

INFINITESIMAL CHOW DILOGARITHM

SİNAN ÜNVER

Abstract. Let C_2 be a smooth and projective curve over the ring of dual numbers of a field k . Given non-zero rational functions f, g , and h on C_2 , we define an invariant $\rho(f \wedge g \wedge h) \in k$. This is an analog of the real analytic Chow dilogarithm and the extension to non-linear cycles of the additive dilogarithm of [11]. Using this construction we state and prove an infinitesimal version of the strong reciprocity conjecture [5]. Also using ρ , we define an infinitesimal regulator on algebraic cycles of weight two which generalizes Park's construction in the case of cycles with modulus [8].

1. INTRODUCTION

Let k be a field of characteristic 0 and $k_n := k[t]/(t^n)$, with k_2 being the ring of dual numbers over k . One expects an abelian category of motives over any scheme over k , and in particular over k_2 . Goncharov while considering the degeneration of motives attached to hyperbolic manifolds of odd dimension expected that motives over the dual numbers should have properties similar to euclidean scissors congruence class groups [4, p. 616]. Bloch and Esnault further expected that the real analytic dilogarithm should degenerate to an additive dilogarithm which would give an analog of the volume map on the euclidean scissors congruence class in three dimensions. This insight was realised in the construction of an additive dilogarithm for the weight two motivic cohomology complex based on the localisation sequence in their fundamental paper [2]. Later an additive dilogarithm for the Bloch complex was constructed in [11]. In a precise sense, the current paper is an extension of the construction of [11] to the non-linear case. The construction in this non-linear case is much more involved and requires many new ideas. In the introduction, we will try to explain the main ideas behind the construction.

Suppose that X/\mathbb{C} is a smooth, projective curve over \mathbb{C} and f_1, f_2 , and f_3 are rational functions on X . Letting

$$r_2(f_1, f_2, f_3) := \text{Alt}_3\left(\frac{1}{6} \log |f_1| \cdot d \log |f_2| \wedge d \log |f_3| - \frac{1}{2} \log |f_1| \cdot d \arg f_2 \wedge d \arg f_3\right),$$

with the property that $dr_2(f_1, f_2, f_3) = \text{Re}(\text{Alt}_3(d \log(f_1) \wedge d \log(f_2) \wedge d \log(f_3)))$, one has, up to a constant multiple, the Chow dilogarithm map [5, p. 4]: $\rho_{\mathbb{R}} : \Lambda^3 \mathbb{C}(X)^{\times} \rightarrow \mathbb{R}$ given by

$$(1.0.1) \quad \rho_{\mathbb{R}}(f_1 \wedge f_2 \wedge f_3) := \int_{X(\mathbb{C})} r_2(f_1, f_2, f_3).$$

The main construction of this paper is the analog ρ of this map for \mathbb{R} replaced with k_2 , which is expected in the spirit of the above discussion about the analogy between mixed tate motives over k_2 and euclidean scissors congruence class groups. In the main set-up of this paper X/\mathbb{C} is replaced with a smooth and projective curve C_2/k_2 .

The main challenge in constructing such a map is to find an analog of the integral (1.0.1) for a curve C_2/k_2 . A related problem is that the integrand involves taking the real part of an algebraic form. Therefore, we have to find an algebraic construction in the context of motives over k_2 which will be an analog of this. We have encountered such a function in [11]: $\log^{\circ} : k[[t]]^{\times} \rightarrow k[[t]]$ defined as $\log^{\circ}(a) := \log(a/a(0))$. We would like to think of \log° as the analog of a branch of the logarithm or as the real part of the logarithm. In order to find an expression which is an analog of (1.0.1), we are again motivated by the construction in [11]: the construction in this paper restricted to the case of $\mathbb{P}_{k_2}^1$ and a triple of linear fractional transformations gives the

construction in [11]. The precise statement can be found in Theorem 3.5.6. Even though both the arguments in [11] and the ones in the current paper use the idea of liftings of the functions to some thickening of k_2 , the approach in [11] is computational, here it is more geometric. We briefly outline the main idea.

In [11], we defined the additive dilogarithm $li_2 : B_2(k_2) \rightarrow k$, by $li_2(\{s + at\}_2) = -\frac{a^3}{2s^2(1-s)^2}$. In order to show that this function indeed satisfies the functional equation for the dilogarithm and deduce its basic properties, we interpreted this formula as follows. Let $\ell : \Lambda_{\mathbb{Z}}^2 k[[t]]^{\times} \rightarrow k$ denote the map (2.4.5), which in this case takes the form $\ell(u \wedge v) := \log^{\circ}(u)[t^2] \cdot \log^{\circ}(v)[t] - \log^{\circ}(u)[t] \cdot \log^{\circ}(v)[t^2]$, where for $a \in k[[t]]$, $a[t^n]$ denotes the coefficient of t^n in a . Then the map $B_2(k[[t]]) \rightarrow k$ defined by sending $\{\alpha\}_2$ to $\ell(1 - \alpha) \wedge \alpha$, factors through the canonical projection $B_2(k[[t]]) \rightarrow B_2(k_2)$, to give li_2 .

Let us now turn to the context of this paper of a curve C_2/k_2 . When considering general rational functions on C_2 , we put a slight restriction on the type of singularities that our functions can have. We allow those functions on C_2 which are locally a product of a unit and a power of an element that reduces to a uniformizer on the closed fiber. The precise definition is in Definition 3.1.1. For a choice of liftings of uniformizers \mathcal{P}_2 , we call a rational function on C_2 , \mathcal{P}_2 -good in §3.2, if it satisfies this hypothesis with respect to the uniformizers in \mathcal{P}_2 . Letting $k(C_2, \mathcal{P}_2)^{\times}$ denote the set of rational functions which are \mathcal{P}_2 -good, the infinitesimal Chow dilogarithm ρ is a map $\rho : \Lambda^3 k(C_2, \mathcal{P}_2)^{\times} \rightarrow k$.

The main idea in constructing ρ is as follows. If C_2/k_2 has a global lifting $\tilde{C}/k[[t]]$ to a smooth and projective curve and the functions (f, g, h) have global liftings $(\tilde{f}, \tilde{g}, \tilde{h})$ to functions on \tilde{C} which are good with respect to a system of uniformizers $\tilde{\mathcal{P}} := \{\tilde{\pi}_c | c \in |\tilde{C}_s|\}$ on \tilde{C} that lift \mathcal{P}_2 then

$$(1.0.2) \quad \rho(f, g, h) = \sum_{c \in |\tilde{C}_s|} \text{Tr}_k(\ell(\text{res}_{\tilde{\pi}_c}(\tilde{f} \wedge \tilde{g} \wedge \tilde{h}))),$$

where $\text{res}_{\tilde{\pi}_c}$ is the residue map (2.4.4).

The existence of a global lifting of C_2 together with good liftings of the functions f, g, h is too much to ask for. Even if we knew that such liftings exist, we need to show that the expression (1.0.2) is independent of the choices of the liftings which is a non-trivial task. In the following, we do not assume the existence of any lifting. Instead, we choose local liftings of C_2 at all of its points, including the generic point. These are known to exist by the smoothness assumption. On the generic lifting we choose arbitrary liftings of the functions; on the local liftings we choose good liftings of the functions. The general expression for ρ is then given in Definition 3.2.2. Sections 3.1 and 3.2 are devoted to showing that this sum makes sense and is independent of all the choices.

In Definition 3.2.2, there is a differential form ω which measures the defect of choosing different liftings. We would like to think of different liftings as the analogs of the different branches of the functions under consideration and ω as measuring the difference between the choice of two different branches. This differential is in some sense the main technical object. The map ω attaches a 1-form to the following data: smooth affine schemes $\mathcal{A}_i/k[[t]]$ of relative dimension one, an isomorphism $\chi : \mathcal{A}_1/(t^2) \xrightarrow{\sim} \mathcal{A}_2/(t^2)$ and triples of functions (f_i, g_i, h_i) in \mathcal{A}_i^{\times} whose reductions modulo t^2 map to each other via χ . The 1-form is given by the formula (2.3.1). However, this description depends on many choices and it is of fundamental importance for the construction of the regulator ρ to show that this construction of ω is independent of all the choices. These are proven in Lemma 2.3.1 and Lemma 2.3.3. It is at this point that we have to assume that our objects have dimension at most 1. The statements about independence of the choices are not true without this assumption. The residue property (P8) in §2.4.5 allows us to compute the residue of ω when both of its arguments are good liftings. This property is used when proving that the regulator ρ does not depend on the choice of local liftings in Proposition 3.1.2.

The proof of the infinitesimal version of the strong reciprocity conjecture uses the relative Bloch group relation (P11). In order to state this relation, we first construct a relative version of the standard Bloch group B_2 , and a relative version of the part of the weight three Bloch complex

in degrees 2 and 3. The Proposition 2.4.5 then states that ω defines a map from the Milnor part of the relative Bloch complex of weight three. Here and the discussions below regarding the strong reciprocity conjecture, we would like to emphasize that we consider only the part of the Bloch complex of weight three. There is in fact a term in degree one which contains a version of the Bloch group B_3 , but this term has no consequence for the present paper since the strong reciprocity only refers to the terms of degree 2 and 3. Defining the analog of B_3 would take us too far outside of the material of this paper, this question will be dealt with in more generality in a future work.

The construction of the infinitesimal Chow dilogarithm with the desired properties is in fact more or less equivalent to the infinitesimal version of the strong reciprocity conjecture of Goncharov [5], by using the results of [11] on the structure of $B_2(k_2)$:

Theorem 3.4.4. *The infinitesimal part of the residue map*

$$\text{res}_{|C|} : \Gamma(k(C_2, \mathcal{P}_2), 3) \rightarrow \Gamma(k, 2)^\circ$$

is homotopic to 0. More precisely, there is a map $h : \Lambda^3 k(C_2, \mathcal{P}_2)^\times \rightarrow B_2(k_2)^\circ$ which makes the diagram

$$\begin{array}{ccc} B_2(k(C_2, \mathcal{P}_2)) \otimes k(C_2, \mathcal{P}_2)^\times & \xrightarrow{\Delta} & \Lambda^3 k(C_2, \mathcal{P}_2)^\times \\ \downarrow \text{res}_{|C|} & \nearrow h & \downarrow \text{res}_{|C|} \\ B_2(k_2)^\circ & \xrightarrow{\delta^\circ} & (\Lambda^2 k_2^\times)^\circ \end{array}$$

commute and has the property that $h(k_2^\times \wedge \Lambda^2 k(C_2, \mathcal{P}_2)^\times) = 0$.

In the last section we use the discussions of this paper to construct a regulator for algebraic cycles of weight two over $k[[t]]$. That this regulator vanishes on boundaries is a direct consequence of the definition. We prove that if two smooth cycles are the same modulo t^2 then they have the same regulator value and also we prove that the regulator vanishes on products. Finally in §4.4.5, we compare our construction in the case of cycles to the construction of Park in the case of cycles with modulus 2. Here we emphasize that we use the strong sup modulus when defining the additive Chow groups.

There are many questions that are left unanswered in this paper. The most important one in our opinion is the generalization of the above construction to higher weights, moduli and to characteristic p . In characteristic p , even in weight two and modulus two, we expect a new component for our Chow dilogarithm which is of arithmetic nature. Another fundamental question is to understand the relation of the theory of this paper with that of [6], which was in fact one of our main motivations for this paper. These will be the subjects of future work. A related question is to study the regulator on algebraic cycles in more detail and define an equivalence relation on cycles over $k[[t]]$ which would give a construction of the motivic cohomology complex for truncated polynomial rings in terms of algebraic cycles. We will pursue this approach in our joint work with J. Park.

Outline. The paper is organized as follows. After a recollection of basic facts about splittings in §2.1, in the remaining part of §2, we construct the map ω . Its uniqueness is proven in §2.2, and existence in §2.3. The fundamental properties of ω are proven in §2.4.

In §3.1 and §3.2, we give the construction of the regulator ρ . In §3.3 and 3.4, we prove the infinitesimal version of the strong reciprocity conjecture. In §3.5, we relate the construction of this paper to that of [11].

In §4, we apply the construction to the case of algebraic cycles of weight two over $k[[t]]$, and construct an infinitesimal regulator on these cycles.

Notation. For a functor F from k -algebras to abelian groups, let $F(k_2)^\circ$ denote the direct summand of $F(k_2)$ which is the kernel of the projection $F(k_2) \rightarrow F(k)$ corresponding to the k -algebra homomorphism $k_2 = k[t]/(t^2) \rightarrow k$ which sends t to 0. If \mathcal{A} is a $k[[t]]$ algebra then we denote $\mathcal{A}/(t^n)$ by $\mathcal{A}|_{t^n}$. Similarly, if $f \in \mathcal{A}$, then we let $f|_{t^n}$ denote its image in $\mathcal{A}|_{t^n}$.

Acknowledgements. The author thanks J. Park very much for many discussions related to the contents of this paper, especially regarding §4. He thanks H. Esnault for mathematical

discussions and for being a wonderful host to him as an Humboldt Experienced Researcher at Freie Universität, where part of this work was carried out. Finally, he would like to thank the Humboldt Foundation for support.

2. ALGEBRAIC CONSTRUCTION AND PROPERTIES OF THE CANONICAL 1-FORM

In this section we would like to present an algebraic construction of the 1-form that we referred to above. We give this construction in a slightly more general context than is necessary for our purposes for the reasons that it makes the construction more transparent and this general construction will be needed for future generalisations of this work.

2.1. Splittings. In the first subsection, we review the notion of a *splitting*, the existence of which is guaranteed by the infinitesimal lifting properties of smooth affine schemes. This notion enables us to make explicit computations by choosing local coordinates.

For a ring A and an ideal I of A , let $\underline{A} := A/I$, a *splitting* of the projection $\pi : A \rightarrow \underline{A}$, is a map $\varphi : \underline{A} \rightarrow A$ such that $\pi \circ \varphi = id$. In the special case that we are interested in, the existence of a splitting is guaranteed by Grothendieck [7, III-Corollaire 5.6]:

Lemma 2.1.1. *Let k be a field, A a noetherian k -algebra, and $I := (x)$ an ideal of A , such that x is not a zero-divisor in A , A is complete with respect to the I -adic topology and \underline{A} is smooth over k . Then there is a continuous isomorphism $\underline{A}[[t]] \rightarrow A$, which sends t to the element x and such that the composition $\underline{A} \hookrightarrow \underline{A}[[t]] \rightarrow A \rightarrow \underline{A}$ is the identity map on \underline{A} .*

Remark 2.1.2. (i) It is easy to see that for any ring B , there is a 1-1 correspondence between splittings $\varphi : B \rightarrow B[[t]]$ of $\pi : B[[t]] \rightarrow B$ and automorphisms $\alpha : B[[t]] \rightarrow B[[t]]$ such that $B \hookrightarrow B[[t]] \xrightarrow{\alpha} B[[t]] \xrightarrow{\pi} B$ is the identity and $\alpha(t) = t$.

(ii) Using Lemma 2.1.1, the above remark generalizes to any A and I which satisfy the hypothesis of the lemma. Namely, there is a 1-1 correspondence between splittings $\varphi : \underline{A} \rightarrow A$ of $\pi : A \rightarrow \underline{A}$ and isomorphisms $\alpha : \underline{A}[[t]] \rightarrow A$ such that $\alpha(t) = x$, and $\underline{A} \hookrightarrow \underline{A}[[t]] \xrightarrow{\varphi} A \xrightarrow{\pi} \underline{A}$ is the identity. Given φ the corresponding α is obtained by restricting to \underline{A} .

(iii) The splittings behave functorially with respect to localisation. More precisely, if $\varphi : \underline{A} \hookrightarrow A$ is a splitting and $g \in \underline{A}$, then φ induces a splitting $\varphi_g : \underline{A}_g \hookrightarrow A_{\varphi(g)}$.

We will use the second remark above constantly in order to choose a local system of coordinates. Given a splitting φ , there is a unique isomorphism $\bar{\varphi} : \underline{A}[[t]] \rightarrow A$ as above such that $\bar{\varphi}|_{\underline{A}} = \varphi$. We will use $\bar{\varphi}^{-1} : A \rightarrow \underline{A}[[t]]$, as a local chart.

