 6.2.2. *For ε sufficiently close to 0 in $\mathbb{V}(N_{\mathfrak{D}/\overline{\mathfrak{X}}}(\mathfrak{x}))_\eta \setminus (\mathfrak{D}_{(\mathfrak{x})})_\eta$ let*

$${}_\epsilon\gamma(\psi)_{\psi(\varepsilon)}(E, \nabla)$$

denote the above isomorphism

$$E(\psi(\varepsilon)) \xrightarrow{\ \sim\ } \varphi^*(E, \nabla)(\varepsilon)(= \overline{E}(\mathfrak{x}_\eta)),$$

for $(E, \nabla) \in \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p)$.

Notation 6.2.3. *For* $\mathfrak{x} \in \mathfrak{D}(\mathbb{Z}_p)$, $u \in N^{\times}_{\mathfrak{D}_\eta/\overline{\mathfrak{X}}_\eta}(\mathfrak{x}_\eta)$, $y \in \mathfrak{X}_\eta(\mathbb{Q}_p)$, *and* $x := \mathfrak{x}_\eta$ *we let*

$$ {}_u\overline{c}(\psi)_y \in {}_u\overline{P}_{dR,y}(\mathbb{Q}_{p,st}) $$

denote the isomorphism between the fiber functors $\omega(y) \otimes \mathbb{Q}_{p,st}$ *and* $\omega(u) \otimes \mathbb{Q}_{p,st}$ *on* $\mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p)$ *given by the limit*

$$ \lim_{\varepsilon \to 0} ({}_u c_\varepsilon \cdot {}_\varepsilon\gamma(\psi)_{\psi(\varepsilon)} \cdot {}_{\psi(\varepsilon)}c_y), $$

over $\varepsilon \in \mathcal{U} \setminus (\mathfrak{D}_{(\mathfrak{x})})_\eta$, *where* ${}_u c_\varepsilon$ *denotes the canonical frobenius invariant path satisfying (6.1.1) between* ε *and* u *in* $\mathbb{V}(N_{\mathfrak{D}/\overline{\mathfrak{X}}}(\mathfrak{x}))_\eta \setminus (\mathfrak{D}_{(\mathfrak{x})})_\eta$, *and* ${}_{\psi(\varepsilon)}c_y$ *denotes the similar crystalline invariant path between* y *and* $\psi(\varepsilon)$ *in* $\mathfrak{X}_\eta$.

6.3. Well-definedness of ${}_u\overline{c}(\psi)_y$**.** Several remarks are in order to explain the definition and notation above. The question of the existence of the limit in Notation 6.2.3 above is completely local. Therefore we will assume without loss of generality that

$$ \psi : \mathcal{U}^\circ_{log} \subseteq D(0,1^-)^r_{log} \to D(0,1^-)^n_{log} \subseteq \overline{\mathfrak{X}}_{\eta,log} $$

is a closed immersion of logarithmic analytic spaces, where both spaces are endowed with the log structures associated to the divisor $z_1 \cdots z_r = 0$, and $D(0,1^-)^n$ reduces to a single point (namely the reduction of $\mathfrak{x}$) in the special fiber. Furthermore

$$ \psi(0) = 0 \ \text{ and } \ \frac{\psi_i}{z_i} \in 1 + \mathfrak{m}_0, \ \text{ for } \ 1 \le i \le r $$

with $\psi := (\psi_1, \cdots, \psi_n)$ and $\mathfrak{m}_0$, the maximal ideal in the local ring at 0. Again, without loss of generality, we will assume that $(E, \nabla)|_{D(0,1^-)^n}$ is the vector bundle with connection associated to a vector space V and r commuting nilpotent operators $N_1, \cdots, N_r$. Therefore we will view every path as an automorphism of $V_{\mathbb{Q}_{p,st}}$.

Note that for any $a_1, a_2 \in \mathfrak{X}_\eta(\mathbb{Q}_p)$

$$ {}_u c_\varepsilon \cdot {}_\varepsilon\gamma(\psi)_{\psi(\varepsilon)} \cdot {}_{\psi(\varepsilon)}c_{a_2} = {}_u c_\varepsilon \cdot {}_\varepsilon\gamma(\psi)_{\psi(\varepsilon)} \cdot {}_{\psi(\varepsilon)}c_{a_1} \cdot {}_{a_1}c_{a_2} $$

therefore when trying to prove the existence of

$$ \lim_{\varepsilon \to 0} ({}_u c_\varepsilon \cdot {}_\varepsilon\gamma(\psi)_{\psi(\varepsilon)} \cdot {}_{\psi(\varepsilon)}c_y) $$

we will assume without loss of generality that $y \in D(0,1^-)^n$. Let $y := (y_1, \cdots, y_n)$, $u := (u_1, \cdots, u_r)$ and $\varepsilon := (\varepsilon_1, \cdots, \varepsilon_r)$.

Lemma 6.3.1. *Using the notation above,*

$$ {}_\varepsilon\gamma(\psi)_{\psi(\varepsilon)} : V_{\mathbb{Q}_{p,st}} = \psi^*(E)(\varepsilon)_{\mathbb{Q}_{p,st}} \to \varphi^*(E,\nabla)(\varepsilon)_{\mathbb{Q}_{p,st}} = V_{\mathbb{Q}_{p,st}} $$

is given by

$$ \prod_{1 \le i \le r} \exp(N_i \log(\frac{\varepsilon_i}{\psi_i(\varepsilon)})), $$

for ε *sufficiently small.*

Proof. Because of the assumption on ψ, the expression in the statement makes sense for ε sufficiently small, and is the identity map at 0. By direct computation, one checks that the map induces a morphism of vector bundles with connection:

$$ \psi^*(E,\nabla) = \psi^*(V, N_1, \cdots, N_r) \to (V, N_1, \cdots, N_r) = \varphi^*(E, \nabla) $$

which is the identity map at the origin and hence is the unique such map $\gamma(\psi)$. □

Because of the computation of the frobenius invariant path on $(\mathbb{G}_m)^r$ in the example following Theorem 6.1.1 and on small discs using the unipotent nearby cycles functor, the limit that we are interested in is

$$\begin{aligned}&\lim_{\varepsilon\to 0}(\prod_{1\le i\le r}\exp(N_i\log\frac{u_i}{\varepsilon_i})\cdot\prod_{1\le i\le r}\exp(N_i\log(\frac{\varepsilon_i}{\psi_i(\varepsilon)}))\cdot\prod_{1\le i\le r}\exp(N_i\log\frac{\psi_i(\varepsilon)}{y_i}))\\ =&\prod_{1\le i\le r}\exp(N_i\log(\frac{u_i}{y_i})).\end{aligned}$$

Therefore the path ${}_u\overline{c}(\psi)_y$ is well-defined.

6.4. Frobenius invariance of ${}_u\overline{c}(\psi)_y$. We would like to show that ${}_u\overline{c}(\psi)_y$ is invariant under frobenius. Since

$${}_u\overline{c}(\psi)_a = {}_u\overline{c}(\psi)_b\cdot {}_bc_a$$

and

$$F_*({}_u\overline{c}(\psi)_a) = F_*({}_u\overline{c}(\psi)_b)\cdot F_*({}_bc_a) = F_*({}_u\overline{c}(\psi)_b)\cdot {}_bc_a,$$

we will assume without loss of generality that we are in the local situation above with $y\in D(0,1^-)^n$.

Let $\mathcal{F}$ be a local lifting of frobenius to $\overline{\mathfrak{X}}/\mathbb{Z}_p$ near $\mathfrak{x}$. Choose local coordinates as in §6.3 with the additional property that $\mathcal{F}(D(0,1^-)^n)\subseteq D(0,1^-)^n$. Let $\mathcal{P}(\mathcal{F})$ denote the principal part of $\mathcal{F}$ as in §4.1.1.