In general, the splittings as above are not unique. We prove uniqueness in a case that we need, in order to handle the 0-dimensional cycles. This result is well-known, but we give a proof since it is easier than finding a suitable reference.

Lemma 2.1.3. *Suppose that k is a perfect field, and A and I are as in Lemma 2.1.1 with \underline{A} a finite extension of k . Then there exists a unique splitting $\varphi : \underline{A} \rightarrow A$.*

Proof. Since \underline{A} is smooth over k , the existence follows from Lemma 2.1.1. For any k -algebra \tilde{B} with a square-zero ideal I , and a k -algebra homomorphism $\phi : \underline{A} \rightarrow \tilde{B}/I$, the set of liftings of ϕ to a k -algebra homomorphism $\tilde{\phi} : \underline{A} \rightarrow \tilde{B}$ is a pseudo-torsor under $Hom_{\underline{A}}(\Omega_{\underline{A}/k}^1, I)$. Since under the assumptions $\Omega_{\underline{A}/k}^1 = 0$, there is at most one lifting. Applying this inductively to the rings $\tilde{B} = A/I^n$, starting with the identity map for $n = 1$ gives the result. \square

2.2. Uniqueness of the canonical 1-form ω . Let $A := k[[t]]$, we will be dealing with a full subcategory \mathcal{C}_A^1 of the category of A -algebras. The objects of this category consist of A -algebras \mathcal{A} , which are smooth of relative dimension 0 or 1, are an integral domain and have the property that the residue fields of the closed points of $\mathcal{A}/(t)$ are one of the following type: a finite extension of k , if the relative dimension is 1; a finitely generated field of transcendence degree 1 over k , or a field of Laurent series $k'((s))$ over a finite extension k' of k , if the relative dimension is 0. Note that the condition on the residue fields of the closed points is satisfied if \mathcal{A} is an algebra of finite type over A or is the localisation or completion at a point in the closed fiber of such an algebra.

If \mathcal{A} is an object of \mathcal{C}_A^1 , we denote the quotient $\mathcal{A}/(t)$ by $\underline{\mathcal{A}}$. If we have an isomorphism $\chi : \tilde{\mathcal{A}}|_{t^2} \rightarrow \hat{\mathcal{A}}|_{t^2}$, and a pair (\tilde{p}, \hat{p}) , where $\tilde{p} := (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\hat{p} := (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ such that $\tilde{y}_i \in \tilde{\mathcal{A}}^\times$, $\hat{y}_i \in \hat{\mathcal{A}}^\times$, and $\chi(\tilde{y}_i|_{t^2}) = \hat{y}_i|_{t^2}$ in $\hat{\mathcal{A}}|_{t^2}$, for $1 \leq i \leq 3$, then we will construct a canonical element in $\Omega_{\underline{\mathcal{A}}/k}^1$. We describe this construction in detail in the next few sections.

Given an isomorphism $\chi : \tilde{\mathcal{A}}|_{t^2} \rightarrow \hat{\mathcal{A}}|_{t^2}$, we let $\beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi)$ denote the abelian group of pairs $(\tilde{y}, \hat{y}) \in \tilde{\mathcal{A}}^\times \times \hat{\mathcal{A}}^\times$ such that $\chi(\tilde{y}|_{t^2}) = \hat{y}|_{t^2}$. We let $\times^n \beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi)$ denote the n -fold cartesian product of $\beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi)$ with itself. Note that this last group can also be thought of as the group of pairs (\tilde{p}, \hat{p}) of n -tuples of elements of $\tilde{\mathcal{A}}^\times$ and $\hat{\mathcal{A}}^\times$ such that if $\tilde{p} = (\tilde{y}_1, \dots, \tilde{y}_n)$ and $\hat{p} = (\hat{y}_1, \dots, \hat{y}_n)$, then $\chi(\tilde{y}_i|_{t^2}) = \hat{y}_i|_{t^2} \in \hat{\mathcal{A}}|_{t^2}$, for all $1 \leq i \leq n$. We abbreviate this last condition as $\chi(\tilde{p}|_{t^2}) = \hat{p}|_{t^2}$. We will freely switch between these two notations without mentioning it.

We will construct a map

$$\omega(\cdot, \cdot, \chi) : \times^3 \beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi) \rightarrow \Omega_{\underline{\mathcal{A}}/k}^1,$$

with the following properties.

(P1) *Normalising condition/definition.* Let k'/k be a finite extension. Suppose that $\hat{\mathcal{A}} = \tilde{\mathcal{A}} = k'((s))[[t]]$ and $\chi = id$. Then because of the assumptions $\hat{y}_i - \tilde{y}_i \in (t^2)$, for $1 \leq i \leq 3$. Therefore, we can write uniquely $\hat{y}_i = \alpha_{0i} e^{t\alpha_{1i} + t^2\hat{\alpha}_{2i} + \dots}$ and $\tilde{y}_i = \alpha_{0i} e^{t\alpha_{1i} + t^2\tilde{\alpha}_{2i} + \dots}$, with $\alpha_{ji}, \hat{\alpha}_{ki}, \tilde{\alpha}_{ki} \in k'((s))$, for $0 \leq j \leq 1, 2 \leq k$, and $1 \leq i \leq 3$.

Then we have

$$(2.2.1) \quad \tilde{\omega}(\tilde{p}, \hat{p}, id) := \sum_{\sigma \in S_3} (-1)^\sigma \alpha_{1\sigma(1)} (\tilde{\alpha}_{2\sigma(3)} - \hat{\alpha}_{2\sigma(3)}) \cdot d \log(\alpha_{0\sigma(2)}) \in \Omega_{k'((s))/k}^1.$$

(P2) *Completion.* Suppose that we have $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ as above together with an isomorphism $\chi : \tilde{\mathcal{A}}|_{t^2} \rightarrow \hat{\mathcal{A}}|_{t^2}$, and pairs (\tilde{p}, \hat{p}) , where $\tilde{p} := (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\hat{p} := (\hat{y}_1, \hat{y}_2, \hat{y}_3)$ such that $\tilde{y}_i \in \tilde{\mathcal{A}}^\times$, $\hat{y}_i \in \hat{\mathcal{A}}^\times$, and $\chi(\tilde{y}_i|_{t^2}) = \hat{y}_i|_{t^2}$ in $\hat{\mathcal{A}}|_{t^2}$, for $1 \leq i \leq 3$.

Let $\tilde{\mathcal{B}}$ and $\hat{\mathcal{B}}$ denote the completions of $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ along the closed fiber. Note that χ induces an isomorphism

$$\tilde{\mathcal{B}}|_{t^2} \simeq \tilde{\mathcal{A}}|_{t^2} \rightarrow \hat{\mathcal{A}}|_{t^2} \simeq \hat{\mathcal{B}}|_{t^2},$$

which we will denote by ψ . Let \tilde{q} and \hat{q} denote the images of \tilde{p} and \hat{p} in the respective completions $\tilde{\mathcal{B}}$ and $\hat{\mathcal{B}}$. Then we have

$$\omega(\tilde{p}, \hat{p}, \chi) = \omega(\tilde{q}, \hat{q}, \psi)$$

(P3) *Functoriality.* Suppose that we have a diagram

$$\begin{array}{ccc} \tilde{\mathcal{A}}_1 & \xrightarrow{\chi_1} & \hat{\mathcal{A}}_1 \\ f \downarrow & & g \downarrow \\ \tilde{\mathcal{A}}_2 & \xrightarrow{\chi_2} & \hat{\mathcal{A}}_2, \end{array}$$

where χ_i are isomorphisms, and such that there exists a closed point in $\tilde{\mathfrak{p}}_2 \in \text{Spec } \underline{\hat{\mathcal{A}}}_2$, with the property that g induces an isomorphism from the localisation at (t) of the completion of $\hat{\mathcal{A}}_1$ at $g^{-1}(\tilde{\mathfrak{p}}_2)$ to the localisation at (t) of the completion of $\hat{\mathcal{A}}_2$ at $\tilde{\mathfrak{p}}_2$. Note that this also implies that f induces an isomorphism from the localisation at (t) of the completion of $\tilde{\mathcal{A}}_1$ at $(\chi_2 \circ f)^{-1}(\tilde{\mathfrak{p}}_2)$ to the localisation at (t) of the completion of $\tilde{\mathcal{A}}_2$ at $\chi_2^{-1}(\tilde{\mathfrak{p}}_2)$.

Let $(\tilde{p}, \hat{p}) \in \times^3 \beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi_1|_{t^2})$, then we have

$$\underline{g}_*(\omega(\tilde{p}, \hat{p}, \chi_1|_{t^2})) = \omega(f(\tilde{p}), g(\hat{p}), \chi_2|_{t^2}),$$

where $\underline{g}_* : \Omega_{\underline{\hat{\mathcal{A}}}_1/k}^1 \rightarrow \Omega_{\underline{\hat{\mathcal{A}}}_2/k}^1$ is the map induced by g .

Proposition 2.2.1. *The properties (P1)-(P3) uniquely determine ω .*

Proof. Suppose that we would like to determine the value of $\omega(\tilde{p}, \hat{p}, \chi)$, for a general $(\tilde{p}, \hat{p}, \chi)$ as above. By property (P2), we will assume that $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ are complete with respect to the ideal defining the closed fiber.

Next note that by smoothness and completeness, we have isomorphisms $\tilde{\mathcal{A}} \xrightarrow{\sim} \tilde{\mathcal{A}}[[t]]$ and $\hat{\mathcal{A}} \xrightarrow{\sim} \hat{\mathcal{A}}[[t]]$, which are the identity maps modulo (t) . Through these isomorphisms it follows that χ corresponds to an isomorphism $\tilde{\mathcal{A}}[[t]]/(t^2) \xrightarrow{\sim} \hat{\mathcal{A}}[[t]]/(t^2)$ and therefore can be lifted, non-uniquely, to an isomorphism $\bar{\chi} : \tilde{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$. Choosing isomorphisms $\tilde{\phi} : \tilde{\mathcal{A}} \xrightarrow{\sim} B[[t]]$ and $\hat{\phi} : \hat{\mathcal{A}} \xrightarrow{\sim} B[[t]]$ and using (P3) with $\chi_2 := \hat{\phi} \circ \bar{\chi} \circ \tilde{\phi}^{-1}$, we assume without loss of generality that $\tilde{\mathcal{A}} = \hat{\mathcal{A}} = B[[t]]$.

The completion of the local ring of $B[[t]]$ at a closed point is of the form $k'[[s]][[t]]$ for a finite extension k'/k of k , and its localisation at (t) is $k'((s))[[t]]$. By the functoriality (P3) with respect to the pair of inclusions $B[[t]] \rightarrow k'((s))[[t]]$ and the injectivity of the map $\Omega_{B/k}^1 \rightarrow \Omega_{k'((s))/k}^1$, we reduce to the case when $\tilde{\mathcal{A}} = \hat{\mathcal{A}} = k'((s))[[t]]$, and an arbitrary A -automorphism $\bar{\chi}$ of $k'((s))[[t]]$.

Finally, taking all the rings in (P3) to be $k'((s))[[t]]$, $\chi_1 = f = \bar{\chi}$ and $\chi_2 = g = id$, by (P3), we reduce to the situation in (P1), where the value of the 1-form is given by an explicit formula. This finishes the proof. \square

In the next sections we will first prove the existence of ω with the properties (P1) and (P3) and then prove that it satisfies the further properties (P3)-(P10)

2.3. Existence of ω and proof of the properties (P1)-(P3).

2.3.1. *Construction of $\omega(\tilde{p}, \hat{p}, \chi)$.* Let $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ be objects of \mathcal{C}_A^1 , and $\chi : \tilde{\mathcal{A}}/(t^2) \rightarrow \hat{\mathcal{A}}/(t^2)$, an isomorphism over A , and $(\tilde{p}, \hat{p}) \in \times^3\beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi)$ as above. We will define $\omega(\tilde{p}, \hat{p}, \chi)$ by first making some choices. Later, we will show that this definition is independent of all the choices.

First, we replace $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ with their completions along the closed fiber. Note that this does not change $\hat{\mathcal{A}}$ and hence $\Omega_{\hat{\mathcal{A}}/k}^1$ in which ω takes values. Similarly, χ and the triples of points \tilde{p} and \hat{p} give corresponding points on the completions. Therefore, without loss of generality, we will assume that $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ are complete with respect to (t) . From now on, in this section, we will always assume that the A -algebras under consideration are complete with respect to the ideal (t) . By the smoothness assumptions we have *non-canonical* isomorphisms $\tilde{\mathcal{A}} \simeq \tilde{\mathcal{A}}[[t]]$ and $\hat{\mathcal{A}} \simeq \hat{\mathcal{A}}[[t]]$.

Let $\bar{\chi} : \tilde{\mathcal{A}} \xrightarrow{\sim} \hat{\mathcal{A}}$ be any lifting of χ , and $\varphi : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ be any splitting of the canonical projection. Denote by $\bar{\varphi}$ the corresponding isomorphism $\tilde{\mathcal{A}}[[t]] \xrightarrow{\sim} \hat{\mathcal{A}}$. We then we define $\omega(\tilde{p}, \hat{p}, \chi)$ as follows:

$$\omega(\tilde{p}, \hat{p}, \chi) := \Omega(\bar{\varphi}^{-1}(\bar{\chi}(\tilde{p})), \bar{\varphi}^{-1}(\hat{p})),$$

where we will define Ω below.

Let (\tilde{q}, \hat{q}) be a pair with $\tilde{q}, \hat{q} \in \times^3\beta(\hat{\mathcal{A}}[[t]])$ and are congruent modulo (t^2) . Let us write $\tilde{q} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\hat{q} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$. Then by assumption $\tilde{y}_i - \hat{y}_i \in (t^2)$, for all $1 \leq i \leq 3$. We write $\tilde{y}_i = \alpha_{0i} e^{t\alpha_{1i} + t^2\hat{\alpha}_{2i} + \dots}$ and $\hat{y}_i = \alpha_{0i} e^{t\alpha_{1i} + t^2\hat{\alpha}_{2i} + \dots}$, with $\alpha_{ji}, \hat{\alpha}_{ki}, \tilde{\alpha}_{ki} \in \hat{\mathcal{A}}$, for $0 \leq j \leq 1$, $2 \leq k$, and $1 \leq i \leq 3$, as in (P1). We then define

$$(2.3.1) \quad \Omega(\tilde{q}, \hat{q}) := \sum_{\sigma \in S_3} (-1)^\sigma \alpha_{1\sigma(1)} (\tilde{\alpha}_{2\sigma(3)} - \hat{\alpha}_{2\sigma(3)}) \cdot d \log(\alpha_{0\sigma(2)}) \in \Omega_{\hat{\mathcal{A}}/k}^1,$$

exactly as in (P1).

This construction a priori depends on $\bar{\chi}$ and φ . Therefore, we will denote the ω defined above temporarily by $\omega_{\bar{\chi}, \varphi}$. We will show below that, in fact, it is independent of $\bar{\chi}$ and φ . In order to show independence with respect to the choice of the splitting φ , we will start with the following lemma which deals with the split case.

Lemma 2.3.1. *Let $\mathcal{A} := k'((s))[[t]]$, where k'/k is a finite extension. Suppose that $(\tilde{p}, \hat{p}) \in \times^3\beta(\mathcal{A}, \mathcal{A}, id)$, and $\psi : \mathcal{A} \rightarrow \mathcal{A}$ be any continuous A -algebra isomorphism which induces the identity map on $\underline{\mathcal{A}}$. Then*

$$\Omega(\tilde{p}, \hat{p}) = \Omega(\psi(\tilde{p}), \psi(\hat{p})).$$

Proof. Since ψ is assumed to be an A -algebra homomorphism, it is completely determined by its restriction to $k'((s))$. Since ψ is identity on k , reduces to identity modulo (t) , and $\Omega_{k'/k}^1 = 0$, ψ is identity on k' . Therefore, it is completely determined by its value on s . Suppose that $\psi(s) = s + at + bt^2 + O(t^3)$, with $a, b \in k'((s))$. Then we have that $\psi(\alpha_0 e^{\alpha_1 t + \alpha_2 t^2}) =$

$$\begin{aligned}
 &= (\alpha_0 + \alpha'_0(at + bt^2) + \frac{\alpha''_0}{2}(at)^2) e^{(\alpha_1 + \alpha'_1 at) + \alpha_2 t^2} + O(t^3) \\
 &= \alpha_0 e^{\frac{\alpha'_0}{\alpha_0} at + (\frac{\alpha'_0}{\alpha_0} b + \frac{\alpha''_0}{2\alpha_0} a^2 - \frac{1}{2}(\frac{\alpha'_0}{\alpha_0} a)^2) t^2} e^{\alpha_1 t + (\alpha'_1 a + \alpha_2) t^2} + O(t^3) \\
 (2.3.2) \quad &= \alpha_0 e^{(\alpha_1 + \frac{\alpha'_0}{\alpha_0} a)t + (\alpha_2 + \alpha'_1 a + \frac{\alpha'_0}{\alpha_0} b + \frac{\alpha''_0}{2\alpha_0} a^2 - \frac{1}{2}(\frac{\alpha'_0}{\alpha_0} a)^2) t^2} + O(t^3),
 \end{aligned}$$

where for a function $f \in k'((s))$, we let $f' \in k'((s))$ denote the derivative with respect to s . More precisely, $f' \in k'((s))$ is the unique series such that $df = f' ds$ in $\hat{\Omega}_{k'((s))/k}^1$.

Then using (2.3.2) and (2.3.1) the difference $\Omega(\psi(\tilde{p}), \psi(\hat{p})) - \Omega(\tilde{p}, \hat{p})$ is given by

$$\begin{aligned}
 &\sum_{\sigma \in S_3} (-1)^\sigma a \cdot \frac{\alpha'_{0\sigma(1)}}{\alpha_{0\sigma(1)}} \cdot (\tilde{\alpha}_{2\sigma(3)} - \hat{\alpha}_{2\sigma(3)}) \cdot d \log(\alpha_{0\sigma(2)}) \\
 &= \sum_{\sigma \in S_3} (-1)^\sigma a \cdot \frac{\alpha'_{0\sigma(1)}}{\alpha_{0\sigma(1)}} \cdot (\tilde{\alpha}_{2\sigma(3)} - \hat{\alpha}_{2\sigma(3)}) \cdot \frac{\alpha'_{0\sigma(2)}}{\alpha_{0\sigma(2)}} ds = 0.
 \end{aligned}$$

□

Corollary 2.3.2. *If $\bar{\chi} : \tilde{\mathcal{A}} \xrightarrow{\sim} \hat{\mathcal{A}}$ is any lifting of χ , and $\varphi, \varphi' : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ are two splittings of the canonical projection $\hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ then*

$$\omega_{\bar{\chi}, \varphi'} = \omega_{\bar{\chi}, \varphi}.$$

Proof. Note that $\omega_{\bar{\chi}, \varphi}(\tilde{p}, \hat{p}, \chi) = \Omega(\bar{\varphi}^{-1}(\bar{\chi}(\tilde{p})), \bar{\varphi}^{-1}(\hat{p}))$ and $\omega'_{\bar{\chi}, \varphi}(\tilde{p}, \hat{p}, \chi) = \Omega(\bar{\varphi}'^{-1}(\bar{\chi}(\tilde{p})), \bar{\varphi}'^{-1}(\hat{p}))$. Therefore, letting $\tilde{q} := \bar{\varphi}^{-1}(\bar{\chi}(\tilde{p}))$ and $\hat{q} := \bar{\varphi}^{-1}(\hat{p}) \in \hat{\mathcal{A}}[[t]]$ and $\psi := \bar{\varphi}'^{-1} \circ \bar{\varphi}$, we need to show that $\Omega(\tilde{q}, \hat{q}) = \Omega(\psi(\tilde{q}), \psi(\hat{q}))$. This is exactly the statement of the above lemma when $\hat{\mathcal{A}} \simeq k'((s))$. However, we can always reduce to this case by choosing a closed point of $\hat{\mathcal{A}}$ and noting that the field of fractions of the completion of $\hat{\mathcal{A}}$ at this closed point is isomorphic to $k'((s))$ and the induced map $\Omega_{\hat{\mathcal{A}}/k}^1 \rightarrow \Omega_{k'((s))/k}^1$ is an injection. □

Because of the corollary above, from now on we will use the notation $\omega_{\bar{\chi}}$ instead of $\omega_{\bar{\chi}, \varphi}$. Next we show independence with respect to the lifting $\bar{\chi}$ of χ .

Lemma 2.3.3. *If $\bar{\chi}$ and $\bar{\chi}'$ are liftings of χ then $\omega_{\bar{\chi}} = \omega_{\bar{\chi}'}$. Therefore, ω is well-defined and does not depend on φ and $\bar{\chi}$.*

Proof. Using a splitting $\varphi : \hat{\mathcal{A}} \rightarrow \hat{\mathcal{A}}$ as in the definition of ω , we reduce to the case when $\tilde{\mathcal{A}} = \hat{\mathcal{A}} = \hat{\mathcal{A}}[[t]]$ and $\chi = id$. Using the argument at the end of the proof of Corollary 2.3.2, we reduce further to the case when $\hat{\mathcal{A}} = k'((s))$. If $\bar{\chi}$ and $\bar{\chi}'$ are two liftings of id , we need to prove that:

$$\Omega(\bar{\chi}(\tilde{p}), \hat{p}) = \Omega(\bar{\chi}'(\tilde{p}), \hat{p}).$$

Letting $\bar{\chi}' = \iota \circ \bar{\chi}$, we have $\iota|_{t^2} = id$ and we need to prove that for any \tilde{p} and \hat{p} such that $\tilde{p}|_{t^2} = \hat{p}|_{t^2}$, we have

$$\Omega(\iota(\tilde{p}), \hat{p}) = \Omega(\tilde{p}, \hat{p}).$$

We have $\iota(s) = s + bt^2 + O(t^3)$, with $b \in k'((s))$, and by the formula (2.3.2), we have

$$\iota(\alpha_0 e^{\alpha_1 t + \alpha_2 t^2}) = \alpha_0 e^{\alpha_1 t + (\alpha_2 + \frac{\alpha'_0}{\alpha_0} b) t^2} + O(t^3).$$

Then the difference $\Omega(\iota(\tilde{p}), \hat{p}) - \Omega(\tilde{p}, \hat{p})$ is given by

$$\sum_{\sigma \in S_3} (-1)^\sigma \alpha_{1\sigma(1)} \cdot (b \frac{\alpha'_{0\sigma(3)}}{\alpha_{0\sigma(3)}}) \cdot d \log(\alpha_{0\sigma(2)}) = b \sum_{\sigma \in S_3} (-1)^\sigma \alpha_{1\sigma(1)} \cdot \frac{\alpha'_{0\sigma(3)}}{\alpha_{0\sigma(3)}} \frac{\alpha'_{0\sigma(2)}}{\alpha_{0\sigma(2)}} ds = 0.$$

This finishes the proof of the lemma. \square

2.4. The additional properties (P4)-(P10) of the map ω . In this section, we will show that the construction above satisfies the following fundamental properties which will be needed below.

2.4.1. (P4) Interchange of components. If we interchange \tilde{p} and \hat{p} then they are related by the following formula

$$\underline{\chi}_*(\omega(\hat{p}, \tilde{p}, \chi^{-1})) = -\omega(\tilde{p}, \hat{p}, \chi),$$

where $\underline{\chi}_* : \Omega_{\underline{\hat{A}}/k}^1 \rightarrow \Omega_{\underline{\tilde{A}}/k}^1$ is the map induced by χ .

Proof. First note that Ω has the property that $\Omega(\tilde{q}, \hat{q}) = -\Omega(\hat{q}, \tilde{q})$. Let $\bar{\chi} : \tilde{\hat{A}} \xrightarrow{\sim} \hat{\tilde{A}}$ be a lifting of χ , and $\varphi : \hat{\tilde{A}} \rightarrow \tilde{\hat{A}}$ a splitting of the canonical projection as above. Then we have, by definition, $\omega(\tilde{p}, \hat{p}, \chi) = \Omega(\bar{\varphi}^{-1}(\bar{\chi}(\tilde{p})), \bar{\varphi}^{-1}(\hat{p}))$.

On the other hand, in order to compute $\omega(\hat{p}, \tilde{p}, \chi^{-1})$, we can use the splitting $\psi := \bar{\chi}^{-1} \circ \varphi \circ \underline{\chi}$ from $\tilde{\hat{A}}$ to $\tilde{\tilde{A}}$, and the lifting $\bar{\chi}^{-1}$ of χ^{-1} . Note that $\bar{\psi}$ can be described as follows. Let $\bar{\chi}$ denote the isomorphism from $\tilde{\hat{A}}[[t]]$ to $\hat{\tilde{A}}[[t]]$, induced by $\underline{\chi} : \tilde{\hat{A}} \rightarrow \hat{\tilde{A}}$. Then $\bar{\psi} = \bar{\chi}^{-1} \circ \bar{\varphi} \circ \bar{\chi}$ and we have $\omega(\hat{p}, \tilde{p}, \chi^{-1}) =$

$$\Omega(\bar{\psi}^{-1}(\bar{\chi}^{-1}(\hat{p})), \bar{\psi}^{-1}(\tilde{p})) = \Omega(\bar{\chi}^{-1}(\bar{\varphi}^{-1}(\hat{p})), \bar{\chi}^{-1}(\bar{\varphi}^{-1}(\bar{\chi}(\tilde{p}))).$$

By transport of structure, $\underline{\chi}_*$ of the last expression is $\Omega(\bar{\varphi}^{-1}(\hat{p}), \bar{\varphi}^{-1}(\bar{\chi}(\tilde{p})))$. Therefore,

$$\underline{\chi}_*(\omega(\hat{p}, \tilde{p}, \chi^{-1})) = \Omega(\bar{\varphi}^{-1}(\hat{p}), \bar{\varphi}^{-1}(\bar{\chi}(\tilde{p}))) = -\Omega(\bar{\varphi}^{-1}(\bar{\chi}(\tilde{p})), \bar{\varphi}^{-1}(\hat{p})) = -\omega(\tilde{p}, \hat{p}, \chi),$$

which finishes the proof. \square

2.4.2. (P5) Transitivity. If we have $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_3 and $\chi_{12} : \mathcal{A}_1|_{t^2} \rightarrow \mathcal{A}_2|_{t^2}$ and $\chi_{23} : \mathcal{A}_2|_{t^2} \rightarrow \mathcal{A}_3|_{t^2}$ then

$$\underline{\chi}_{23*}(\omega(p_1, p_2, \chi_{12})) + \omega(p_2, p_3, \chi_{23}) = \omega(p_1, p_3, \chi_{13}),$$

where $\chi_{13} = \chi_{23} \circ \chi_{12}$

Proof. First note that $\Omega(q_1, q_2) + \Omega(q_2, q_3) = \Omega(q_1, q_3)$. Let $\bar{\chi}_{12} : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is a lifting of χ_{12} and $\bar{\chi}_{23} : \mathcal{A}_2 \rightarrow \mathcal{A}_3$ is a lifting of χ_{23} . Then $\bar{\chi}_{13} := \bar{\chi}_{23} \circ \bar{\chi}_{12}$ is a lifting of χ_{13} . Let $\varphi : \underline{\mathcal{A}}_3 \rightarrow \underline{\mathcal{A}}_3$ be a splitting. Then we have

$$(2.4.1) \quad \omega(p_2, p_3, \chi_{23}) = \Omega(\bar{\varphi}^{-1}(\bar{\chi}_{23}(p_2)), \bar{\varphi}^{-1}(p_3)).$$

Note that $\varphi_2 := \bar{\chi}_{23}^{-1} \circ \varphi \circ \underline{\chi}_{23} : \underline{\mathcal{A}}_2 \rightarrow \underline{\mathcal{A}}_2$ is a splitting with $\bar{\varphi}_2 = \bar{\chi}_{23}^{-1} \circ \bar{\varphi} \circ \bar{\chi}_{23}$. Then we have

$$\omega(p_1, p_2, \chi_{12}) = \Omega(\bar{\varphi}_2^{-1}(\bar{\chi}_{12}(p_1)), \bar{\varphi}_2^{-1}(p_2)) = \Omega(\bar{\chi}_{23}^{-1}(\bar{\varphi}^{-1}(\bar{\chi}_{13}(p_1))), \bar{\chi}_{23}^{-1}(\bar{\varphi}^{-1}(\bar{\chi}_{23}(p_2)))),$$

which implies that

$$(2.4.2) \quad \underline{\chi}_{23*}(\omega(p_1, p_2, \chi_{12})) = \Omega(\bar{\varphi}^{-1}(\bar{\chi}_{13}(p_1)), \bar{\varphi}^{-1}(\bar{\chi}_{23}(p_2))).$$

Similarly,

$$(2.4.3) \quad \omega(p_1, p_3, \chi_{13}) = \Omega(\bar{\varphi}^{-1}(\bar{\chi}_{13}(p_1)), \bar{\varphi}^{-1}(p_3)).$$

Summing (2.4.1), (2.4.2), and (2.4.3), gives the result. \square

2.4.3. *(P6) Multilinearity.* The map $\omega(\cdot, \cdot, \chi)$ is multilinear on each of the three components of $\times^3\beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi)$.

Proof. This follows from the same statement for Ω . Namely, if $(\tilde{y}_i, \hat{y}_i) \in \beta(\hat{\mathcal{A}}[[t]], \tilde{\mathcal{A}}[[t]], id)$, and $(\tilde{y}'_i, \hat{y}'_i) \in \beta(\hat{\mathcal{A}}[[t]], \tilde{\mathcal{A}}[[t]], id)$, for $1 \leq i \leq 3$, then

$$\Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3), (\hat{y}_1, \hat{y}_2, \hat{y}_3)) = \Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3), (\hat{y}_1, \hat{y}_2, \hat{y}_3)) + \Omega((\tilde{y}'_1, \tilde{y}_2, \tilde{y}_3), (\hat{y}'_1, \hat{y}_2, \hat{y}_3)),$$

$$\Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}'_3), (\hat{y}_1, \hat{y}_2, \hat{y}_3)) = \Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3), (\hat{y}_1, \hat{y}_2, \hat{y}_3)) + \Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}'_3), (\hat{y}_1, \hat{y}'_2, \hat{y}_3)),$$

and

$$\Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3), (\hat{y}_1, \hat{y}_2, \hat{y}'_3)) = \Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3), (\hat{y}_1, \hat{y}_2, \hat{y}_3)) + \Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}'_3), (\hat{y}_1, \hat{y}_2, \hat{y}'_3)).$$

□

2.4.4. *(P7) Anti-symmetry.* The map $\omega(\cdot, \cdot, \chi)$ is anti-symmetric with respect to the action of S_3 on $\times^3\beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi)$.

Proof. This follows from the same statement for Ω . More precisely, if $(\tilde{y}_i, \hat{y}_i) \in \beta(\hat{\mathcal{A}}[[t]], \tilde{\mathcal{A}}[[t]], id)$, for $1 \leq i \leq 3$, then

$$\Omega((\tilde{y}_{\sigma(1)}, \tilde{y}_{\sigma(2)}, \tilde{y}_{\sigma(3)}), (\hat{y}_{\sigma(1)}, \hat{y}_{\sigma(2)}, \hat{y}_{\sigma(3)})) = (-1)^\sigma \Omega((\tilde{y}_1, \tilde{y}_2, \tilde{y}_3), (\hat{y}_1, \hat{y}_2, \hat{y}_3)),$$

for any $\sigma \in S_3$.

□

Corollary 2.4.1. *The properties (P6) and (P7) imply that $\omega(\cdot, \cdot, \chi)$ induces a map*

$$\omega(\cdot, \cdot, \chi) : \Lambda_{\mathbb{Z}}^3\beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi) \rightarrow \Omega_{\tilde{\mathcal{A}}/k}^1,$$

which we will denote by the same notation.