Then by the definition of the frobenius action, for $(E,\nabla)\in \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p)$ and $(V,N_1,\cdots,N_r)$ associated to that as in §6.3,

$$F_*({}_u\overline{c}_y)(E,\nabla) = {}_u\mathrm{par}_{\mathcal{P}(\mathcal{F})(u)}(\varphi^*(E,\nabla))\cdot({}_u\overline{c}_y)(\mathcal{F}^*(E,\nabla))\cdot {}_{\mathcal{F}(y)}\mathrm{par}_y(E,\nabla),$$

where on the right hand side of the equation the first *par* denotes parallel transport along the connection on $\mathbb{V}(N_{\mathfrak{D}_\mathfrak{x}/\mathfrak{X}}(\mathfrak{x}))_\eta\setminus(\mathfrak{D}_{(\mathfrak{x})})_\eta$ and the second one denotes parallel transport along the connection on $\mathfrak{X}_\eta$. We temporarily omit ψ from the notation. For ε sufficiently small we know that, by §6.3,

$$\begin{aligned}{}_u\overline{c}_y(\mathcal{F}^*(E,\nabla)) &= ({}_uc_\varepsilon\cdot {}_\varepsilon\gamma_{\psi(\varepsilon)}\cdot {}_{\psi(\varepsilon)}c_y)(\mathcal{F}^*(E,\nabla))\\ &= {}_uc_\varepsilon(\varphi^*(\mathcal{F}^*(E,\nabla)))\cdot {}_\varepsilon\gamma_{\psi(\varepsilon)}(\mathcal{F}^*(E,\nabla))\cdot {}_{\psi(\varepsilon)}c_y(\mathcal{F}^*(E,\nabla))\\ &= {}_uc_\varepsilon(\mathcal{P}(\mathcal{F})^*(\varphi^*(E,\nabla)))\cdot {}_\varepsilon\gamma_{\psi(\varepsilon)}(\mathcal{F}^*(E,\nabla))\cdot {}_{\psi(\varepsilon)}c_y(\mathcal{F}^*(E,\nabla)).\end{aligned}$$

The frobenius invariance of ${}_uc_\varepsilon$ implies that

$${}_uc_\varepsilon(\varphi^*(E,\nabla)) =$$

$${}_u\mathrm{par}_{\mathcal{P}(\mathcal{F})(u)}(\varphi^*(E,\nabla))\cdot {}_uc_\varepsilon(\mathcal{P}(\mathcal{F})^*(\varphi^*(E,\nabla)))\cdot {}_{\mathcal{P}(\mathcal{F})(\varepsilon)}\mathrm{par}_\varepsilon(\varphi^*(E,\nabla))$$

and the frobenius invariance of ${}_{\psi(\varepsilon)}c_y$ implies that

$${}_{\psi(\varepsilon)}c_y(E,\nabla) = {}_{\psi(\varepsilon)}\mathrm{par}_{\mathcal{F}(\psi(\varepsilon))}(E,\nabla)\cdot {}_{\psi(\varepsilon)}c_y(\mathcal{F}^*(E,\nabla))\cdot {}_{\mathcal{F}(y)}\mathrm{par}_y(E,\nabla).$$

Putting these together we obtain

$$\begin{aligned}F_*({}_u\overline{c}_y)(E,\nabla) =\ & {}_uc_\varepsilon(\varphi^*(E,\nabla))\cdot {}_\varepsilon\mathrm{par}_{\mathcal{P}(\mathcal{F})(\varepsilon)}(\varphi^*(E,\nabla))\cdot {}_\varepsilon\gamma_{\psi(\varepsilon)}(\mathcal{F}^*(E,\nabla))\cdot\\ & {}_{\mathcal{F}(\psi(\varepsilon))}\mathrm{par}_{\psi(\varepsilon)}(E,\nabla)\cdot {}_{\psi(\varepsilon)}c_y(E,\nabla).\end{aligned}$$

Lemma 6.4.1. *The isomorphism*

$${}_{\varepsilon}\gamma_{\psi(\varepsilon)}(\mathcal{F}^*(E,\nabla)) : V_{\mathbb{Q}_{p,st}} = E(\mathcal{F}(\psi(\varepsilon)))_{\mathbb{Q}_{p,st}} \to \varphi^*(\mathcal{F}^*(E,\nabla))(\varepsilon)_{\mathbb{Q}_{p,st}} = V_{\mathbb{Q}_{p,st}}$$

is given by

$$\prod_{1\leq i\leq r} \exp(N_i \log(\frac{\mathcal{P}(\mathcal{F})_i(\varepsilon)}{(\mathcal{F}\circ\psi)_i(\varepsilon)})),$$

where

$$\mathcal{P}(\mathcal{F})(\varepsilon) = (\mathcal{P}(\mathcal{F})_1(\varepsilon), \cdots, \mathcal{P}(\mathcal{F})_r(\varepsilon))$$

and

$$\mathcal{F}\circ\psi(\varepsilon) = ((\mathcal{F}\circ\psi)_1(\varepsilon), \cdots, (\mathcal{F}\circ\psi)_n(\varepsilon)).$$

Proof. Note that if $\mathcal{F} = (\mathcal{F}_1, \cdots, \mathcal{F}_n)$, then we have $\frac{\mathcal{F}_i}{z_i^p} = a_i + \mathfrak{m}_0$ for some $a_i \in \mathbb{Q}_p^\times$, for all $1 \leq i \leq r$. We have $P(\mathcal{F})(z_1, \cdots, z_r) = (a_1 z_1^p, \cdots, a_r z_r^p)$ and $\frac{\psi_i}{z_i} \in 1 + \mathfrak{m}_0$. Putting all of these together we see that

$$\frac{\mathcal{P}(\mathcal{F})_i}{(\mathcal{F}\circ\psi)_i} \in 1 + \mathfrak{m}_0.$$

This implies that the map in the statement above makes sense for ε sufficiently small.

Note that $\varphi^*(\mathcal{F}^*(E,\nabla))$ is the vector bundle with connection associated to $(V, pN_1, \cdots, pN_r)$. In other words it is

$$(V \otimes_{\mathbb{Q}_p} \mathcal{O}, d - \sum_{1\leq i\leq r} pN_i d\log(z_i)) = (V \otimes_{\mathbb{Q}_p} \mathcal{O}, d - \sum_{1\leq i\leq r} N_i d\log(\mathcal{P}(\mathcal{F})_i)).$$

It can now be checked easily that the above defined map gives an isomorphism $\psi^*(\mathcal{F}^*(E,\nabla)) \to \varphi^*(\mathcal{F}^*(E,\nabla))$, which is the identity at the origin, and hence is the map γ as in the statement of the lemma. □

Now the lemma above gives

$${}_{\varepsilon}\text{par}_{\mathcal{P}(\mathcal{F})(\varepsilon)}(\varphi^*(E,\nabla)) \cdot {}_{\varepsilon}\gamma_{\psi(\varepsilon)}(\mathcal{F}^*(E,\nabla)) \cdot {}_{\mathcal{F}(\psi(\varepsilon))}\text{par}_{\psi(\varepsilon)}(E,\nabla) = {}_{\varepsilon}\gamma_{\psi(\varepsilon)}(E,\nabla)$$

and hence

$$F_*({}_u\overline{c}_y)(E,\nabla) = {}_u c_{\varepsilon}(\varphi^*(E,\nabla)) \cdot {}_{\varepsilon}\gamma_{\psi(\varepsilon)}(E,\nabla) \cdot {}_{\psi(\varepsilon)}c_y(E,\nabla) = {}_u\overline{c}_y(E,\nabla).$$

This finishes the proof of the frobenius invariance of ${}_u\overline{c}_y$.

Corollary 6.4.2. *The path ${}_u\overline{c}_y(\psi)$ does not depend on ψ.*

Proof. Let ψ' be any other map as above that satisfies (4.2.2), (4.2.3), (4.2.4) and (4.2.5). The frobenius invariance above gives

$$F_*(({}_u\overline{c}(\psi')_y)^{-1} \cdot {}_u\overline{c}(\psi)_y) = F_*({}_u\overline{c}(\psi')_y)^{-1} \cdot F_*({}_u\overline{c}(\psi)_y) = ({}_u\overline{c}(\psi')_y)^{-1} \cdot {}_u\overline{c}(\psi)_y.$$

Therefore $({}_u\overline{c}(\psi')_y)^{-1} \cdot {}_u\overline{c}(\psi)_y$, being the frobenius invariant path from y to y, is the trivial path. □

Notation 6.4.3. *We let ${}_uc_y := {}_u\overline{c}(\psi)_y$ for any ψ as above. And if u and v are any two tangential basepoints we let ${}_uc_v := {}_uc_y \cdot {}_yc_v$ for any basepoint y.*

Corollary 6.4.4. *The path ${}_uc_v$ described above is the unique frobenius invariant path from v to u.*

Corollary 6.4.5. *Let* $(\overline{\mathfrak{X}}/\mathbb{Z}_p, \mathfrak{D})$ *be as in the beginning of* §6. *Let* $\mathfrak{x}, \mathfrak{y} \in \mathfrak{D}(\mathbb{Z}_p)$ *and*

$$\mathfrak{u} \in N^{\times}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}(\mathfrak{x}), \;\; \mathfrak{v} \in N^{\times}_{\mathfrak{D}_{\mathfrak{y}}/\overline{\mathfrak{X}}}(\mathfrak{y}).$$

Then the path

$${}_{\mathfrak{u}_\eta}c_{\mathfrak{v}_\eta} \in {}_{\mathfrak{u}_\eta}P_{dR,\mathfrak{v}_\eta}(\mathbb{Q}_{p,st})$$

is in fact defined over $\mathbb{Q}_p$.