2.4.5. *(P8) Residue.* Suppose that $\tilde{\mathcal{A}}^\circ$ and $\hat{\mathcal{A}}^\circ$ are objects of \mathcal{C}_A^1 whose reductions $\tilde{\mathcal{A}}^\circ$ and $\hat{\mathcal{A}}^\circ$ are discrete valuation rings. Let $\tilde{\mathcal{A}}$ and $\hat{\mathcal{A}}$ be their localisations at the prime ideal (t) . Suppose that $\chi : \tilde{\mathcal{A}}^\circ|_{t^2} \xrightarrow{\sim} \hat{\mathcal{A}}^\circ|_{t^2}$. Note that this gives, after localisation, an isomorphism $\chi : \tilde{\mathcal{A}}|_{t^2} \xrightarrow{\sim} \hat{\mathcal{A}}|_{t^2}$. Suppose that $\hat{\pi} \in \hat{\mathcal{A}}^\circ$ (resp. $\tilde{\pi} \in \tilde{\mathcal{A}}^\circ$), such that its reduction to $\tilde{\mathcal{A}}^\circ$ (resp. $\hat{\mathcal{A}}^\circ$) is a uniformizer. Assume that $\chi(\hat{\pi}|_{t^2}) = \tilde{\pi}|_{t^2}$ in $\hat{\mathcal{A}}^\circ|_{t^2}$.

Definition 2.4.2. We let $\beta(\hat{\mathcal{A}}^\circ, \hat{\pi}) := \{\hat{y} \in \hat{\mathcal{A}}^\times | \hat{y} = \hat{\pi}^a u, \text{ for some } a \in \mathbb{Z}, \text{ and } u \in (\hat{\mathcal{A}}^\circ)^\times\}$ and $\times^n\beta(\hat{\mathcal{A}}^\circ, \hat{\pi})$, the n -fold product of $\beta(\hat{\mathcal{A}}^\circ, \hat{\pi})$ with itself. We say that $\hat{y} \in \hat{\mathcal{A}}$ is $\hat{\pi}$ -good if $\hat{y} \in \beta(\hat{\mathcal{A}}^\circ, \hat{\pi})$.

Note that $\times^n\beta(\hat{\mathcal{A}}^\circ, \hat{\pi}) \subseteq \times^n\hat{\mathcal{A}}^\times$. We let $\times^n\beta(\tilde{\mathcal{A}}^\circ, \hat{\mathcal{A}}^\circ, \chi, \tilde{\pi}, \hat{\pi}) \subseteq \times^n\beta(\tilde{\mathcal{A}}, \hat{\mathcal{A}}, \chi)$ denote the subgroup of those pairs of n -tuples which lie in $\times^n\beta(\tilde{\mathcal{A}}^\circ, \tilde{\pi})$ and $\times^n\beta(\hat{\mathcal{A}}^\circ, \hat{\pi})$. We claim that on this set, the 1-form ω has only logarithmic singularities at the closed point of $\text{Spec } \tilde{\mathcal{A}}^\circ$:

$$\omega : \times^3\beta(\tilde{\mathcal{A}}^\circ, \hat{\mathcal{A}}^\circ, \chi, \tilde{\pi}, \hat{\pi}) \rightarrow \Omega_{\tilde{\mathcal{A}}^\circ/k}^1 \subseteq \Omega_{\tilde{\mathcal{A}}/k}^1,$$

where, if $\hat{\pi}$ denotes the reduction of $\hat{\pi}$ to $\tilde{\mathcal{A}}$, then $\Omega_{\tilde{\mathcal{A}}^\circ/k}^1 := (\Omega_{\tilde{\mathcal{A}}^\circ/k}^1 + \tilde{\mathcal{A}}^\circ \cdot d \log(\hat{\pi})) \subseteq \Omega_{\tilde{\mathcal{A}}/k}^1$. In the remainder of this section, we will prove the above statement and give a formula for the residue of ω .

First note that we have the following version:

$$(2.4.4) \quad \text{res}_{\hat{\pi}} : \Lambda^n\beta(\hat{\mathcal{A}}^\circ, \hat{\pi}) \rightarrow \Lambda^{n-1}(\hat{\mathcal{A}}^\circ/(\hat{\pi}))^\times,$$

of the residue map in [3, §1.14]. This map is characterised by the following properties. It is multilinear, anti-symmetric, vanishes when restricted to $\Lambda^n(\hat{\mathcal{A}}^\circ)^\times$; and

$$\text{res}_{\hat{\pi}}(\hat{\pi}, u_2, \dots, u_n) = (u_2, \dots, u_n),$$

where if $u_i \in (\hat{\mathcal{A}}^\circ)^\times$, for $2 \leq i \leq n$, then \underline{u}_i denotes reduction modulo $(\hat{\pi})$. We will need the fact that this construction of $\text{res}_{\hat{\pi}}$ in fact depends only on the ideal $(\hat{\pi})$ generated by $\hat{\pi}$ in $\hat{\mathcal{A}}^\circ$.

Lemma 2.4.3. *If $\hat{\pi}' = v \cdot \hat{\pi}$, for some unit $v \in (\hat{\mathcal{A}}^\circ)^\times$, then $\text{res}_{\hat{\pi}'} = \text{res}_{\hat{\pi}}$.*

Proof. First note that under the assumptions $\beta(\hat{\mathcal{A}}^\circ, \hat{\pi}') = \beta(\hat{\mathcal{A}}^\circ, \hat{\pi})$ and hence the sources and the targets of $res_{\hat{\pi}'}$ and $res_{\hat{\pi}}$ are the same and the statement makes sense. In order to prove the statement, by the multi-linearity, anti-symmetry and the vanishing of both $res_{\hat{\pi}'}$ and $res_{\hat{\pi}}$ on $\Lambda^n(\hat{\mathcal{A}}^\circ)^\times$, we need to check that they agree on elements of the form $(\hat{\pi}', u_2, \dots, u_n)$ with $u_i \in (\hat{\mathcal{A}}^\circ)^\times$. On this element, we have $res_{\hat{\pi}'}(\hat{\pi}', u_2, \dots, u_n) = (\underline{u}_2, \dots, \underline{u}_n)$ and $res_{\hat{\pi}}(\hat{\pi}', u_2, \dots, u_n) = res_{\hat{\pi}}(v \cdot \hat{\pi}', u_2, \dots, u_n) = res_{\hat{\pi}}(v, u_2, \dots, u_n) + res_{\hat{\pi}}(\hat{\pi}', u_2, \dots, u_n) = (\underline{u}_2, \dots, \underline{u}_n)$, which finishes the proof. \square

Let \mathcal{C}_A^0 denote the full subcategory of the category of A -algebras whose objects consist of étale A -algebras which are integral domains with the residue field of the closed point a finite extension k' of k . Given such an A -algebra B , the completion \hat{B} along the ideal (t) is canonically isomorphic to $k'[[t]]$, where k' is the residue field of the closed point, so we write $\psi : \hat{B} \xrightarrow{\sim} k'[[t]]$. Let $[t^2 \wedge t] : \Lambda_k^2 k'[[t]] \rightarrow k'$ denote the map that sends $a \wedge b$ to the coefficient of $t^2 \wedge t$ in $a \wedge b$.

We have a canonical map

$$\ell : \Lambda_{\mathbb{Z}}^2 B^\times \rightarrow \Lambda_{\mathbb{Z}}^2 \hat{B}^\times \xrightarrow{\Lambda^2 \psi} \Lambda_{\mathbb{Z}}^2 k'[[t]]^\times \xrightarrow{\Lambda^2 \log^\circ} \Lambda_k^2 k'[[t]] \xrightarrow{[t^2 \wedge t]} k',$$

for every object B of \mathcal{C}_A^0 . With the above notation, $\hat{\mathcal{A}}^\circ/(\hat{\pi})$ is an object of \mathcal{C}_A^0 , and we have a canonical map:

$$(2.4.5) \quad \ell : \Lambda_{\mathbb{Z}}^2(\hat{\mathcal{A}}^\circ/(\hat{\pi}))^\times \rightarrow k' := \hat{\mathcal{A}}^\circ/(\hat{\pi}, t).$$

Proposition 2.4.4. *If $(\tilde{p}, \hat{p}, \chi) \in \Lambda^3 \beta(\tilde{\mathcal{A}}^\circ, \hat{\mathcal{A}}^\circ, \chi, \tilde{\pi}, \hat{\pi})$, then we have $\omega(\tilde{p}, \hat{p}, \chi) \in \Omega_{\tilde{\mathcal{A}}_{\text{log}}^1/k}^1$ with residue given by*

$$(2.4.6) \quad res_{\tilde{\pi}} \omega(\tilde{p}, \hat{p}, \chi) = \underline{\chi} \circ \ell \circ res_{\tilde{\pi}}(\tilde{p}) - \ell \circ res_{\tilde{\pi}}(\hat{p}).$$

Proof. First, without loss of generality, we replace $\hat{\mathcal{A}}^\circ$ and $\tilde{\mathcal{A}}^\circ$ with their completions at their closed point corresponding to $(\hat{\pi}, t)$ and $(\tilde{\pi}, t)$. Then we choose $\bar{\chi} : \tilde{\mathcal{A}}^\circ \rightarrow \hat{\mathcal{A}}^\circ$ as the unique lifting of χ which sends $\tilde{\pi}$ to $\hat{\pi}$ and is an A -algebra isomorphism. Choosing also an isomorphism $k'[[s]] \simeq \hat{\mathcal{A}}^\circ$, which sends s to $\hat{\pi}$, we reduce the problem to the case when $\hat{\mathcal{A}}^\circ = \tilde{\mathcal{A}}^\circ = k'[[s, t]]$, $\chi = id$ and $\hat{\pi} = \tilde{\pi} = s$.

Let $\tilde{p} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ and $\hat{p} = (\hat{y}_1, \hat{y}_2, \hat{y}_3)$. Then $\omega(\tilde{p}, \hat{p}, \chi) = \Omega(\tilde{p}, \hat{p})$. If $\tilde{y}_i, \hat{y}_i \in k'[[s, t]]^\times$, for all $1 \leq i \leq 3$ then $\Omega(\tilde{p}, \hat{p}) \in \Omega_{k'[[s]]/k}^1$ and hence $res_s \Omega(\tilde{p}, \hat{p}) = 0$. In this case, we also have $res_s \tilde{p} = res_s \hat{p} = 0 \in \Lambda^2 k[[s]]^\times$. Therefore, we have the equality in the (2.4.6) in this case.

By the above property, the multi-linearity and anti-symmetry of both sides of (2.4.6), and the definition of s -goodness, we are reduced to proving the equality in case when $\tilde{p} = (s, \tilde{y}_2, \tilde{y}_3)$ and $\hat{p} = (s, \hat{y}_2, \hat{y}_3)$, with $\tilde{y}_i, \hat{y}_i \in k'[[s, t]]^\times$. We can then write $\hat{y}_i = \alpha_{0i} e^{t\alpha_{1i} + t^2\hat{\alpha}_{2i} + \dots}$ and $\tilde{y}_i = \alpha_{0i} e^{t\alpha_{1i} + t^2\tilde{\alpha}_{2i} + \dots}$, with $\alpha_{1i}, \hat{\alpha}_{ri}, \tilde{\alpha}_{ri} \in k'[[s]]$, for $2 \leq r$, and $2 \leq i \leq 3$; and $\alpha_{0i} \in k[[s]]^\times$, for $2 \leq i \leq 3$. Then we have $res_s \Omega(\tilde{p}, \hat{p}) = \alpha_{13}(0)(\tilde{\alpha}_{22}(0) - \hat{\alpha}_{22}(0)) - \alpha_{12}(0)(\tilde{\alpha}_{23}(0) - \hat{\alpha}_{23}(0))$

$$= (\alpha_{13}(0)\tilde{\alpha}_{22}(0) - \alpha_{12}(0)\tilde{\alpha}_{23}(0)) - (\alpha_{13}(0)\hat{\alpha}_{22}(0) - \alpha_{12}(0)\hat{\alpha}_{23}(0)).$$

On the other hand, $res_s(\tilde{p}) = (\alpha_{02}(0)e^{t\alpha_{12}(0) + t^2\tilde{\alpha}_{22}(0) + \dots}) \wedge (\alpha_{03}(0)e^{t\alpha_{13}(0) + t^2\tilde{\alpha}_{23}(0) + \dots}) \in \Lambda_{\mathbb{Z}}^2 k'[[t]]^\times$, and hence $(\Lambda^2 \log^\circ)(res_s(\tilde{p})) = (t\alpha_{12}(0) + t^2\tilde{\alpha}_{22}(0) + \dots) \wedge (t\alpha_{13}(0) + t^2\tilde{\alpha}_{23}(0) + \dots)$. This implies that $\ell(res_s(\tilde{p})) = \tilde{\alpha}_{22}(0)\alpha_{13}(0) - \alpha_{12}(0)\tilde{\alpha}_{23}(0)$. Similarly, $\ell(res_s(\hat{p})) = \hat{\alpha}_{22}(0)\alpha_{13}(0) - \alpha_{12}(0)\hat{\alpha}_{23}(0)$. Therefore, we have

$$res_s \Omega(\tilde{p}, \hat{p}) = \ell(res_s(\tilde{p})) - \ell(res_s(\hat{p})),$$

which finishes the proof of the proposition. \square

2.4.6. (P9) Difference relation. Suppose that we have objects $\tilde{\mathcal{A}}_i$ and $\hat{\mathcal{A}}_i$ in \mathcal{C}_A^1 , for $i = 1, 2$, as above, together with isomorphisms $\alpha_i : \tilde{\mathcal{A}}_i/(t^2) \xrightarrow{\sim} \hat{\mathcal{A}}_i/(t^2)$, for $i = 1, 2$, and $\chi : \tilde{\mathcal{A}}_2/(t^2) \xrightarrow{\sim} \hat{\mathcal{A}}_1/(t^2)$, and $\psi : \hat{\mathcal{A}}_2/(t^2) \xrightarrow{\sim} \hat{\mathcal{A}}_1/(t^2)$, with $\alpha_1 \circ \chi = \psi \circ \alpha_2$. Moreover, suppose that $(\tilde{p}_2, \tilde{p}_1) \in \times^3 \beta(\tilde{\mathcal{A}}_2, \tilde{\mathcal{A}}_1, \chi)$, $(\hat{p}_2, \hat{p}_1) \in \times^3 \beta(\hat{\mathcal{A}}_2, \hat{\mathcal{A}}_1, \psi)$, and $(\tilde{p}_i, \hat{p}_i) \in \times^3 \beta(\tilde{\mathcal{A}}_i, \hat{\mathcal{A}}_i, \alpha_i)$, for $i = 1, 2$. Then we have

$$\alpha_{1*}(\omega(\tilde{p}_2, \tilde{p}_1, \chi)) - \omega(\hat{p}_2, \hat{p}_1, \psi) = \underline{\psi}_*(\omega(\tilde{p}_2, \hat{p}_2, \alpha_2)) - \omega(\tilde{p}_1, \hat{p}_1, \alpha_1).$$

Proof. In order to prove the identity above, first, without loss of generality, we replace all the rings above with their completions at a closed point in their special fibers such that these points correspond to each other under the given isomorphisms α_1, α_2, χ and ψ . Choose liftings, $\bar{\alpha}_i$ and $\bar{\chi}$. Since $\alpha_1 \circ \chi = \psi \circ \alpha_2$, $\bar{\alpha}_1 \circ \bar{\chi} \circ \bar{\alpha}_2^{-1}$ is a lifting of ψ , and we can choose $\bar{\psi} := \bar{\alpha}_1 \circ \bar{\chi} \circ \bar{\alpha}_2^{-1}$ as the lifting of ψ . With this choice, we have $\bar{\alpha}_1 \circ \bar{\chi} = \bar{\psi} \circ \bar{\alpha}_2$.

Let $\varphi : \hat{\mathcal{A}}_1 \rightarrow \hat{\mathcal{A}}_1$ be a splitting. Then $\bar{\psi}^{-1} \circ \varphi \circ \underline{\psi} : \hat{\mathcal{A}}_2 \rightarrow \hat{\mathcal{A}}_2$ and $\bar{\alpha}_1^{-1} \circ \varphi \circ \underline{\alpha}_1 : \hat{\mathcal{A}}_1 \rightarrow \hat{\mathcal{A}}_1$, with the corresponding isomorphisms given by

$$\bar{\psi}^{-1} \circ \varphi \circ \underline{\psi} : \hat{\mathcal{A}}_2[[t]] \rightarrow \hat{\mathcal{A}}_2 \quad \text{and} \quad \bar{\alpha}_1^{-1} \circ \varphi \circ \underline{\alpha}_1 : \hat{\mathcal{A}}_1[[t]] \rightarrow \hat{\mathcal{A}}_1.$$

By the definition of ω , we have

$$\begin{aligned} \omega(\tilde{p}_1, \hat{p}_1, \alpha_1) &= \Omega(\bar{\varphi}^{-1}(\bar{\alpha}_1(\tilde{p}_1)), \bar{\varphi}^{-1}(\hat{p}_1)), \\ \omega(\hat{p}_2, \hat{p}_1, \psi) &= \Omega(\bar{\varphi}^{-1}(\bar{\psi}(\hat{p}_2)), \bar{\varphi}^{-1}(\hat{p}_1)), \\ \omega(\tilde{p}_2, \hat{p}_2, \alpha_2) &= \Omega((\bar{\psi}^{-1} \circ \bar{\varphi} \circ \underline{\psi})^{-1}(\bar{\alpha}_2(\tilde{p}_2)), (\bar{\psi}^{-1} \circ \bar{\varphi} \circ \underline{\psi})^{-1}(\hat{p}_2)), \end{aligned}$$

hence

$$\underline{\psi}_*(\omega(\tilde{p}_2, \hat{p}_2, \alpha_2)) = \Omega((\bar{\psi}^{-1} \circ \bar{\varphi})^{-1}(\bar{\alpha}_2(\tilde{p}_2)), (\bar{\psi}^{-1} \circ \bar{\varphi})^{-1}(\hat{p}_2)) = \Omega(\bar{\varphi}^{-1}(\bar{\psi}(\bar{\alpha}_2(\tilde{p}_2))), \bar{\varphi}^{-1}(\bar{\psi}(\hat{p}_2)));$$

and

$$\omega(\tilde{p}_2, \tilde{p}_1, \chi) = \Omega((\bar{\alpha}_1^{-1} \circ \bar{\varphi} \circ \underline{\alpha}_1)^{-1}(\bar{\chi}(\tilde{p}_2)), (\bar{\alpha}_1^{-1} \circ \bar{\varphi} \circ \underline{\alpha}_1)^{-1}(\tilde{p}_1)),$$

hence

$$\alpha_{1*}(\omega(\tilde{p}_2, \tilde{p}_1, \chi)) = \Omega((\bar{\alpha}_1^{-1} \circ \bar{\varphi})^{-1}(\bar{\chi}(\tilde{p}_2)), (\bar{\alpha}_1^{-1} \circ \bar{\varphi})^{-1}(\tilde{p}_1)) = \Omega(\bar{\varphi}^{-1}(\bar{\psi}(\bar{\alpha}_2(\tilde{p}_2))), \bar{\varphi}^{-1}(\bar{\alpha}_1(\tilde{p}_1))).$$

The above expressions, together with the basic identity,

$$\Omega(\tilde{q}_2, \tilde{q}_1) - \Omega(\hat{q}_2, \hat{q}_1) = \Omega(\tilde{q}_2, \hat{q}_2) - \Omega(\tilde{q}_1, \hat{q}_1),$$

finishes the proof of the property above. \square

2.4.7. (P10) *Modulus.* The map $\omega(\cdot, \cdot, \chi)$ depends on \tilde{p} and \hat{p} only modulo (t^3) .

Proof. The statement follows from the same statement for Ω , which is seen directly from its definition. \square

2.4.8. (P11) *Relative Bloch group relation.* Suppose that B is a local ring with infinite residue field and $I \subsetneq B$ an ideal, we will define a relative Bloch group of (B, I) . Let us write $\beta(B, I) := \{(\tilde{b}, \hat{b}) \mid \tilde{b} - \hat{b} \in I \text{ and } \tilde{b}, \hat{b} \in B^\times\}$, and $B^b := \{b \in B \mid b \cdot (1 - b) \in B^\times\}$. Let $\beta^b(B, I)$ denote the intersection $(B^b \times B^b) \cap \beta(B, I)$ inside $B^\times \times B^\times$.

Let us define the relative Bloch group $B_2(B, I)$ as the abelian group generated by the symbols $\{(\tilde{b}, \hat{b})\}_2$ for every $(\tilde{b}, \hat{b}) \in \beta^b(B, I)$, modulo the relations generated by the analog of the five term relation for the dilogarithm:

$$\{(\tilde{x}, \hat{x})\}_2 - \{(\tilde{y}, \hat{y})\}_2 + \{(\tilde{y}/\tilde{x}, \hat{y}/\hat{x})\}_2 - \left\{ \left(\frac{1 - \tilde{x}^{-1}}{1 - \tilde{y}^{-1}}, \frac{1 - \hat{x}^{-1}}{1 - \hat{y}^{-1}} \right) \right\}_2 + \left\{ \left(\frac{1 - \tilde{x}}{1 - \tilde{y}}, \frac{1 - \hat{x}}{1 - \hat{y}} \right) \right\}_2$$

for every $(\tilde{x}, \hat{x}), (\tilde{y}, \hat{y}) \in \beta^b(B, I)$ such that $(\tilde{x} - \tilde{y}, \hat{x} - \hat{y}) \in \beta(B, I)$. As in the classical case, we obtain a complex

$$\delta : B_2(B, I) \rightarrow \Lambda^2 \beta(B, I),$$

which sends (\tilde{b}, \hat{b}) to $(1 - \tilde{b}, 1 - \hat{b}) \wedge (\tilde{b}, \hat{b})$. This map induces the following map:

$$\Delta : B_2(B, I) \otimes \beta(B, I) \rightarrow \Lambda^3 \beta(B, I),$$

which sends $\{(\tilde{x}, \hat{x})\}_2 \otimes (\tilde{y}, \hat{y})$ to $\delta(\{(\tilde{x}, \hat{x})\}_2) \wedge (\tilde{y}, \hat{y})$.

Let $B := k'((s))[[t]]$, and $I = (t^2)$, then the two above definitions of β are related as follows: $\beta(B, (t^2)) = \beta(B, B, id)$, where $id : B/(t^2) \rightarrow B/(t^2)$ is the identity map. In particular, we have a map $\omega : \Lambda^3 \beta(B, (t^2)) \rightarrow \Omega_{\underline{B}/k}^1$.

Proposition 2.4.5. *With $B := k'((s))[[t]]$, ω vanishes on $\Delta(B_2(B, (t^2)) \otimes \beta(B, (t^2))) \subseteq \Lambda^3 \beta(B, (t^2))$.*

Proof. We need to show that ω vanishes on terms such as

$$((1 - \tilde{f}, \tilde{f}, \tilde{g}), (1 - \hat{f}, \hat{f}, \hat{g})),$$

with $(\tilde{f}, \hat{f}) \in \beta^b(B, (t^2))$ and $(\tilde{g}, \hat{g}) \in \beta(B, (t^2))$.

By the transitivity property (P5), it is enough to show that ω vanishes on the following terms:

$$(2.4.7) \quad ((1 - \tilde{f}, \tilde{f}, \tilde{g}), (1 - \tilde{f}, \tilde{f}, \hat{g}))$$

and

$$(2.4.8) \quad ((1 - \tilde{f}, \tilde{f}, \hat{g}), (1 - \hat{f}, \hat{f}, \hat{g})).$$

In order to prove that ω vanishes on (2.4.7), let $\tilde{f} = \alpha_0 e^{\alpha_1 t + \tilde{\alpha}_2 t^2 + \dots}$, $\tilde{g} := \beta_0 e^{\beta_1 t + \tilde{\beta}_2 t^2 + \dots}$ and $\hat{g} := \beta_0 e^{\beta_1 t + \hat{\beta}_2 t^2 + \dots}$. Then we have

$$1 - \tilde{f} = (1 - \alpha_0) e^{\frac{\alpha_0 \alpha_1}{\alpha_0 - 1} t + (\frac{\alpha_0 \tilde{\alpha}_2}{\alpha_0 - 1} - \frac{\alpha_0 \alpha_1^2}{2(\alpha_0 - 1)^2}) t^2 + \dots}.$$

Then property (P1) gives that ω evaluated on (2.4.7) is equal to

$$\left(\frac{\alpha_0 \alpha_1}{\alpha_0 - 1} \cdot d \log(\alpha_0) - \alpha_1 \cdot d \log(1 - \alpha_0) \right) \cdot (\tilde{\beta}_2 - \hat{\beta}_2) = 0.$$

In order to prove that ω vanishes on (2.4.8), let $\hat{f} = \alpha_0 e^{\alpha_1 t + \hat{\alpha}_2 t^2 + \dots}$. Again by (P1), we compute that ω evaluated on (2.4.8) is equal to

$$\begin{aligned} & \left(\frac{\alpha_0}{\alpha_0 - 1} (\tilde{\alpha}_2 - \hat{\alpha}_2) \alpha_1 - \frac{\alpha_0 \alpha_1}{\alpha_0 - 1} (\tilde{\alpha}_2 - \hat{\alpha}_2) \right) d \log(\beta_0) \\ & + \left(d \log(1 - \alpha_0) (\tilde{\alpha}_2 - \hat{\alpha}_2) - \frac{\alpha_0}{\alpha_0 - 1} (\tilde{\alpha}_2 - \hat{\alpha}_2) d \log(\alpha_0) \right) \beta_1 = 0. \end{aligned}$$

This finishes the proof. \square

Corollary 2.4.6. *Suppose that \mathcal{A} is an object of \mathcal{C}_A^1 , and $(\tilde{p}, \hat{p}) = ((1 - \tilde{f}, \tilde{f}, \tilde{g}), (1 - \hat{f}, \hat{f}, \hat{g}))$ with $(\tilde{p}, \hat{p}) \in \times^3 \beta(\mathcal{A}, \mathcal{A}, id)$. Then $\omega(\tilde{p}, \hat{p}, id) = 0$.*

Proof. This follows from the above proposition after completing at a closed point in the special fiber and then localizing at the prime ideal (t) . \square

3. INFINITESIMAL REGULATOR AND THE STRONG RECIPROCITY THEOREM

In this section, we will give the construction of $\rho(f \wedge g \wedge h)$, where f, g and h are rational functions on a smooth proper curve C_2 over $k[t]/(t^2)$. The construction will rely on the map ω of the previous section. In the first subsection, we will give a local definition which depends on certain choices. In the next subsection, we will define the global version which is independent of all the choices.

3.1. Construction of ε . In this subsection we will define a map ε which will be defined on the triples of functions on a generic lifting of the ring under consideration.

3.1.1. Notion of goodness. Let $S := \text{Spec}(A)$, $S_n := \text{Spec}(A_n)$, with $A_n := k[[t]]/(t^n)$. Let $\mathcal{C}_{A_2}^1$ denote the following full subcategory of the category of A_2 -algebras. The objects of $\mathcal{C}_{A_2}^1$ consist of A_2 -algebras \mathcal{A}_2 such that \mathcal{A}_2 is a smooth A_2 -algebra of relative dimension 0 or 1, \mathcal{A}_2 is an integral domain, the natural multiplication by t map induces an isomorphism $\mathcal{A}_2 \xrightarrow{\sim} (t)$ of A_2 -modules and the residue fields of the closed points of \mathcal{A}_2 are one of the following type: a finite extension of k , if the relative dimension is 1; a finitely generated field of transcendence degree 1 over k , or a field of Laurent series $k'((s))$ over a finite extension k' of k , if the relative dimension is 0. Note that these algebras \mathcal{A}_2 are precisely the algebras that are obtained as $\mathcal{A}/(t^2)$ with \mathcal{A} an object of \mathcal{C}_A^1 .

In the rest of this section, assume that \mathcal{A}_2° is an object of $\mathcal{C}_{A_2}^1$ such that \mathcal{A}_2° is a discrete valuation ring. We call an element $\pi_2 \in \mathcal{A}_2^\circ$, a *uniformizer*, if its reduction $\pi_1 \in \mathcal{A}_2^\circ$, is a

uniformizer, in the usual sense. This is equivalent to requiring that the natural map from $k'[[\pi_2, t]]/(t^2)$ to the completion of \mathcal{A}_2° at the maximal ideal being an isomorphism, where k' is the residue field of the closed point. Let \mathcal{A}_2 denote the localization of \mathcal{A}_2° at the prime ideal (t) , similar to the notation in §2.4.5. If B is a general A -algebra with $\underline{B} = B/(t)$ an integral domain, we also use the notation B_η to denote the localisation of B at the prime ideal (t) .

Definition 3.1.1. We let $\beta(\mathcal{A}_2^\circ, \pi_2) := \{y \in \mathcal{A}_2^\circ \mid y = \pi_2^a u, \text{ for some } a \in \mathbb{Z}, \text{ and } u \in (\mathcal{A}_2^\circ)^\times\}$ and say that $y \in \mathcal{A}_2$ is π_2 -good if $y \in \beta(\mathcal{A}_2^\circ, \pi_2)$.

Note that if $y \in (\mathcal{A}_2^\circ)^\times$, then it is π_2 -good for any choice of a uniformizer π_2 . On the other hand, if for example $\mathcal{A}_2^\circ = k[[s, t]]/(t^2)$ and $f = (s + t)/s \in \mathcal{A}_2 = k((s))[t]/(t^2)$ then f is not good for any choice of a uniformizer.

3.1.2. *Fixed generic lifting.* Let $\tilde{\mathcal{A}}$ be an object in \mathcal{C}_A^1 together with a fixed isomorphism

$$\alpha : \tilde{\mathcal{A}}_2 := \tilde{\mathcal{A}}/(t^2) \xrightarrow{\sim} \mathcal{A}_2.$$

Assume further that $\tilde{\mathcal{A}}$ is complete with respect to the ideal (t) . Note that $\tilde{\mathcal{A}}$ is a lifting of \mathcal{A}_2 only and we do *not* assume that $\tilde{\mathcal{A}}$ is the localisation of a lifting of \mathcal{A}_2° . Let now

$$\beta(\tilde{\mathcal{A}}, \alpha^{-1}(\pi_2)) := \{y \in \tilde{\mathcal{A}}^\times \mid \alpha(y|_{t^2}) \in \beta(\mathcal{A}_2^\circ, \pi_2)\},$$

denote the elements in $\tilde{\mathcal{A}}$ whose reductions modulo (t^2) are $\alpha^{-1}(\pi_2)$ -good. Note that we do *not* require that the elements in $\tilde{\mathcal{A}}$ are good with respect to a uniformizer. In fact such a requirement would not have made sense since there is neither a fixed integral structure nor a choice of a uniformizer on this integral structure on $\tilde{\mathcal{A}}$. There is such a structure only modulo (t^2) . This is a crucial point in what follows.

We will define a function

$$\varepsilon : \times^3 \beta(\tilde{\mathcal{A}}, \alpha^{-1}(\pi_2)) \rightarrow k',$$

where k' is the residue field of the closed point of \mathcal{A}_2° .