Proof. The proof is exactly as in the remark following Theorem 6.1.1. We only need to remark that when the tangential basepoints have finite reduction as above then F_* is defined over $\mathbb{Q}_p$. □

6.5. Alternative description of ${}_uc_y$**.** Let the notation be as in §6.2 with $\mathfrak{y} \in \mathfrak{X}(\mathbb{Z}_p)$, $\mathfrak{x} \in \mathfrak{D}(\mathbb{Z}_p)$ and $\mathfrak{u} \in N^{\times}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}(\mathfrak{x})$; $x := \mathfrak{x}_\eta$, $y := \mathfrak{y}_\eta$ and $u := \mathfrak{u}_\eta$. Given a $\mathbb{Q}_p$-vector space V and an operator $T : V_{\mathbb{Q}_{p,st}} \to V_{\mathbb{Q}_{p,st}}$, let $\mathring{T} : V \to V$ denote the map induced by T on $V = V_{\mathbb{Q}_{p,st}}/l(p)V_{\mathbb{Q}_{p,st}}$. Let $(E, \nabla) \in \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p)$, ψ and $\mathcal{U}^\circ$ be as in §6.2.

Let

$$\alpha : (\psi^*\overline{E})|_{\mathcal{U}^\circ} \to \overline{E}(x) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{U}^\circ}$$

be any isomorphism of the vector bundles, such that $\alpha(0)$ is the identity map.

Consider the following composition

$$\alpha(p^N u) \cdot {}_{\psi(p^N u)}\mathring{c}_y(E, \nabla) : E(y) \to E(\psi(p^N u)) \to \overline{E}(x).$$

Lemma 6.5.1. *With the notation as above,*

$$\lim_{N \to \infty} \alpha(p^N u) \cdot {}_{\psi(p^N u)}\mathring{c}_y(E, \nabla) = {}_uc_y(E, \nabla).$$

Proof. The existence and the value of the limit on the left side of the equality is independent of the choice of α with the property that $\alpha(0)$ is the identity map. Below we will show the equality in the statement of the lemma for a specific choice of α.

We know, by Lemma 6.2.1, that there is an isomorphism

$$\psi^*(\overline{E}, \nabla)|_{\mathcal{U}^\circ} \to \varphi^*(E, \nabla)|_{\mathcal{U}^\circ}$$

which is the identity map on the fibers at 0. Let

$$\alpha : \psi^*\overline{E}|_{\mathcal{U}^\circ} \to \overline{E}(x) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{U}^\circ}$$

denote the underlying isomorphism of the vector bundles. We will use this choice of α below.

Then for N sufficiently large, by the definition in §6.2, we see that

$${}_uc_y(E, \nabla) = {}_uc_{p^N u}(\varphi^*(E, \nabla)) \cdot \alpha(p^N u) \cdot {}_{\psi(p^N u)}c_y(E, \nabla).$$

By the finite reduction assumptions on y and u, Corollary 6.4.5 implies that

$${}_uc_y(E, \nabla) : E(y)_{\mathbb{Q}_{p,st}} \to \overline{E}(x)_{\mathbb{Q}_{p,st}}$$

is in fact induced from a map from $E(y) \to \overline{E}(x)$, i.e.

$${}_uc_y(E, \nabla) = {}_u\mathring{c}_y(E, \nabla) \otimes id_{\mathbb{Q}_{p,st}}.$$

We will suppress the symbol $id_{\mathbb{Q}_{p,st}}$ from now on. Therefore we have

$${}_uc_y(E, \nabla) = {}_u\mathring{c}_y(E, \nabla) = {}_u\mathring{c}_{p^N u}(\varphi^*(E, \nabla)) \cdot \alpha(p^N u) \cdot {}_{\psi(p^N u)}\mathring{c}_y(E, \nabla).$$

By the computation of the frobenius invariant path in $\mathbb{G}_m^r$ in the Example in §6.1, we see that if $\varphi^*(E,\nabla)$ corresponds to the vector bundle with connection on $\mathbb{G}_m^r \simeq \mathbb{V}(N_{\mathfrak{D}_\mathfrak{r}/\overline{\mathfrak{X}}})_\eta \setminus (\mathfrak{D}_{(\mathfrak{r})})_\eta$ associated to $(\overline{E}(y), N_1, \cdots, N_r)$ then

$$ {}_u c_{p^N u}(\varphi^*(E,\nabla)) = \prod_{1\le i\le r} \exp(\log(p^{-N})N_i) = \prod_{1\le i\le r} \exp(-Nl(p)\cdot N_i), $$

and

$$ {}_u \mathring{c}_{p^N u}(\varphi^*(E,\nabla)) = id. $$

Hence for N sufficiently large

$$ \alpha(p^N u) \cdot_{\psi(p^N u)} \mathring{c}_y(E,\nabla) = {}_u c_y(E,\nabla), $$

for this special choice of α, which proves the lemma. We emphasize that for a general α one *needs* to, in fact, pass to the limit. □

6.6. Description of ${}_u c_y$ using the de Rham fiber functor. Use the above notation, but with the additional assumption that $H^1(\overline{\mathfrak{X}}_\eta, \mathcal{O}) = 0$. Note that in this case if s is any (tangential) basepoint, there is a natural isomorphism from $\omega(dR)$ to $\omega(s)$. Denote this by ${}_s e(dR)_{\omega(dR)}$, its inverse by ${}_{\omega(dR)} e(dR)_s$, and let ${}_t e(dR)_s := {}_t e(dR)_{\omega(dR)} \cdot {}_{\omega(dR)} e(dR)_s$.

Lemma 6.6.1. *With u, y and ψ be as in §6.5, the path*

$$ {}_{\omega(dR)} e(dR)_u \cdot {}_u c_y \cdot {}_y e(dR)_{\omega(dR)} \in \pi_1(\mathfrak{X}_\eta, \omega(dR)) $$

is equal to the limit

$$ \lim_{N\to\infty} {}_{\omega(dR)} e(dR)_{\psi(p^N u)} \cdot {}_{\psi(p^N u)} \mathring{c}_y \cdot {}_y e(dR)_{\omega(dR)}. $$

Proof. Let $(E,\nabla) \in \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p)$. Then the expression in the limit evaluated at (E,∇) is the automorphism of $\Gamma(\overline{\mathfrak{X}}_\eta, \overline{E})$ that is the composition of the isomorphisms

$$ \Gamma(\overline{\mathfrak{X}}_\eta, \overline{E}) \longrightarrow E(y) \xrightarrow{{}_{\psi(p^N u)}\mathring{c}_y} E(\psi(p^N u)) \longrightarrow \Gamma(\overline{\mathfrak{X}}_\eta, \overline{E}), $$

where the first and the last isomorphisms are the ones induced by the canonical isomorphism

$$ \overline{E} \xrightarrow{\beta} \Gamma(\overline{\mathfrak{X}}_\eta, \overline{E}) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\overline{\mathfrak{X}}_\eta}. $$

Letting for $\alpha : \psi^*\overline{E}|_{\mathcal{U}^\circ} \to \overline{E}(x) \otimes_{\mathbb{Q}_p} \mathcal{O}_{\mathcal{U}^\circ}$, following the notation in §6.5, the composition

$$ (\beta(x)^{-1} \otimes (id)) \circ \psi^*(\beta) $$

the expression in the limit evaluated at (E,∇) takes the form

$$ {}_{\omega(dR)} e(dR)_u \cdot \alpha(p^N u) \cdot {}_{\psi(p^N u)} \mathring{c}_y(E,\nabla) \cdot {}_y e(dR)_{\omega(dR)}. $$

Then the statement follows immediately from Lemma 6.5.1. □

6.7. Change of tangential basepoints. In order to see that changing the tangential basepoints by multiplication with roots of unity have no effect in the crystalline-de Rham theory, we need the following lemma.