In order to do this, first we choose a lifting $\tilde{\mathcal{B}}^\circ$ of \mathcal{A}_2° . In other words, $\tilde{\mathcal{B}}^\circ$ is an object of \mathcal{C}_A^1 , complete with respect to (t) , together with a fixed isomorphism

$$\tilde{\gamma} : (\tilde{\mathcal{B}}^\circ)_2 := \tilde{\mathcal{B}}^\circ/(t^2) \xrightarrow{\sim} \mathcal{A}_2^\circ.$$

Let $\tilde{\pi}$ be a uniformizer on $\tilde{\mathcal{B}}^\circ$ such that $\tilde{\gamma}(\tilde{\pi}|_{t^2}) = \pi_2$. Moreover let $(\tilde{\mathcal{B}}^\circ)_\eta$ denote the localization of $\tilde{\mathcal{B}}^\circ$ at the ideal (t) . Then $(\tilde{\mathcal{B}}^\circ)_\eta$ is a lifting of \mathcal{A}_2 together with the isomorphism $\tilde{\gamma}_\eta$, which is the localization of $\tilde{\gamma}$:

$$\tilde{\gamma}_\eta : (\tilde{\mathcal{B}}^\circ)_\eta/(t^2) = (\tilde{\mathcal{B}}^\circ)_{2,\eta} \xrightarrow{\sim} \mathcal{A}_2.$$

Because of the choice of a uniformizer $\tilde{\pi}$ on $\tilde{\mathcal{B}}^\circ$, we have the notion of $\tilde{\pi}$ -goodness for elements of $(\tilde{\mathcal{B}}^\circ)_\eta$, the set of which we denoted by $\beta(\tilde{\mathcal{B}}^\circ, \tilde{\pi})$ in Definition 2.4.2.

Now suppose that we start with $\tilde{p} \in \times^3 \beta(\tilde{\mathcal{A}}, \alpha^{-1}(\pi_2))$. By assumption, $\alpha(\tilde{p}|_{t^2}) \in \times^3 \beta(\mathcal{A}_2^\circ, \pi_2)$. On the other hand, since $\tilde{\gamma}(\tilde{\pi}|_{t^2}) = \pi_2$, there is an element, not unique, say $\tilde{q} \in \times^3 \beta(\tilde{\mathcal{B}}^\circ, \tilde{\pi})$, such that $\tilde{\gamma}_\eta(\tilde{q}|_{t^2}) = \alpha(\tilde{p}|_{t^2})$. This implies that $(\tilde{p}, \tilde{q}) \in \times^3 \beta(\tilde{\mathcal{A}}, (\tilde{\mathcal{B}}^\circ)_\eta, \tilde{\gamma}_\eta^{-1} \circ \alpha)$. Therefore, by the construction of the previous section, we have

$$\omega(\tilde{p}, \tilde{q}, \tilde{\gamma}_\eta^{-1} \circ \alpha) \in \Omega_{\tilde{\mathcal{B}}/k}^1,$$

where $\tilde{\mathcal{B}} := (\tilde{\mathcal{B}}^\circ)_\eta/(t)$. If we let $\tilde{\mathcal{B}}^\circ := (\tilde{\mathcal{B}}^\circ)/(t)$ then by assumption $\tilde{\mathcal{B}}^\circ$ is a discrete valuation ring with field of fractions $\tilde{\mathcal{B}}$. Therefore, the residue $\text{res}(\omega(\tilde{p}, \tilde{q}, \tilde{\gamma}_\eta^{-1} \circ \alpha))$ of $\omega(\tilde{p}, \tilde{q}, \tilde{\gamma}_\eta^{-1} \circ \alpha)$ is a well-defined element of the residue field $\tilde{\mathcal{B}}^\circ/(\tilde{\pi})$ of the closed point of $\tilde{\mathcal{B}}^\circ$. This construction depends on the liftings $\tilde{\mathcal{B}}^\circ$ and \tilde{q} , and the liftings $\tilde{\mathcal{A}}$ and \tilde{p} . Recall the residue map

$$\text{res} : \Lambda^3 \beta(\tilde{\mathcal{B}}^\circ, \tilde{\pi}) \rightarrow \Lambda^2(\tilde{\mathcal{B}}^\circ/(\tilde{\pi}))^\times$$

in (2.4.4) and the map ℓ

$$\ell : \Lambda^2(\tilde{\mathcal{B}}^\circ/(\tilde{\pi}))^\times \rightarrow \tilde{\mathcal{B}}^\circ/(\tilde{\pi}).$$

We define

$$\varepsilon_{\tilde{q}}(\tilde{p}) := \ell(\text{res}(\tilde{q})) + \text{res}(\omega(\tilde{p}, \tilde{q}, \tilde{\gamma}_\eta^{-1} \circ \alpha)).$$

We first show that the expression above is independent of the choice of local liftings \tilde{q} .

Proposition 3.1.2. *Suppose that we have another lifting $\widehat{\mathcal{B}}^\circ$ of \mathcal{A}_2° , with the isomorphism*

$$\hat{\gamma} : (\widehat{\mathcal{B}}^\circ)_2 := \widehat{\mathcal{B}}^\circ / (t^2) \xrightarrow{\sim} \mathcal{A}_2^\circ,$$

$\hat{\pi}$, a uniformizer on $\widehat{\mathcal{B}}^\circ$ such that $\hat{\gamma}(\hat{\pi}|_{t^2}) = \pi_2$, and $\hat{q} \in \beta(\widehat{\mathcal{B}}^\circ, \hat{\pi})$, a $\hat{\pi}$ -good lifting of \tilde{p} , i.e. $\hat{\gamma}_\eta(|\hat{q}_{t^2}) = \alpha(\tilde{p}|_{t^2})$, then we have

$$\hat{\gamma}(\varepsilon_{\hat{q}}(\tilde{p})) = \underline{\gamma}(\varepsilon_{\tilde{q}}(\tilde{p})),$$

where $\underline{\gamma}$ (resp. $\hat{\gamma}$) is the map from $\widetilde{\mathcal{B}}^\circ / (\hat{\pi})$ (resp. $\widehat{\mathcal{B}}^\circ / (\hat{\pi})$) to $\mathcal{A}_2^\circ / (t, \pi_2) = k'$ induced by $\tilde{\gamma}$ (resp. $\hat{\gamma}$).

Proof. By the transitivity property (P5) applied on the generic point, we have

$$\tilde{\gamma}_{\eta,*}(\omega(\tilde{p}, \tilde{q}, \tilde{\gamma}_\eta^{-1} \circ \alpha)) + \hat{\gamma}_{\eta,*}\omega(\tilde{q}, \hat{q}, \hat{\gamma}_\eta^{-1} \circ \tilde{\gamma}_\eta) = \hat{\gamma}_{\eta,*}\omega(\tilde{p}, \hat{q}, \hat{\gamma}_\eta^{-1} \circ \alpha).$$

Taking residues in the above expression, the statement that we would like to prove reduces to:

$$\underline{\gamma} \circ \ell(\text{res}_{\hat{\pi}}(\tilde{q})) - \hat{\gamma} \circ \ell(\text{res}_{\hat{\pi}}(\hat{q})) = \hat{\gamma} \circ \text{res}_{\hat{\pi}}(\omega(\tilde{q}, \hat{q}, \hat{\gamma}^{-1} \circ \tilde{\gamma})).$$

Note that in the last expression since \hat{q} and \tilde{q} are good liftings, i.e.

$$(\tilde{q}, \hat{q}) \in \times^3 \beta(\widetilde{\mathcal{B}}^\circ, \widehat{\mathcal{B}}^\circ, \hat{\gamma}^{-1} \circ \tilde{\gamma}, \hat{\pi}, \hat{\pi}),$$

the residue $\text{res}_{\hat{\pi}}(\omega(\tilde{q}, \hat{q}, \hat{\gamma}^{-1} \circ \tilde{\gamma}))$ can be computed in terms of ℓ , by the residue property (P8), and hence gives the above expression. \square

Definition 3.1.3. With the notation as above we define $\varepsilon : \times^3 \beta(\tilde{\mathcal{A}}, \alpha^{-1}(\pi_2)) \rightarrow k'$, by the formula

$$\varepsilon(\tilde{p}) := \underline{\gamma}(\varepsilon_{\tilde{q}}(\tilde{p})) = \underline{\gamma}(\ell(\text{res}(\tilde{q})) + \text{res}(\omega(\tilde{p}, \tilde{q}, \tilde{\gamma}_\eta^{-1} \circ \alpha))).$$

By the previous proposition, this definition is independent of the choice of the local good lifting \tilde{q} . In fact, by the multi-linearity and the anti-symmetry properties of ω , ℓ and res , ε induces a map

$$\varepsilon : \Lambda^3 \beta(\tilde{\mathcal{A}}, \alpha^{-1}(\pi_2)) \rightarrow k'.$$

3.2. The construction of ρ . Suppose that C_2/S_2 is a smooth and proper curve. We let $C := C_2 \times_{S_2} S_1$ be its special fiber, η , the generic point of C , and $|C|$, the set of closed points of C . Similarly, for an open subscheme $U_2 \subseteq C_2$, we let U to be the special fiber. We call an element of $k(C_2)^\times := \mathcal{O}_{C_2, \eta}^\times$, a rational function on C_2 . We call $\mathcal{P}_2 = \{\pi_{2,c} | c \in |C|\}$ a *system of uniformizers*, if for every $c \in |C|$, $\pi_{2,c} \in \hat{\mathcal{O}}_{C_2, c}$ is a uniformizer. Here $\hat{\mathcal{O}}_{C_2, c}$ is the completion of the local ring $\mathcal{O}_{C_2, c}$ at its maximal ideal. We say that a rational function f is \mathcal{P}_2 -good, if f is $\pi_{2,c}$ -good for every $c \in |C|$. We denote the group of \mathcal{P}_2 -good rational functions by $k(C_2, \mathcal{P}_2)^\times$. Similarly, we say that $p = (f, g, h)$ is \mathcal{P}_2 -good, if all of f, g , and h are. If we let $|p| := |\text{div}(f|_C)| \cup |\text{div}(g|_C)| \cup |\text{div}(h|_C)| \subseteq |C|$, then p is $\pi_{2,c}$ -good for $c \notin |p|$, for any choice of a uniformizer $\pi_{2,c}$. On the other hand, for example, a triple of functions of the form $(s, s + t, \cdot)$ is not $\pi_{2,c}$ -good on the spectrum of $k[[s, t]]/(t^2)$, for any choice of a uniformizer π_2 .

Fix $p = (f, g, h) \in \times^3 k(C_2, \mathcal{P}_2)^\times$. Choose a lifting $\tilde{\mathcal{A}}$ of $\mathcal{A}_2 := \mathcal{O}_{C_2, \eta}$, as in §3.1, together with a fixed isomorphism

$$\alpha : \tilde{\mathcal{A}}_2 := \tilde{\mathcal{A}} / (t^2) \xrightarrow{\sim} \mathcal{A}_2.$$

Choose any lifting \tilde{p} of p to $\tilde{\mathcal{A}}$. For any $c \in |C|$, we apply the construction in §3.1, with $\tilde{\mathcal{A}}_2^\circ := \hat{\mathcal{O}}_{C_2, c}$. Since, by assumption, p is \mathcal{P}_2 -good, we have, for any $c \in |C|$, $\tilde{p} \in \times^3 \beta(\tilde{\mathcal{A}}, \alpha^{-1}(\pi_{2,c}))$. Therefore corresponding to c , we have $\varepsilon(\tilde{p}; c) \in k(c)$. For any element y in $k(c)$, let $\text{Tr}_k(y)$ denote its trace from $k(c)$ to k .

We define $\rho(\tilde{p})$ as:

$$(3.2.1) \quad \rho(\tilde{p}) = \sum_{c \in |C|} \text{Tr}_k \varepsilon(\tilde{p}; c) \in k.$$

We will need the following proposition in order to show that this sum makes sense, and to show that it defines the function that we are seeking.

Proposition 3.2.1. *For any lifting \tilde{p} , the sum in the definition (3.2.1) of $\rho(\tilde{p})$ is finite. Moreover, for any other lifting \hat{p} , we have $\rho(\tilde{p}) = \rho(\hat{p})$.*

Proof. First, let us choose an affine open set $U_2 \subseteq C_2$ such that U_2 has a lifting \tilde{U} and $p|_{U_2}$ has a lifting \tilde{p}_U to \tilde{U} , which is good with respect to a system of parameters $\mathcal{P}_2(\tilde{U})$ on \tilde{U} which lift $\mathcal{P}_2|_{U_2}$. More precisely, \tilde{U} is an affine scheme whose ring of functions \tilde{A}_U is an object of \mathcal{C}_A^1 together with an isomorphism

$$\alpha_{U_2} : \tilde{A}_U|_{t^2} \xrightarrow{\sim} \mathcal{A}_{2,U},$$

where $\mathcal{A}_{2,U}$ is the ring of functions of U_2 ; $\mathcal{P}_2(\tilde{U})$ is a system of parameters on \tilde{A}_U which lift $\alpha_{U_2}^{-1}(\mathcal{P}_2|_{U_2})$; and $\tilde{p}_U \in \tilde{A}_U$ is a lifting of $\alpha_{U_2}^{-1}(p|_{U_2})$.

Then on the generic point, we have the generic lifting $\tilde{p}_{U,\eta}$. Let us look at the summands that appear in the expression (3.2.1) for $\rho(\tilde{p}_{U,\eta})$. These are the terms

$$\text{Tr}_k \varepsilon(\tilde{p}_{U,\eta}; c) = \text{Tr}_k(\ell(\text{res}_c(\tilde{q}_c)) + \text{res}_c \omega(\tilde{p}_{U,\eta}, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_{U_2,\eta})),$$

where \tilde{q}_c is any local good choice of lifting of p at c . Again to be more precise, $\tilde{\mathcal{B}}_c^\circ$ is an object of \mathcal{C}_A^1 , together with an isomorphism

$$\tilde{\gamma}_c : \tilde{\mathcal{B}}_c^\circ|_{t^2} \xrightarrow{\sim} \hat{\mathcal{O}}_{C_2,c},$$

and \tilde{q}_c is a lifting of $\tilde{\gamma}_{c,\eta}^{-1}(p_\eta)$, which is good with respect to a uniformizer in $\tilde{\mathcal{B}}_c^\circ$ which lifts $\tilde{\gamma}_c^{-1}(\pi_{2,c})$.

In case $c \in U$, $\tilde{p}_{U,\eta}$, by our assumption, is a good lifting at c . Since \tilde{q}_c is also a good lifting, by the residue property (P8) we have

$$\text{Tr}_k \text{res}_c \omega(\tilde{p}_{U,\eta}, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_{U_2,\eta}) = \text{Tr}_k \ell(\text{res}_c \tilde{p}_U) - \text{Tr}_k \ell(\text{res}_c \tilde{q}_c)$$

and therefore $\text{Tr}_k \varepsilon(\tilde{p}_{U,\eta}; c) = \text{Tr}_k \ell(\text{res}_c \tilde{p}_U)$. Since \tilde{p}_U is a good lifting of $p|_{U_2}$, right hand side is non-zero only when $c \in |p|$. Hence for $c \in U$, the terms $\varepsilon(\tilde{p}_{U,\eta}; c)$ are non-zero for only finitely many c . Since $C \setminus U$ is a finite set, we conclude that the sum (3.2.1) for $\rho(\tilde{p}_{U,\eta})$ is finite.