If

$$w \in N^{\times}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}(\mathfrak{x}) = \prod_{1\le i\le r} N^{\times}_{\mathfrak{D}_i/\overline{\mathfrak{X}}}(\mathfrak{x}),$$

where $\mathfrak{D}_i$, for $1 \le i \le r$, are the components of $\mathfrak{D}$ passing through $\mathfrak{x}$, let $w := (w_1, \cdots, w_r)$.

Lemma 6.7.1. *With notation as above, if $u, v \in N^{\times}_{\mathfrak{D}_{\mathfrak{x}}/\overline{\mathfrak{X}}}(\mathfrak{x})$ is such that $v_i/u_i \in \mathbb{Z}_p^{\times}$ is a root of unity for every $1 \le i \le r$ then*

$${}_v c_y = {}_u c_y.$$

Proof. First, note that if $(E, \nabla) \in \mathrm{Mic}_{uni}(\mathfrak{X}_\eta/\mathbb{Q}_p)$ then ${}_u c_y(E, \nabla)$ and ${}_v c_y(E, \nabla)$ are both isomorphims from $E(y)_{\mathbb{Q}_{p,st}}$ to $\overline{E}(x)_{\mathbb{Q}_{p,st}}$.

The local description of ${}_u c_y$ and ${}_v c_y$, by using a map ψ and the local rigid analytic trivialization $(\overline{E}(x), N_1, \cdots, N_r)$ of (E, ∇) as in §6.3 gives immediately that

$${}_v c_y = \prod_{1\le i\le r} \exp(\log(\frac{v_i}{u_i})N_i) \cdot {}_u c_y.$$

Since $\log(\zeta) = 0$, if ζ is a root of unity, the statement follows. □

6.8. p-adic multi-zeta values. (§4.3, [31]) For a smooth variety $X/\mathbb{Q}_p$ and a (tangential) basepoint x on X, let $\mathcal{U}_{dR}(X, x)$ denote the universal enveloping algebra of the Lie algebra $\mathrm{Lie}\,\pi_{1,dR}(X, x)$ and $\widehat{\mathcal{U}}_{dR}(X, x)$ denote its completion with respect to the augmentation ideal (§4.2, [31]). It is a co-commmutative Hopf algebra and its topological dual is the Hopf algebra of functions on $\pi_{1,dR}(X, x)$.

From now on, let $\mathfrak{X}/\mathbb{Z}_p$ be $\mathbb{G}_m \setminus \{1\}/\mathbb{Z}_p$, $\overline{\mathfrak{X}} = \mathbb{P}^1$ and $X := \mathfrak{X}_\eta$. Let e_0, e_1, and $e_\infty \in \mathrm{Lie}\,\pi_{1,dR}(X/\mathbb{Q}_p, t_{01})$ denote the residues corresponding to the points $0, 1, \infty$ in $\overline{X}$ respectively (§5.3). Then $\mathrm{Lie}\,\pi_{1,dR}(X, t_{01}) \simeq \mathrm{Lie}\,\langle\langle e_0, e_1, e_\infty\rangle\rangle/(e_0 + e_1 + e_\infty)$ (§4.3, [31]) and $\widehat{\mathcal{U}}_{dR}(X, t_{01})$ is isomorphic to the ring of associative formal power series on e_0 and e_1 with the co-product Δ given by $\Delta(e_0) = 1 \otimes e_0 + e_0 \otimes 1$, and $\Delta(e_1) = 1 \otimes e_1 + e_1 \otimes 1$. By the duality above, $\mathbb{Q}_p$-rational points of $\pi_{1,dR}(X, t_{01})$ correspond to associative formal power series a in e_0 and e_1 with coefficients in $\mathbb{Q}_p$, whose constant term is 1 and satisfies $\Delta(a) = a \otimes a$.

We let

$$g := {}_{t_{01}}e(dR)_{t_{10}} F_*({}_{t_{10}}e(dR)_{t_{01}}) \in \pi_{1,dR}(X, t_{01})(\mathbb{Q}_p) \subseteq \mathbb{Q}_p\langle\langle e_0, e_1\rangle\rangle.$$

Using this series, we define the p-adic multi-values $\zeta_p(s_k, \cdots, s_1)$ as

$$g[e_0^{s_k-1}e_1 \cdots e_0^{s_1-1}e_1] =: p^{\sum s_i}\zeta_p(s_k, \cdots, s_1).$$

In turn, these values determine g.

7. Drinfel'd-Ihara relations

In this section, we prove that the series g satisfies the Drinfel'd-Ihara relations. These, of course, imply relations on p-adic multi-zeta values. The most tricky relation is the 5-cycle relation whose proof constitutes most of the section. We start with the 2-cycle and 3-cycle relations.

7.1. 2-cycle relation. Letting $\gamma := {}_{t_{10}}e(dR)_{t_{01}}$, and

$$g := g(e_0, e_1) = \gamma^{-1} \cdot F_*(\gamma) \in \mathbb{Q}_p\langle\langle e_0, e_1\rangle\rangle,$$

we would like to see that

$$g(e_1, e_0)g(e_0, e_1) = 1. \tag{7.1.1}$$

Let τ be the automorphism of X that maps z to $1 - z$. Then

$$\tau(t_{01}) = t_{10},\ \tau_*(\gamma) = \gamma^{-1},\ \tau_*(e_0) = \gamma \cdot e_1 \cdot \gamma^{-1},\ \tau_*(e_1) = \gamma \cdot e_0 \cdot \gamma^{-1}.$$

We have

$$\begin{aligned}\gamma \cdot g(e_1, e_0) \cdot \gamma^{-1} &= \tau_*(g(e_0, e_1)) = \tau_*(\gamma^{-1}) \cdot \tau_*(F_*\gamma)\\ &= \tau_*(\gamma^{-1}) \cdot F_*(\tau_*(\gamma)) = \gamma \cdot F_*(\gamma^{-1}).\end{aligned}$$

Therefore

$$g(e_1, e_0) = F_*(\gamma^{-1}) \cdot \gamma = g(e_0, e_1)^{-1}.$$

7.2. 3-cycle relation. In this subsection, we will prove that

$$g(e_\infty, e_0)g(e_1, e_\infty)g(e_0, e_1) = 1. \tag{7.2.1}$$

Let $\delta := {}_{t_{\infty 0}}e(dR)_{t_{01}}$, $r := {}_{t_{1\infty}}e(dR)_{t_{10}}$, and $q := r \cdot \gamma = {}_{t_{1\infty}}e(dR)_{t_{01}}$.
First note that:

Lemma 7.2.1. $F_*(r) = r$.