Now let \hat{p}_η and \tilde{p}_η be arbitrary generic liftings. In other words, we have isomorphisms

$$\hat{\alpha} : \hat{\mathcal{A}}|_{t^2} \xrightarrow{\sim} \mathcal{A}_2$$

and

$$\tilde{\alpha} : \tilde{\mathcal{A}}|_{t^2} \xrightarrow{\sim} \mathcal{A}_2,$$

and elements $\hat{p}_\eta \in \hat{\mathcal{A}}$ and $\tilde{p}_\eta \in \tilde{\mathcal{A}}$ which lift $\hat{\alpha}^{-1}(p_\eta)$ and $\tilde{\alpha}^{-1}(p_\eta)$. The differences $\varepsilon(\tilde{p}_\eta; c) - \varepsilon(\hat{p}_\eta; c)$ can be computed by choosing a good local lifting \tilde{q}_c . Namely, we have

$$\begin{aligned} \varepsilon(\tilde{p}_\eta; c) - \varepsilon(\hat{p}_\eta; c) &= \tilde{\gamma}_c(\varepsilon_{\tilde{q}_c}(\tilde{p}_\eta)) - \tilde{\gamma}_c(\varepsilon_{\tilde{q}_c}(\hat{p}_\eta)) = \tilde{\gamma}_c(\varepsilon_{\tilde{q}_c}(\tilde{p}_\eta) - \varepsilon_{\tilde{q}_c}(\hat{p}_\eta)) \\ &= \tilde{\gamma}_c(\text{res}(\omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \tilde{\alpha}_c)) - \text{res}(\omega(\hat{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \tilde{\alpha}_c))). \end{aligned}$$

In the last expression, $\tilde{\alpha}_c$ (resp. $\hat{\alpha}_c$) is the map induced by $\tilde{\alpha}$ (resp. $\hat{\alpha}$) after completing at c ; and $\tilde{\gamma}_{c,\eta}$ is the map induced by $\tilde{\gamma}_c$ after localising at η . By the transitivity property (P5),

$$\omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \tilde{\alpha}_c) - \omega(\hat{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \hat{\alpha}_c) = (\tilde{\gamma}_{c,\eta}^{-1} \circ \tilde{\alpha}_c)_* \omega(\tilde{p}_\eta, \hat{p}_\eta, \hat{\alpha}_c^{-1} \circ \tilde{\alpha}_c).$$

Therefore, after taking residues, we have

$$\varepsilon(\tilde{p}_\eta; c) - \varepsilon(\hat{p}_\eta; c) = \hat{\alpha}_c(\text{res} \omega(\tilde{p}_\eta, \hat{p}_\eta, \hat{\alpha}_c^{-1} \circ \tilde{\alpha}_c)) = \text{res}(\hat{\alpha}_{c,*}(\omega(\tilde{p}_\eta, \hat{p}_\eta, \hat{\alpha}_c^{-1} \circ \tilde{\alpha}_c)))$$

By the functoriality property (P3), $\hat{\alpha}_{c,*} \omega(\tilde{p}_\eta, \hat{p}_\eta, \hat{\alpha}_c^{-1} \circ \tilde{\alpha}_c)$ is the restriction to the formal neighbourhood of c , of the rational 1-form $\hat{\alpha}_*(\omega(\tilde{p}_\eta, \hat{p}_\eta, \hat{\alpha}^{-1} \circ \tilde{\alpha}))$. Using this in the last equality above, and taking traces, we obtain:

$$(3.2.2) \quad \text{Tr}_k \varepsilon(\tilde{p}_\eta; c) - \text{Tr}_k \varepsilon(\hat{p}_\eta; c) = \text{Tr}_k \text{res}_c \hat{\alpha}_*(\omega(\tilde{p}_\eta, \hat{p}_\eta, \hat{\alpha}^{-1} \circ \tilde{\alpha})),$$

for every $c \in |C|$.

Since the rational 1-form $\hat{\alpha}_*(\omega(\tilde{p}_\eta, \hat{p}_\eta, \hat{\alpha}^{-1} \circ \tilde{\alpha}))$ has only finitely many singularities, by the identity (3.2.2), the finiteness of the sum for $\rho(\tilde{p}_\eta)$ and for $\rho(\hat{p}_\eta)$ are equivalent. Since we know the finiteness of the sum for one lifting, namely for $\tilde{p}_{U,\eta}$, we therefore know it for all the liftings. This proves the first statement.

In order to prove the second statement, we note that by summing (3.2.2) above over all $c \in |C|$:

$$\rho(\tilde{p}_\eta) - \rho(\hat{p}_\eta) = \sum_{c \in |C|} \text{Tr}_k \text{res}_c \hat{\alpha}_*(\omega(\tilde{p}_\eta, \hat{p}_\eta, \hat{\alpha}^{-1} \circ \tilde{\alpha})).$$

Since the sum of the residues of a 1-form on a curve is 0, we have $\rho(\tilde{p}_\eta) = \rho(\hat{p}_\eta)$. \square

Definition 3.2.2. For $p \in \times^3 k(C_2, \mathcal{P}_2)^\times$, we define the value of the regulator ρ on p as

$$(3.2.3) \quad \rho(p) := \sum_{c \in |C|} \text{Tr}_k(\ell(\text{res}_c(\tilde{q}_c)) + \text{res}_c \omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c)),$$

where \tilde{p}_η is a lifting of p_η , through a generic lifting $\tilde{\mathcal{A}}$ and an isomorphism $\alpha : \tilde{\mathcal{A}}/(t^2) \xrightarrow{\sim} \mathcal{O}_{C_2,\eta}$; and \tilde{q}_c is a local good choice of a lifting, through a local lifting $\tilde{\mathcal{B}}_c^\circ$ and an isomorphism $\tilde{\gamma}_c : \tilde{\mathcal{B}}_c^\circ/(t^2) \xrightarrow{\sim} \hat{\mathcal{O}}_{C_2,c}$. By the above Proposition 3.1.2 and Proposition 3.2.1 this sum is finite and is independent of the choice of a generic lifting \tilde{p}_η and the choice of good local liftings $\{\tilde{q}_c | c \in C\}$.

By the multi-linearity and the anti-symmetry of the functions appearing in the definition of ρ , we see that ρ induces a map

$$\rho : \Lambda^3 k(C_2, \mathcal{P}_2)^\times \rightarrow k.$$

The following corollary gives a more explicit expression for $\rho(p)$. Choose a cover $\{U_i\}_{1 \leq i \leq r}$ of C_2 by open sets such that there are liftings \tilde{U}_i of U_i to smooth schemes over S whose ring of regular functions lie in \mathcal{C}_A^1 , and functions \tilde{p}_i on \tilde{U}_i which are good liftings of $p|_{U_i}$, with respect to a system of uniformizers which lift that on U_i . If X is any subset of U_i , let $|X|$ denote the set of closed points in X and we let $r_X(\tilde{p}_i)$ denote the sum of the local contributions $\text{Tr}_k \ell(\text{res}_x(\tilde{p}_i))$, for $x \in |X|$. Note that since \tilde{p}_i is a good lifting of p on U_i , these contributions are possibly non-zero only for $x \in |X| \cap |p|$. Similarly, if ω is any rational 1-form on U_i , we let ω_X denote the sum of the residues $\sum_{x \in |X|} \text{Tr}_k \text{res}_x \omega$.

Corollary 3.2.3. *With notation as above, let $U'_i := U_i \setminus \cup_{i+1 \leq j \leq r} U_j$, for $i < r$. Then we have*

$$(3.2.4) \quad \rho(p) = r_{U_r}(\tilde{p}_r) + \sum_{1 \leq i \leq r-1} (r_{U'_i}(\tilde{p}_i) + \omega_{U'_i}(\tilde{p}_r, \tilde{p}_i, \chi_{U_i \cap U_r})),$$

where $\chi_{U_i \cap U_r}$ denotes the isomorphism between $(\tilde{U}_r|_{U_i \cap U_r})|_{t^2} \xrightarrow{\sim} (\tilde{U}_i|_{U_i \cap U_r})|_{t^2}$.

Proof. In order to compute $\rho(p)$ we need to choose a generic lifting of p and a collection of good local liftings at each closed point. We let $\tilde{p}_{r,\eta}$ be the generic lifting. We choose the good local liftings \tilde{q}_c as follows. If $c \in |U_r|$, let $\tilde{q}_c = \tilde{p}_{U_r,c}$. Otherwise, there is a unique $1 \leq i \leq r-1$ such that $c \in |U'_i|$ and we let $\tilde{q}_c := \tilde{p}_{i,c}$. Then we have

$$\rho(p) = \rho(\tilde{p}_{r,\eta}) = \sum_{c \in |C|} \text{Tr}_k \varepsilon_{\tilde{q}_c}(\tilde{p}_{r,\eta}) = \sum_{C \in |U_r|} \text{Tr}_k \varepsilon_{\tilde{p}_{r,c}}(\tilde{p}_{r,\eta}) + \sum_{1 \leq i \leq r-1} \sum_{c \in |U'_i|} \text{Tr}_k \varepsilon_{\tilde{p}_{i,c}}(\tilde{p}_{r,\eta}).$$

The statement then follows after noting that $\text{Tr}_k \varepsilon_{\tilde{p}_{r,c}}(\tilde{p}_{r,\eta}) = \text{Tr}_k \ell(\text{res}_c \tilde{p}_r)$. \square

3.3. Bloch group relation. In this section, we will construct a residue map from a weight 3 infinitesimal Bloch complex to a weight 2 infinitesimal Bloch complex and study the relation of ρ to this map of complexes. The results of this section will be used in the proof of the infinitesimal strong reciprocity conjecture.

3.3.1. *Infinitesimal Bloch group of a discrete valuation ring and the residue map.* In this subsection we follow the notation in §3.1.1. We first define the Bloch group $B_2(\mathcal{A}_2^\circ, \pi_2)$.

Definition 3.3.1. Let $\beta^b(\mathcal{A}_2^\circ, \pi_2) := \{x|x, 1-x \in \beta(\mathcal{A}_2^\circ, \pi_2)\}$. We define the Bloch group $B_2(\mathcal{A}_2^\circ, \pi_2)$ to be the quotient of the free abelian group generated by the symbols $\{x\}_2$, with $x \in \beta^b(\mathcal{A}_2^\circ, \pi_2)$, modulo the relations generated by $\{x\}_2 - \{y\}_2 + \{(y/x)\}_2 - \{(1-x^{-1})/(1-y^{-1})\}_2 + \{(1-x)/(1-y)\}_2$, for $x, y \in \beta^b(\mathcal{A}_2^\circ, \pi_2)$, with $x-y \in \beta(\mathcal{A}_2^\circ, \pi_2)$. As usual this gives us a complex $B_2(\mathcal{A}_2^\circ, \pi_2) \xrightarrow{\delta} \Lambda^2 \beta(\mathcal{A}_2^\circ, \pi_2)$, where $\delta(\{x\}_2) := (1-x) \wedge x$.

This induces part of a weight 3 complex: $B_2(\mathcal{A}_2^\circ, \pi_2) \otimes \beta(\mathcal{A}_2^\circ, \pi_2) \xrightarrow{\Delta} \Lambda^3 \beta(\mathcal{A}_2^\circ, \pi_2)$, where Δ sends $\{x\}_2 \otimes y$ to $\delta(x) \wedge y$. We also have a residue map that gives a commutative diagram

$$\begin{array}{ccc} B_2(\mathcal{A}_2^\circ, \pi_2) \otimes \beta(\mathcal{A}_2^\circ, \pi_2) & \xrightarrow{\Delta} & \Lambda^3 \beta(\mathcal{A}_2^\circ, \pi_2) \\ \text{res} \downarrow & & \text{res} \downarrow \\ B_2(\mathcal{A}_2^\circ/(\pi_2)) & \xrightarrow{\delta} & \Lambda^2(\mathcal{A}_2^\circ/(\pi_2))^\times, \end{array}$$

as in [3]. The residue map on $\Lambda^3 \beta(\mathcal{A}_2^\circ, \pi_2)$ is defined exactly as in §2.4.5. The residue map on $B_2(\mathcal{A}_2^\circ, \pi_2) \otimes \beta(\mathcal{A}_2^\circ, \pi_2)$ is characterised by the following properties:

- (i) $\text{res}(\{x\}_2 \otimes y) = 0$, if $x \notin (\mathcal{A}_2^\circ)^b := \{a|a, 1-a \in (\mathcal{A}_2^\circ)^\times\}$ and
- (ii) $\text{res}(\{x\}_2 \otimes y) = n\{\bar{x}\}_2$, if $x \in (\mathcal{A}_2^\circ)^b$ and $y = \pi_2^n u$, with $u \in (\mathcal{A}_2^\circ)^\times$, and \bar{x} is the reduction of x modulo (π_2) .

3.3.2. *Global version.* Let us now assume that C_2/S_2 is a smooth and proper curve, C denote its special fiber and \mathcal{P}_2 a system of uniformizers on C_2 . For a finite extension k'/k and $n \geq 1$, we write $k'_n := k'[t]/(t^n)$. Note that for every $c \in |C|$, there is a choice of a uniformizer $\pi_{2,c} \in \mathcal{P}_2$, and this gives a quotient $\mathcal{O}_{C_2,c}/(\pi_{2,c})$, which is canonically isomorphic to $k(c)_2$.

Definition 3.3.2. Let $k(C_2, \mathcal{P}_2)^b := \{x|x, 1-x \in k(C_2, \mathcal{P}_2)^\times\}$. Then we define the Bloch group $B_2(k(C_2), \mathcal{P}_2)$ as the quotient of the free abelian group generated by the symbols $\{x\}_2$, with $x \in k(C_2, \mathcal{P}_2)^b$, modulo the relations generated by $\{x\}_2 - \{y\}_2 + \{(y/x)\}_2 - \{(1-x^{-1})/(1-y^{-1})\}_2 + \{(1-x)/(1-y)\}_2$, for $x, y \in k(C_2, \mathcal{P}_2)^b$, with $x-y \in k(C_2, \mathcal{P}_2)^\times$. This gives a complex $B_2(k(C_2), \mathcal{P}_2) \xrightarrow{\delta} \Lambda^2 k(C_2), \mathcal{P}_2)^\times$.

We have natural maps $k(C_2, \mathcal{P}_2)^\times \rightarrow \beta(\mathcal{O}_{C_2,c}, \pi_{2,c})$ and $B_2(k(C_2), \mathcal{P}_2) \rightarrow B_2(\mathcal{O}_{C_2,c}, \pi_{2,c})$. Combining these with the sum of the residue maps in §3.3.1, corresponding to $\mathcal{A}_2^\circ = \mathcal{O}_{C_2,c}$, over all $c \in |C|$ and identifying $\mathcal{O}_{C_2,c}/(\pi_{2,c})$ with $k(c)_2$, we obtain a commutative diagram:

$$\begin{array}{ccc} B_2(k(C_2), \mathcal{P}_2) \otimes k(C_2, \mathcal{P}_2)^\times & \xrightarrow{\Delta} & \Lambda^3 k(C_2, \mathcal{P}_2)^\times \\ \text{res} \downarrow & & \text{res} \downarrow \\ \bigoplus_{c \in |C|} B_2(k(c)_2) & \xrightarrow{\delta} & \bigoplus_{c \in |C|} \Lambda^2 k(c)_2^\times. \end{array}$$

We constructed an additive dilogarithm map $li_2 : B_2(k_2) \rightarrow k$, for any field k of characteristic 0 in [11]. Let us briefly recall this function. It is given by the explicit formula

$$li_2(\{s+at\}_2) = -\frac{a^3}{2s^2(1-s)^2},$$

for $s \in k^b := k^\times \setminus \{1\}$ and $a \in k$, on the generators of $B_2(k_2)$. It has also the following description using the notation of §2.4.5: taking $B := k[[t]]$ as in §2.4.5, we have a map $\ell : \Lambda^2 k[[t]]^\times \rightarrow k$. The composition $B_2(k[[t]]) \xrightarrow{\delta} \Lambda^2 k[[t]]^\times \xrightarrow{\ell} k$, then factors through the natural projection $B_2(k[[t]]) \rightarrow B_2(k_2)$ to give $li_2 : B_2(k_2) \rightarrow k$.

Let $li_{2,c}$ denote the additive dilogarithm from $B_2(k(c)_2)$ to $k(c)$, and $li_{2,|C|}$ denote the composition $\bigoplus_{c \in |C|} B_2(k(c)_2) \xrightarrow{\oplus li_{2,c}} \bigoplus_{c \in |C|} k(c) \xrightarrow{\sum \text{Tr}_{k(c)/k}} k$. Then we have the following relation of this function with the regulator ρ .

Proposition 3.3.3. *The functions $li_{2,|C|} \circ res$ and $\rho \circ \Delta$ from $B_2(k(C_2, \mathcal{P}_2)) \otimes k(C_2, \mathcal{P}_2)^\times$ to k are equal.*

Proof. We need to show that both of the functions agree on a general element $\{f\}_2 \otimes g \in B_2(k(C_2, \mathcal{P}_2)) \otimes k(C_2, \mathcal{P}_2)^\times$. We have $\Delta(\{f\}_2 \otimes g) = (1-f) \wedge f \wedge g$, and hence $\rho \circ \Delta(\{f\}_2 \otimes g) = \rho(p)$, with $p = (1-f, f, g)$. In order to compute $\rho(p)$, we need to choose a generic lifting of p_η and good local liftings of p_c , for every $c \in |C|$.