Proof. Note that t_{10} and $t_{1\infty}$ are both tangent vectors at 1 with the property that $t_{1\infty} = -t_{10}$. Let $T_1(\overline{X})$ denote the tangent space of $\overline{X}$ at 1. By the definition of the frobenius action on tangential basepoints, all we need to show is that if

$$r' := {}_{-1}e(dR)_1$$

denotes the de Rham path from 1 to -1 in $\mathbb{G}_m \simeq T_1(\overline{X}) \setminus \{0\}$ then

$$F_*(r') = r'.$$

Given

$$(E, \nabla) := (V \otimes_{\mathbb{Q}_p} \mathcal{O}, d - N\frac{dz}{z}) \in \mathrm{Mic}_{uni}(\mathbb{G}_m/\mathbb{Q}_p),$$

where N is a nilpotent operator on V, and let $\mathcal{F}(z) = z^p$ be the lifting of the frobenius on the special fiber to the (log) compactification of $\mathbb{G}_m/\mathbb{Q}_p$,

$$\begin{aligned}F_*(r')(E, \nabla) &= {}_{-1}\mathrm{par}_{\mathcal{F}(-1)}(E, \nabla) \cdot \mathcal{F}_*(r')(E, \nabla) \cdot {}_{\mathcal{F}(1)}\mathrm{par}_1(E, \nabla)\\ &= \exp(\log((-1)^{p-1}) \cdot N) = id_V,\end{aligned}$$

since $\log(-1)^{p-1} = 0$. This proves the claim. □

This implies that

$$q^{-1} \cdot F_*(q) = \gamma^{-1} \cdot r^{-1} \cdot F_*(r) \cdot F_*(\gamma) = \gamma^{-1} \cdot F_*(\gamma) = g.$$

Let ω be the automorphism of X that sends z to $\frac{1}{1-z}$. Then

$$\omega(t_{01}) = t_{1\infty}, \quad \omega(t_{1\infty}) = t_{\infty 0},$$

$$\delta = \omega_*(q) \cdot q,$$

$$\omega_*(e_0) = q \cdot e_1 \cdot q^{-1}, \quad \omega_*(e_1) = q \cdot e_\infty \cdot q^{-1},$$

$$\omega_*^2(e_0) = \omega_*^2(q)^{-1} \cdot e_\infty \cdot \omega_*^2(q), \quad \omega_*^2(e_1) = \omega_*^2(q)^{-1} \cdot e_0 \cdot \omega_*^2(q).$$

Applying frobenius to

$$\omega_*^2(q) \cdot \omega_*(q) \cdot q = 1$$

we obtain

$$\begin{aligned}
1 &= \omega_*^2(F_*q) \cdot \omega_*(F_*q) \cdot F_*q = \omega_*^2(F_*q) \cdot \omega_*(F_*q) \cdot (q \cdot q^{-1}) \cdot F_*q \\
&= \omega_*^2(F_*q) \cdot \omega_*(F_*q) \cdot q \cdot g = \omega_*^2(F_*q) \cdot \omega_*(q) \cdot \omega_*(q^{-1} \cdot F_*q) \cdot q \cdot g \\
&= \omega_*^2(F_*q) \cdot \omega_*(q) \cdot \omega_*(g) \cdot q \cdot g \\
&= \omega_*^2(F_*q) \cdot \omega_*(q) \cdot q \cdot (q^{-1} \cdot \omega_*(g) \cdot q) \cdot g \\
&= \omega_*^2(q) \cdot \omega_*^2(q^{-1} \cdot F_*q) \cdot \omega_*^2(q)^{-1} \cdot (q^{-1} \cdot \omega_*(g) \cdot q) \cdot g \\
&= (\omega_*^2(q) \cdot \omega_*^2(g) \cdot \omega_*^2(q)^{-1}) \cdot (q^{-1} \cdot \omega_*(g) \cdot q) \cdot g \\
&= g(e_\infty, e_0) \cdot g(e_1, e_\infty) \cdot g(e_0, e_1).
\end{aligned}$$

7.3. 5-cycle relation. In this section, with notation as in §5.3, we will prove the 5-cycle relation

$$g(e_{23}, e_{34})g(e_{40}, e_{01})g(e_{12}, e_{23})g(e_{34}, e_{40})g(e_{01}, e_{12}) = 1. \tag{7.3.1}$$

7.3.1. *The description of ${}_sc_t$.* We will use the identification

$$M_{0,5} \simeq (\mathbb{G}_m \setminus \{1\})^2 \setminus \{(z_1, z_2) | z_1 z_2 = 1\} \subseteq \mathbb{A}^2. \tag{7.3.2}$$

We will start this section by expressing the frobenius invariant path ${}_sc_t$ on $M_{0,5}$ from t to s, where t is the tangent vector $(1,1)$ at the point $(0,0)$ and s is the tangent vector $(-1,1)$ at the point $(1,0)$, where to specify tangent vectors we use the identification of the tangent space of any point in $\mathbb{A}^2$ with $\mathbb{A}^2$.

By Lemma 6.6.1, using the obvious choices for ψ's, we see that

$$_{\omega(dR)}e(dR)_s \cdot {}_sc_t \cdot {}_te(dR)_{\omega(dR)} \in \pi_1(M_{0,5}, \omega(dR)) \tag{7.3.3}$$

is equal to

$$\lim_{N\to\infty} {}_{\omega(dR)}e(dR)_{(1-p^N,p^N)} \cdot {}_{(1-p^N,p^N)}\mathring{c}_{(p^N,p^N)} \cdot {}_{(p^N,p^N)}e(dR)_{\omega(dR)}. \tag{7.3.4}$$

Let $X_N := \mathbb{A}^1 \setminus \{0, 1, p^{-N}\}$, and $i_N : X_N \to M_{0,5}$, the inclusion that sends z to (z, p^N). The exact sequences of de Rham fundamental groups:

$$1 \to \pi_1(X_N, \omega_{dR}) \to \pi_1(M_{0,5}, \omega_{dR}) \to \pi_1(X, \omega_{dR}) \to 1 \tag{7.3.5}$$

given by Lemma 5.2.3, canonically identify the $\pi_{1,dR}(X_N, \omega(dR))$'s for all N. Note that Lie $\pi_{1,dR}(X_N, \omega(dR)) = \text{Lie}\ \langle\langle e_0, e_1, e_{p^{-N}}\rangle\rangle$ and the identification between the fundamental groups for varying N is the obvious one. We will denote the image of $e_{p^{-N}}$ in this identification by $e_{p^{-\infty}}$, in order to emphasize its independence of N.

Let $c(X_N)$ denote the frobenius invariant path on X_N. We have by functoriality $i_{N,*}({}_{1-p^N}c_{p^N}(X_N)) = {}_{(1-p^N,p^N)}c_{(p^N,p^N)}$ and hence

$$i_{N,*}({}_{\omega(dR)}e(dR)_{1-p^N} \cdot {}_{1-p^N}\mathring{c}_{p^N}(X_N) \cdot {}_{p^N}e(dR)_{\omega(dR)})$$

is equal to

$$_{\omega(dR)}e(dR)_{(1-p^N,p^N)} \cdot {}_{(1-p^N,p^N)}\mathring{c}_{(p^N,p^N)} \cdot {}_{(p^N,p^N)}e(dR)_{\omega(dR)}.$$

Therefore in order to compute (7.3.3), using (7.3.4), we have to compute the limit of the paths ${}_{1-p^N}\mathring{c}_{p^N}(X_N)$, viewed as group-like elements in $\mathbb{Q}_p\langle\langle e_0, e_1, e_{p^{-\infty}}\rangle\rangle$:

Proposition 7.3.1. *Viewed as an element of* $\mathbb{Q}_p\langle\langle e_0, e_1, e_{p-\infty}\rangle\rangle$, *the limit*

$$\lim_{N\to\infty} {}_{\omega(dR)}e(dR)_{1-p^N} \cdot {}_{1-p^N}\mathring{c}_{p^N}(X_N) \cdot {}_{p^N}e(dR)_{\omega(dR)},$$

in fact, lies in $\mathbb{Q}_p\langle\langle e_0, e_1\rangle\rangle$.

We will postpone the proof of this proposition to the end of this section, after first making some remarks on Coleman integration which will be needed in the proof. Let c denote the limit in the statement above. In the course of the proof we will omit the concatenation of paths with ${}_*e(dR)_{\omega(dR)}$ or ${}_{\omega(dR)}e(dR)_*$ from the notation, for simplicity.

If $m := e_{i_1}\cdots e_{i_n}$, with $e_{i_j} \in \{e_0, e_1, e_{p-\infty}\}$, we need to prove that if m includes $e_{p-\infty}$ as a factor then $c[m] = 0$.

Lemma 7.3.2. *In order to prove Proposition 7.3.1, it is sufficient to show that* $c[m] = 0$ *for any* m *which ends with* $e_{p-\infty}$.

Proof. Note that $\mathbb{Q}_p\langle\langle e_0, e_1, e_{p-\infty}\rangle\rangle$ is a complete Hopf algebra with the co-product given by

$$\Delta(e_i) = 1\otimes e_i + e_i \otimes 1, \quad \text{for } e_i \in \{e_0, e_1, e_{p-\infty}\},$$

and c is group-like, i. e., the constant term of c is 1 and

$$\Delta(c) = c\otimes c. \tag{7.3.6}$$

Now suppose that we know that $c[m_0 e_{p-\infty}] = 0$, for every monomial m_0 in the e_i's. Assume that $c[m' e_{p-\infty} m''] = 0$, if the length of m'' is less than or equal to n. In order to show that $c[m' e_{p-\infty} m''] = 0$ for m'' of length $n+1$, let us look at the coefficient of $(m' e_{p-\infty})\otimes m''$ in (7.3.6). This coefficient is $c[m' e_{p-\infty}]\cdot c[m''] = 0$ on the right hand side. On the left hand side, it is the sum of $c[m' e_{p-\infty} m'']$ and of terms of the form $c[m_1 e_{p-\infty} m_2]$ with length of m_2 less than $n+1$, and hence equal to 0 by the induction hypothesis. Therefore $c[m' e_{p-\infty} m''] = 0$. ☐

7.3.2. *Coleman integration.* Given $x \in X_N(\mathbb{Q}_p)$ consider the fundamental torsor of paths ${}_*P_{dR,x}(X_N)$ from x to a variable point. The frobenius invariant path ${}_*c_x(X_N)$ is a section of ${}_*P_{dR,x}(X_N)$ which induces the frobenius invariant path ${}_*\mathring{c}_x(X_N)$ of Besser [3] corresponding to the branch of the logarithm with $\log(p) = 0$. For each y we view ${}_yP_{dR,x}(X_N)(\mathbb{Q}_p)$ inside $\widehat{\mathcal{U}}_{dR}(X_N, \omega(dR)) = \mathbb{Q}_p\langle\langle e_0, e_1, e_{p-\infty}\rangle\rangle$, using the usual identification of (tangential) basepoints with the de Rham basepoint.

Note that if we fix a monomial m as above, then we get a map from the underlying analytic space $X_{N,an}$ of X_N to $\mathbb{C}_p$ given by

$$y \to {}_y\mathring{c}_x(X_N)[m],$$

which is only *locally* analytic. Since the section $\mathring{c}$, is frobenius invariant by construction, the map ${}_*\mathring{c}_x(X_N)[m]$ is induced by an *abstract Coleman function* in the sense of Definition 4.1 and Definition 4.10 of [3]. Let us clarify this below.

Let $D(a, r)$ and $D(a, r^-)$ denote, respectively, the closed and open discs of radius r around a and, for $0 < r < 1$, let

$$U_r := D(0, 1/r) \setminus (D(0, r^-)\cup D(1, r^-)).$$

Then following the notation of Besser [3], the proof of (Theorem 5.7, [3]) implies that there is an $0 < r < 1$ such that the restiction of ${}_*\mathring{c}_x(X_N)[m]$ to U_r gives an element in $M(U_r)$, the ring of functions on U_r which was originally defined by Coleman in [5]. Some of the properties of $M(U_r)$ is that:

(i) for every $f \in M(U_r)$, there is a $g \in M(U_r)$ such that $dg = fdz$;

(ii) if $g \in M(U_r)$ satisfies $dg = 0$, then g is constant.

(iii) if f is a rigid analytic function on U_r that does not vanish on U_r then $\log(f) \in M(U_r)$ (here we fix the branch of the logarithm with $\log(p) = 0$, (p.40, [3])).

Hence for any $x \in U_r$ and $f \in M(U_r)$, the notation $\int_x^* f dz$ can be defined as the unique function $g \in M(U_r)$ as in (i) above with the properties that $dg = f\,dz$ and $g(x) = 0$. As usual one defines the iterated integral

$$\int_x^y f_n dz \circ \cdots \circ f_2 dz \circ f_1 dz$$

for $f_i \in M(U_r)$ inductively as,

$$\int_x^y f_n dt \cdot \big(\int_x^t f_{n-1} dz \circ \cdots \circ f_2 dz \circ f_1 dz \big).$$

Lemma 7.3.3. *Let $m := e_{i_n} \cdots e_{i_2} e_{i_1}$ be a monomial. If $i \in \{0, 1, p^{-\infty}\}$, let $\omega_{i,N} := \omega_i = \frac{dz}{z-i}$, if $i = 0, 1$ and $\omega_{p^{-\infty},N} := \frac{dz}{z-p^{-N}}$ on X_N. Then on X_N,*

$${}_y\mathring{c}_x(X_N)[m] = \int_x^y \omega_{i_n,N} \circ \cdots \circ \omega_{i_2,N} \circ \omega_{i_1,N}$$

for $x, y \in U_r$.

Proof of the lemma. Fix $x \in U_r$ and think of both sides as functions of y. Since both sides are in $M(U_r)$, to see the equality one only needs to check that they have the same derivative because of the property (ii) above, which is easy to check for c is locally nothing other than the parallel transport. □

Let $D(a, r_1^-, r_2^-)$ denote the open annulus around a of points z such that $r_1 < |z - a| < r_2$. If $A(D(a, r_1^-, r_2^-))$ denotes the space of rigid analytic functions on $D(a, r_1^-, r_2^-)$, let

$$A_{log}(D(a, r_1^-, r_2^-)) := A(D(a, r_1^-, r_2^-))[\log(z - a)],$$

the polynomial ring in $\log(z - a)$ over $A(D(a, r_1^-, r_2^-))$. This ring has the following property: if $f \in A(D(a, r_1^-, r_2^-))$ such that f does not vanish on $D(a, r_1^-, r_2^-)$ then $\log(f) \in A_{log}(D(a, r_1^-, r_2^-))$, (Corollary 2.2.a, [5]).

By the local description of the frobenius invariant path that we gave in Example (i) of §6.1, if we fix x and consider ${}_z\mathring{c}_x(X_N)[m]$ as a function of $z \in D(1, 0^-, 1^-)$ then it is an element of $A_{log}(D(1, 0^-, 1^-))$. By the rigidity of rigid analytic functions the restriction

$$A_{log}(D(1, 0^-, 1^-)) \to A_{log}(D(1, r^-, 1^-))$$

is injective. Therefore the restriction of ${}_*\mathring{c}_x(X_N)[m]$ to $D(1, 0^-, 1^-)$ is completely determined by its restriction to $D(1, r^-, 1^-) \subseteq U_r$. Since we know that the restriction of ${}_*\mathring{c}_x(X_N)[m]$ to U_r is given by the iterated integral in $M(U_r)$ as above, this idea gives a way to compute ${}_z\mathring{c}_x(X_N)$ for $x \in U_r$ and z as close to 1 as desired.

Applying the same argument to ${}_x\mathring{c}_w(X_N)$ for w close to 0, using the injection

$$A_{log}(D(0, 0^-, 1^-)) \to A_{log}(D(0, r^-, 1^-))$$

gives a way to compute ${}_x\mathring{c}_w(X_N)$. Using the fact that ${}_z\mathring{c}_x(X_N) \cdot {}_x\mathring{c}_w(X_N) = {}_z\mathring{c}_w(X_N)$ this gives a way to compute ${}_z\mathring{c}_w(X_N)$ for z near 1 and w near 0.

We will now apply this idea to compute the limit of ${}_{1-p^N}\mathring{c}_{p^N}(X_N)$, which we denoted by c.

Let $m = e_{i_n} \cdots e_{i_2} e_{i_1}$. Then

$$ {}_y\mathring{c}_x(X_N)[m] = \int_x^y \omega_{i_n,N} \circ \cdots \circ \omega_{i_2,N} \circ \omega_{i_1,N}, $$

for $x, y \in U_r$ as in the lemma above. We continue to denote the unique extension of this iterated integral, which satisfies the logarithmic singularities condition near 0 and 1 above, to $D(0, 1/r) \setminus \{0, 1\}$ by the same notation. Then the expression of ${}_y\mathring{c}_x(X_N)[m]$ as an iterated integral is valid for all $x, y \in D(0, 1/r) \setminus \{0, 1\}$.

Notation 7.3.4. *For $\varphi \in A(D(0, 1/r))$, the ring of rigid analytic functions on $D(0, 1/r)$, and $\alpha \in D(0, 1/r)$ we let $P_\alpha(\varphi) \in A(D(0, 1/r))$ be the unique primitive of φ with $P_\alpha(\varphi)(\alpha) = 0$.*

We will need a couple of lemmas to finish the proof of the proposition.

Lemma 7.3.5. *Let $(\varphi_N(z))$ be a sequence of functions in $A(D(0, 1/r))$, which is uniformly convergent to 0 on $D(0, 1/r)$. Then the sequences*

$$ (P_0(\varphi_N)(z)),\ \left(\frac{P_0(\varphi_N)(z)}{z - p^{-N}}\right) \ \text{ and } \ \left(\frac{P_a(\varphi_N)(z)}{z - a}\right) $$

for $a = 0, 1$, are uniformly convergent to 0 on $D(0, 1/r)$.

Proof of the lemma. If $f(z) = \sum_{n\geq 0} a_n z^n \in A(D(0, 1))$, then $\sup_{D(0,1)}|f(z)| = \max_{n\geq 0}|a_n|$, ([28]). Using this and noting that $D(0, 1/r) = D(1, 1/r)$, easily gives the statements above. □

Let us look at the following two statements that both depend on a nonnegative integer k :

$A(k)$: If $(\varphi_N(z))$ is a sequence of functions in $A(D(0, 1/r))$ which uniformly converges to 0 on $D(0, 1/r)$ and if $m := e_{i_n} \cdots e_{i_1}$ is any monomial in which $e_{p^{-\infty}}$ appears less than or equal to $k-$times then

$$ \lim_{N\to\infty} \int_{p^N}^{1-p^N} \omega_{i_n,N} \circ \cdots \circ \omega_{i_1,N} \circ \varphi_N(z)dz = 0. $$

$B(k)$: If $m := e_{i_n} \cdots e_{i_1}$ is any monomial in which $e_{p^{-\infty}}$ appears at least once and less than or equal to k-times then

$$ \lim_{N\to\infty} \int_{p^N}^{1-p^N} \omega_{i_n,N} \circ \cdots \circ \omega_{i_1,N} = 0, $$

i.e. $c[m] = 0$.

The proposition we are proving can be restated as $B(k)$, for all $k \geq 1$. We will prove this by proving the following three staments:

(i) $A(0)$ is true.

(ii) $A(k)$ implies $B(k + 1)$, for all $k \geq 0$.

(iii) $B(k)$ implies $A(k)$, for all $k \geq 1$.

Claim 7.3.6. $A(0)$ *is true.*

Proof of the claim. We will prove this by induction on the length of m, which does not contain any $e_{p^{-\infty}}$'s.

The $m = 1$ case follows immediately from $\int_{p^N}^{1-p^N} \varphi_N(z)dz = P_0(\varphi_N)(1 - p^N) - P_0(\varphi_N)(p^N)$, which converges to 0 since $P_0(\varphi_N)$ converges uniformly to 0 by Lemma 7.3.5.

Let $m = e_{i_n} \cdots e_{i_1}$, not contain any $e_{p^{-\infty}}$'s, and asume that we know the statement for m of length less than n. Then note that

$$\int_{p^N}^{1-p^N} \omega_{i_n} \circ \cdots \omega_{i_2} \circ \omega_{i_1} \circ \varphi_N(z)dz = \int_{p^N}^{1-p^N} \omega_{i_n} \circ \cdots \omega_{i_2} \circ (\omega_{i_1} P_{p^N}(\varphi_N)(z)),$$

and using $P_{p^N}(\varphi_N)(z) = P_{i_1}(\varphi_N)(z) + P_0(\varphi_N)(i_1) - P_0(\varphi_N)(p^N)$, the limit we need to compute turns out to be

$$\int_{p^N}^{1-p^N} \omega_{i_n} \circ \cdots \omega_{i_2} \circ \frac{P_{i_1}(\varphi_N)(z)}{z - i_1} + (P_0(\varphi_N)(i_1) - P_0(\varphi_N)(p^N)) \int_{p^N}^{1-p^N} \omega_{i_n} \circ \cdots \circ \omega_{i_1}.$$

The limit of the first integral as $N \to \infty$ is zero by the induction hypothesis since $(\frac{P_{i_1}(\varphi_N)(z)}{z-i_1})$ is uniformly convergent to zero by Lemma 7.3.5. By the same lemma the coefficient in front of the integral $\int_{p^N}^{1-p^N} \omega_{i_n} \circ \cdots \circ \omega_{i_1}$ goes to zero as well. So all that remains is to compute the limit of the last integral. However by the description of the limit of the frobenius invariant path in Lemma 6.6.1, we immediately see that

$$\lim_{N \to \infty} \int_{p^N}^{1-p^N} \omega_{i_n} \circ \cdots \circ \omega_{i_1} = {}_{t_{10}}c_{t_{01}}[m],$$

where as usual ${}_{t_{10}}c_{t_{01}}$ denotes the frobenius invariant path from the standard tangent vector at 0 to the one at 1. □

Claim 7.3.7. *$A(k)$ implies $B(k+1)$, for all $k \geq 0$.*

Proof of the claim. Let $m := e_{i_n} \cdots e_{i_2} e_{i_1}$ be a monomial that contains less than or equal to $k+1$ $e_{p^{-\infty}}$'s. In order to show that $c[m] = 0$, we will assume without loss of generality by the proof of Lemma 7.3.2 that $e_{i_1} = e_{p^{-\infty}}$. Then letting $m = m' e_{p^{-\infty}}$, and $\varphi_N(z) = \frac{1}{z - p^{-N}}$, which is uniformly convergent to 0 on $D(0, 1/r)$ as $N \to \infty$, and applying $A(k)$ to m' and $(\varphi_N(z))$ proves the claim. □

Claim 7.3.8. *$B(k)$ implies $A(k)$ for all $k \geq 1$.*

Proof of the claim. Let $m := e_{i_n} \cdots e_{i_1}$ be a monomial in which $e_{p^{-\infty}}$ appears less than or equal to k times. We will prove $A(k)$, assuming $B(k)$, by induction on the length of m. Let $(\varphi_N(z))$ be a sequence of functions in $A(D(0, 1/r))$ uniformly convergent to 0.

If $m = 1$ then the statement follows from $\int_{p^N}^{1-p^N} \varphi_N(z)dz = P_0(\varphi_N)(1 - p^N) - P_0(\varphi_N)(p^N)$.

In general let m be as above. We are looking at the limit of

$$\int_{p^N}^{1-p^N} \omega_{i_n,N} \circ \cdots \omega_{i_1,N} \circ \varphi_N(z)dz.$$

If $i_1 \in \{0, 1\}$ then proceeding exactly as in Claim 7.3.6, using the induction hypothesis and the fact that $\int_{p^N}^{1-p^N} \omega_{i_n,N} \circ \cdots \circ \omega_{i_1,N}$ goes to 0 (because of $B(k)$) shows the claim.

In case $i_1 = p^{-\infty}$ then $\int_{p^N}^{1-p^N} \omega_{i_n,N} \circ \cdots \omega_{i_1,N} \circ \varphi_N(z)dz =$

$$\int_{p^N}^{1-p^N} \omega_{i_n,N} \circ \cdots \circ \omega_{i_2,N} \circ (\frac{P_0(\varphi_N)(z)}{z - p^{-N}} dz) - (P_0(\varphi_N)(p^N)) \int_{p^N}^{1-p^N} \omega_{i_n,N} \circ \cdots \circ \omega_{i_1,N},$$

and exactly as above this goes to 0. □

Proof of Proposition 7.3.1. The statement in the proposition is equivalent to $B(k)$ for all k. $B(k)$ follows from Claim 7.3.6, 7.3.7 and 7.3.8. □

Since
$$\widehat{\mathcal{U}}(X_N, \omega(dR)) = \mathbb{Q}_p\langle\langle e_0, e_1, e_{p^{-\infty}}\rangle\rangle$$
and
$$\widehat{\mathcal{U}}(X, \omega(dR)) = \mathbb{Q}_p\langle\langle e_0, e_1\rangle\rangle,$$
the obvious inclusion $\mathbb{Q}_p\langle\langle e_0, e_1\rangle\rangle \subseteq \mathbb{Q}_p\langle\langle e_0, e_1, e_{p^{-\infty}}\rangle\rangle$ induces a map

$$i_* : \pi_{1,dR}(X, \omega_{dR}) \to \pi_{1,dR}(X_N, \omega_{dR}) \to \pi_{1,dR}(M_{0,5}, \omega_{dR}). \tag{7.3.7}$$

We would like to stress that this map is *not* induced by a map of varieties.

Corollary 7.3.9. *We have*

$$i_*({}_{\omega_{dR}}e(dR)_{t_{10}}\, {}_{t_{10}}c_{t_{01}}\, {}_{t_{01}}e(dR)_{\omega_{dR}}) = {}_{\omega_{dR}}e(dR)_s\, {}_s c_t\, {}_t e(dR)_{\omega_{dR}}. \tag{7.3.8}$$

In other words, ${}_s c_t$ is the image of the frobenius invariant path between the tangential basepoints at 0 and 1 in X.

Proof. By Proposition 7.3.1, we see that the right hand side of the equation above lies inside the image of i_*. If $\pi : M_{0,5} \to X$ denotes the map that sends (z_1, z_2) to z_1 then $\pi_* \circ i_*$ is the identity map. Since π_* maps ${}_s c_t$ to ${}_{t_{10}}c_{t_{01}}$, the corollary follows. □

Notation 7.3.10. *Let $Z = M_{0,4}$ or $M_{0,5}$. If a and b are tangent vectors in the standard compactification of Z, we let*

$${}_b g_a := {}_{\omega_{dR}}e(dR)_b \cdot F_*({}_b e(dR)_a) \cdot {}_a e(dR)_{\omega_{dR}}.$$

Corollary 7.3.11. *We have*

$$i_*({}_{t_{10}}g_{t_{01}}) = {}_s g_t.$$

Proof. Because of the functoriality of g with respect to $\pi : M_{0,5} \to X$, as in the proof of the corollary above, all we need to see is that the right hand side lies in the image of i_*.

Using the description (7.3.2), let $\pi_2 : M_{0,5} \to X$ denote the second projection. First, we would like to see that ${}_s g_t$ maps to 1 under $\pi_{2,*} : \pi_{1,dR}(M_{0,5}, \omega_{dR}) \to \pi_{1,dR}(X, \omega_{dR})$. Fix an N_0, and let $t(N_0)$ and $s(N_0)$ be the standard tangent vectors at 0 and 1 on X_{N_0}, that correspond to t_{01} and t_{10} under $X_{N_0} \subseteq X$. Then $\pi_{2,*}({}_s g_t) = \pi_{2,*}({}_s g_{s(N_0)} \cdot {}_{s(N_0)}g_{t(N_0)} \cdot {}_{t(N_0)}g_t) = {}_{t_{01}}g_{p^{N_0}} \cdot {}_{p^{N_0}}g_{t_{01}} = 1$.

Therefore, ${}_s g_t \in \mathbb{Q}_p\langle\langle e_0, e_1, e_{p^{-\infty}}\rangle\rangle$. Let $e_i' := {}_t e(dR)_{\omega_{dR}} \cdot e_i \cdot {}_{\omega_{dR}}e(dR)_t$. Then, noting that pulling back by frobenius multiplies residues by p, we have

$$F_*(e_0') = pe_0', \text{ and } F_*(e_1') = ({}_s g_t')^{-1} \cdot pe_1' \cdot {}_s g_t', \tag{7.3.9}$$

where g' denotes the image of g under the obvious identification $\mathbb{Q}_p\langle\langle e_0, e_1, e_{p^{-\infty}}\rangle\rangle \simeq \mathbb{Q}_p\langle\langle e_0', e_1', e_{p^{-\infty}}'\rangle\rangle$.

On the other hand, by Proposition 7.3.1,

$${}_s c_t' := {}_t e(dR)_s \cdot {}_s c_t \in \mathbb{Q}_p\langle\langle e_0', e_1'\rangle\rangle, \tag{7.3.10}$$

and

$$F_*({}_s c_t') = ({}_s g_t')^{-1} \cdot {}_s c_t'. \tag{7.3.11}$$

Suppose that there is a monomial m such that $({}_s g_t')^{-1}[e_{p^{-\infty}}' m] \neq 0$, and let m_0 be such a monomial of smallest length. Then $F_*({}_s c_t')[e_{p^{-\infty}}' m_0] \neq 0$ by (7.3.11). On the other hand, (7.3.9) implies that $F_*({}_s c'_t)[e'_{p-\infty} m_0] = 0$. Therefore $({}_s g'_t)^{-1}[e'_{p-\infty} m] = 0$ for every monomial m, and this implies that $({}_s g'_t)^{-1}$ does not depend on $e'_{p-\infty}$ by the argument in the proof of Lemma 7.3.2. This implies that ${}_s g'_t$ does not depend on $e'_{p-\infty}$ and hence ${}_s g_t$ does not depend on $e_{p-\infty}$. □

Using the notation in §6.8, the previous corollary can be restated as:

Corollary 7.3.12. *Using the coordinates on $M_{0,5}$ as in (7.3.2), let D_0 and D_1 denote the divisors on $\overline{M}_{0,5}$ that correspond to $z_1 = 0$ and $z_1 = 1$, respectively. Then*

$$ {}_s g_t = g(e_{D_0}, e_{D_1}). \tag{7.3.12}$$

7.3.3. *Proof of the 5-cycle relation.* Following the notation in §5.4, we choose tangential basepoints $t_{i_1 i_2, j_1 j_2}$ at $x_{i_1 i_2, j_1 j_2}$ on $\overline{M}_{0,5}$. Notation 7.3.10 then defines elements:

$$ {}_{t_{k_1 k_2, l_1 l_2}} g_{t_{i_1 i_2, j_1 j_2}} \in \pi_{1,dR}(M_{0,5}, \omega(dR)). \tag{7.3.13}$$

Lemma 7.3.13. *The elements in (7.3.13) depend only on the points $x_{i_1 i_2, j_1 j_2}$ and not on the tangent vectors at these points.*

Proof. If u and v are any two of the four tangent vectors above chosen at $x_{i_1 i_2, j_1 j_2}$ then u and v satisfy the hypothesis of Lemma 6.7.1 and hence we have ${}_v c_u = {}_v e(dR)_u$. This gives that $F_*({}_v e(dR)_u) = F_*({}_v c_u) = {}_v c_u = {}_v e(dR)_u$, and hence

$$ {}_v g_u = {}_{\omega_{dR}} e(dR)_v \cdot F_*({}_v e(dR)_u) \cdot {}_u e(dR)_{\omega_{dR}} = 1. $$

□

Because of the previous lemma, it makes sense to let

$$ {}_{k_1 k_2, l_1 l_2} g_{i_1 i_2, j_1 j_2} := {}_{t_{k_1 k_2, l_1 l_2}} g_{t_{i_1 i_2, j_1 j_2}}. $$

Claim 7.3.14. *We have*

$$ {}_{i_1 k_2, j_1 j_2} g_{i_1 i_2, j_1 j_2} = g(e_{i_1 i_2}, e_{i_1 k_2}). $$

Proof. Consider the isomorphism

$$ M_{0,5} \to (\mathbb{G}_m \setminus \{1\})^2 \setminus \{(z_1, z_2) | z_1 z_2 = 1\}, $$

given by sending the point which has representative $\{x_{i_1}, x_{i_2}, x_{j_1}, x_{j_2}, x_{k_2}\}$ to

$$ \left(\frac{x_{i_1} - x_{i_2}}{x_{i_1} - x_{j_1}} \cdot \frac{x_{k_2} - x_{j_1}}{x_{k_2} - x_{i_2}}, \frac{x_{j_2} - x_{j_1}}{x_{j_2} - x_{i_2}} \cdot \frac{x_{k_2} - x_{i_2}}{x_{k_2} - x_{j_1}}\right). $$

Under this isomorphism ${}_{i_1 k_2, j_1 j_2} g_{i_1 i_2, j_1 j_2}$ maps to

$$ {}_s g_t = g(e_{D_0}, e_{D_1}) = g(e_{i_1 i_2}, e_{i_1 k_2}). $$

□

If we apply F_* to the identity:

$$ {}_{34,01} e(dR)_{01,23} \cdot {}_{23,01} e(dR)_{40,23} \cdot {}_{40,23} e(dR)_{40,12} \cdot {}_{40,12} e(dR)_{12,34} \cdot {}_{12,34} e(dR)_{01,34} = 1 $$

and use Claim 7.3.14, we obtain the 5-cycle relation:

$$ g(e_{23}, e_{34}) g(e_{40}, e_{01}) g(e_{12}, e_{23}) g(e_{34}, e_{40}) g(e_{01}, e_{12}) = 1. $$

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