Let \tilde{A} denote a lifting of $\mathcal{O}_{C_2, \eta}$, together with an isomorphism $\alpha : \tilde{A}|_{t^2} \xrightarrow{\sim} \mathcal{O}_{C_2, \eta}$; and let $\tilde{\mathcal{B}}_c^\circ$ be a lifting of $\hat{\mathcal{O}}_{C_2, c}$, together with isomorphisms $\tilde{\gamma}_c : \tilde{\mathcal{B}}_c^\circ|_{t^2} \xrightarrow{\sim} \hat{\mathcal{O}}_{C_2, c}$, for every $c \in |C|$. In order to choose a generic lifting of p_η , choose liftings \tilde{f}_η and \tilde{g}_η of $\alpha^{-1}(f_\eta)$ and $\alpha^{-1}(g_\eta)$ and let $\tilde{p}_\eta := (1 - \tilde{f}_\eta, \tilde{f}_\eta, \tilde{g}_\eta)$. For the local lifting at c , first choose a uniformizer $\tilde{\pi}_c$ on $\tilde{\mathcal{B}}_c^\circ$ lifting the given one $\pi_{2,c}$ on $\hat{\mathcal{O}}_{C_2, c}$. Since g_c is $\pi_{2,c}$ -good, let \tilde{g}_c be any $\tilde{\pi}_c$ -good lifting of g_c . By assumption both f_c and $1-f_c$ are $\pi_{2,c}$ -good. If $c \notin \text{div}(f|_C) \cup \text{div}((1-f)|_C)$, and if \tilde{f}_c is any lifting of f_c then both \tilde{f}_c and $1-\tilde{f}_c$ will be $\tilde{\pi}_c$ -good. If $f|_C$ vanishes at c , choose a $\tilde{\pi}_c$ -good lifting \tilde{f}_c of f_c , then since $(1-f)|_C$ is not 0 at c , $1-\tilde{f}_c$ becomes a $\tilde{\pi}_c$ -good lifting of $1-f_c$. In case, $(1-f)|_C$ vanishes at c , choose \tilde{f}_c such that $1-\tilde{f}_c$ is a $\tilde{\pi}_c$ -good lifting of $1-f_c$, then also \tilde{f}_c becomes a $\tilde{\pi}_c$ -good lifting of f_c . The other remaining case is when $f|_C$ and $(1-f)|_C$ both have a pole at c . In this case, if we let \tilde{f}_c as any $\tilde{\pi}_c$ -good lifting of f_c then $1-\tilde{f}_c$ is a $\tilde{\pi}_c$ -good lifting of $1-f_c$. In each of these cases, we let $\tilde{q}_c := (1 - \tilde{f}_c, \tilde{f}_c, \tilde{g}_c)$ as the good local lifting of p_c .

By definition, $\rho(p) = \sum_{c \in |C|} \text{Tr}_k \varepsilon_{\tilde{q}_c}(\tilde{p}_\eta) = \sum_{c \in |C|} (\text{Tr}_k \ell(res \tilde{q}_c) + \text{Tr}_k res_c \omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c, \eta}^{-1} \circ \alpha_c))$. First we deal with the second summand. Note that

$$res_c \omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c, \eta}^{-1} \circ \alpha_c) = res_c \omega(\alpha_c(\tilde{p}_\eta), \tilde{\gamma}_{c, \eta}(\tilde{q}_c), id),$$

with $\alpha_c(\tilde{p}_\eta) = (1 - \alpha_c(\tilde{f}_\eta), \alpha_c(\tilde{f}_\eta), \alpha_c(\tilde{g}_\eta))$ and $\tilde{\gamma}_{c, \eta}(\tilde{q}_c) = (1 - \tilde{\gamma}_{c, \eta}(\tilde{f}_c), \tilde{\gamma}_{c, \eta}(\tilde{f}_c), \tilde{\gamma}_{c, \eta}(\tilde{g}_c))$. By the construction of $\tilde{f}_\eta, \tilde{g}_\eta, \tilde{f}_c$ and \tilde{g}_c , we see that $(\alpha_c(\tilde{p}_\eta), \tilde{\gamma}_{c, \eta}(\tilde{q}_c)) \in \Lambda^3 \beta(\hat{K}_{C_2, c}, (t^2))$, where $\hat{K}_{C_2, c}$ is the localization of $\hat{\mathcal{O}}_{C_2, c}$ at the generic point of its special fiber. By the above expressions, we have:

$$(\alpha_c(\tilde{p}_\eta), \tilde{\gamma}_{c, \eta}(\tilde{q}_c)) = \Delta(\{(\alpha_c(\tilde{f}_\eta), \tilde{\gamma}_{c, \eta}(\tilde{f}_c))\}_2 \otimes (\alpha_c(\tilde{g}_\eta), \tilde{\gamma}_{c, \eta}(\tilde{g}_c))),$$

with $\{(\alpha_c(\tilde{f}_\eta), \tilde{\gamma}_{c, \eta}(\tilde{f}_c))\}_2 \otimes (\alpha_c(\tilde{g}_\eta), \tilde{\gamma}_{c, \eta}(\tilde{g}_c)) \in B_2(\hat{K}_{C_2, c}, (t^2)) \otimes \beta(\hat{K}_{C_2, c}, (t^2))$. The relative Bloch group relation for ω , namely Corollary 2.4.6, then implies that $\omega(\alpha_c(\tilde{p}_\eta), \tilde{\gamma}_{c, \eta}(\tilde{q}_c), id) = 0$. Hence

$$\rho(p) = \sum_{c \in |C|} \text{Tr}_k \ell(res \tilde{q}_c).$$

Let us first look at the expression $res \tilde{q}_c = res(1 - \tilde{f}_c) \wedge \tilde{f}_c \wedge \tilde{g}_c$. Since $\Delta(\{\tilde{f}_c\}_2 \otimes \tilde{g}_c) = (1 - \tilde{f}_c) \wedge \tilde{f}_c \wedge \tilde{g}_c$, the commutative diagram in §3.3.1, implies that

$$res \tilde{q}_c = \delta \circ res(\{\tilde{f}_c\}_2 \otimes \tilde{g}_c) \in \Lambda^2(\mathcal{O}_{C_2, c}/(\tilde{\pi}_c))^\times = \Lambda^2 k(c)_2^\times,$$

where the last equality is written using the *canonical* isomorphism between the k -algebras $\mathcal{O}_{C_2, c}/(\tilde{\pi}_c)$ and $k(c)_2$. Applying ℓ to the above expression we obtain,

$$\ell(res \tilde{q}_c) = \ell \circ \delta \circ res(\{\tilde{f}_c\}_2 \otimes \tilde{g}_c).$$

On the other hand, $\ell \circ \delta = li_2$, and li_2 depends on its argument only modulo (t^2) , as was described in the paragraph before this proposition. Therefore,

$$\ell(res \tilde{q}_c) = li_{2, c} \circ res(\{f_c\}_2 \otimes g_c),$$

since \tilde{f}_c and \tilde{g}_c are liftings of f_c and g_c . Summing over all $c \in |C|$ and taking traces, we obtain

$$\rho \circ \Delta(\{f\}_2 \otimes g) = \rho(p) = li_{2, |C|} \circ res(\{f\}_2 \otimes g).$$

This finishes the proof. \square

3.4. Strong reciprocity conjecture. In this section, we assume that k is algebraically closed. The following theorem of Suslin is proven in much more generality in [10].

Theorem. (*Suslin Reciprocity*) *Let C/k be a smooth and projective curve over k , then the sum of the residue maps*

$$K_3^M(k(C)) \xrightarrow{\oplus \text{res}_c} \bigoplus_{c \in |C|} K_2^M(k) \xrightarrow{\Sigma} K_2^M(k)$$

is 0.

There are Bloch complexes of weight n defined by Goncharov which conjecturally compute motivic cohomology of weight n . Let K be a field then

$$\Gamma(K, n) : B_n(K) \rightarrow B_{n-1}(K) \otimes K^\times \rightarrow \cdots \rightarrow B_2(K) \otimes \Lambda^{n-2} K^\times \rightarrow \Lambda^n K^\times.$$

The cohomological complex $\Gamma(n, K)$ is concentrated in degrees $[1, n]$, with $B_n(K)$ in degree 1 and conjecturally $H^i(\Gamma(n, K)_\mathbb{Q}) \simeq H_{\mathcal{M}}^i(K, \mathbb{Q}(n))$. Here $H_{\mathcal{M}}^i(K, \mathbb{Q}(n))$ denotes motivic cohomology of weight n . By Voevodsky's description in terms of algebraic cycles and the Bloch-Grothendieck-Riemann-Roch theorem, $H_{\mathcal{M}}^i(K, \mathbb{Q}(n)) \simeq K_{2n-i}(K)_\mathbb{Q}^{(n)}$. This is known when $i = n$, since then $H^n(\Gamma(n, K)) \simeq K_n^M(K)_\mathbb{Q} \simeq K_n(K)_\mathbb{Q}^{(n)}$.

In what follows only the Bloch group B_2 will be relevant. There are several different definitions of these groups and it is not known that these definitions are equivalent. Note that the definition of $B_2(A)$ that we have been using is the free abelian group generated by $\{a\}_2$, with $a \in A^\flat := \{a|a, 1 - a \in A^\times\}$, modulo the 5-term functional equation of the dilogarithm.

We have a residue map $\text{res}_c : \Gamma(k(C), 3) \rightarrow \Gamma(k, 2)$, for every $c \in |C|$. Summing them over all $c \in |C|$, we obtain the commutative diagram

$$\begin{array}{ccc} B_3(k(C)) & \longrightarrow & B_2(k(C)) \otimes k(C)^\times \xrightarrow{\Delta} \Lambda^3 k(C)^\times \\ & & \downarrow \text{res}_{|C|} \qquad \qquad \downarrow \text{res}_{|C|} \\ & & B_2(k) \xrightarrow{\delta} \Lambda^2 k^\times. \end{array}$$

By the Suslin reciprocity theorem $\text{Im}(\text{res}_{|C|}) \subseteq \text{Im}(\delta)$ in $\Lambda^2 k^\times$.

Conjecture. (*Strong reciprocity conjecture*) *The residue map $\text{res}_{|C|} : \Gamma(k(C), 3) \rightarrow \Gamma(k, 2)$, is homotopic to 0. More precisely, there is a map h , with $h(k^\times \wedge \Lambda^2 k(C)^\times) = 0$, which gives a commutative diagram:*

$$\begin{array}{ccc} B_3(k(C)) & \longrightarrow & B_2(k(C)) \otimes k(C)^\times \xrightarrow{\Delta} \Lambda^3 k(C)^\times \\ & & \downarrow \text{res}_{|C|} \qquad \swarrow h \qquad \downarrow \text{res}_{|C|} \\ & & B_2(k) \xrightarrow{\delta} \Lambda^2 k^\times. \end{array}$$

We need some preliminary results before proving the infinitesimal version of the strong reciprocity conjecture.

Proposition 3.4.1. *The sum of the local residue maps $\text{res}_{|C|} : \Lambda^2 k(C_2, \mathcal{P}_2)^\times \rightarrow k_2^\times$ is 0.*

Proof. This last statement can be reduced to two statements since $k_2^\times \simeq k^\times \oplus k$. Fix $f \wedge g \in \Lambda^2 k(C_2, \mathcal{P}_2)^\times$. That the map $\Lambda^2 k(C_2, \mathcal{P}_2)^\times \rightarrow k^\times$ is zero follows from applying the classical Weil reciprocity to C and the functions $f|_C$ and $g|_C$. On the other hand, denote the map $\Lambda^2 k(C_2, \mathcal{P}_2)^\times \rightarrow k$ by $\text{res}_{|C|}^\circ$. It is the sum of res_c° , for $c \in |C|$. Here $\text{res}_c^\circ(f \wedge g) := \text{res}_t \frac{1}{t} \text{res}_{\tilde{\pi}_c} (d \log \tilde{f}_c \wedge d \log \tilde{g}_c)$, where $\tilde{\pi}_c$ is a uniformizer lifting $\pi_{2,c}$ and \tilde{f}_c and \tilde{g}_c are $\tilde{\pi}_c$ -good local liftings of f_c and g_c . By the anti-commutativity of the residues we see that

$$\text{res}_{|C|}^\circ = - \sum_{c \in |C|} \text{res}_c \text{res}_t \left(\frac{1}{t} d \log \tilde{f}_c \wedge d \log \tilde{g}_c \right).$$

On the other hand, we see that $\text{res}_t(\frac{1}{t}d\log\tilde{f}_c \wedge d\log\tilde{g}_c)$ defines a rational 1-form ω on C independent of c . The statement then follows from the fact that the sum of the residues of ω on C being 0. \square

The following is an infinitesimal version of the Suslin reciprocity theorem.

Proposition 3.4.2. *The composition $\Lambda^3 k(C_2, \mathcal{P}_2)^\times \xrightarrow{\text{res}_{|C|}} (\Lambda^2 k_2^\times)^\circ \rightarrow K_2^M(k_2)^\circ$ is 0.*

Proof. There is an isomorphism $K_2^M(k_2)^\circ \simeq \Omega_{k/\mathbb{Q}}^1$, given by sending $\{a_1, a_2\}$ to $\text{res}_t \frac{1}{t} d\log(a_1) \wedge d\log(a_2)$. For any $c \in |C|$, the composition

$$\Lambda^3 k(C_2, \mathcal{P}_2)^\times \xrightarrow{\text{res}_c} (\Lambda^2 k_2^\times)^\circ \rightarrow \Omega_{k/\mathbb{Q}}^1$$

is given by sending $f \wedge g \wedge h$ to $\text{res}_t(\text{res}_{\tilde{\pi}_c}(\frac{1}{t}d\log(\tilde{f}_c) \wedge d\log(\tilde{g}_c) \wedge d\log(\tilde{h}_c)))$, where $\tilde{\pi}_c$ is any uniformizer on a lifting of $\hat{\mathcal{O}}_{C_2, c}$, which lifts $\pi_{2, c}$ and \tilde{f}_c, \tilde{g}_c and \tilde{h}_c are any $\tilde{\pi}_c$ -good local liftings of f, g and h . Therefore, to prove the statement we need to show that the sum

$$(3.4.1) \quad \sum_{c \in |C|} \text{res}_t(\text{res}_{\tilde{\pi}_c}(\frac{1}{t}d\log(\tilde{f}_c) \wedge d\log(\tilde{g}_c) \wedge d\log(\tilde{h}_c))) = - \sum_{c \in |C|} \text{res}_c(\tilde{\omega}_c)$$

is 0, where $\tilde{\omega}_c := \text{res}_t(\frac{1}{t}d\log(\tilde{f}_c) \wedge d\log(\tilde{g}_c) \wedge d\log(\tilde{h}_c))$. On the other hand, ω_c clearly depends only on \tilde{f}_c, \tilde{g}_c and \tilde{h}_c modulo t^2 and hence $\tilde{\omega}_c$ is the restriction of an (absolute) rational 2-form ω on C . Since the last expression in (3.4.1) is the sum of all the residues of the 2-form ω on the curve C , it is equal to 0. \square

Proposition 3.4.3. *The map $\rho : \Lambda^3 k(C_2, \mathcal{P}_2)^\times \rightarrow k$, when restricted to $k_2^\times \wedge \Lambda^2 k(C_2, \mathcal{P}_2)^\times$, is 0.*

Proof. Let $r \in k_2^\times$ and $g, h \in k(C_2, \mathcal{P}_2)^\times$. In order to compute $\rho(r \wedge g \wedge h)$, we need to choose local and generic liftings of r, g , and h . Let us choose a lifting $\tilde{r} \in k[[t]]$ of r . Then we choose the generic lifting and the good local liftings all to be equal to \tilde{r} . We also choose a generic lifting and good local liftings of g and h arbitrarily, satisfying the hypotheses in the definition (3.2.3) of ρ . Let us put $\tilde{r} = r_0 e^{r_1 t + \tilde{r}_2 t^2 + \dots}$, with $r_0 \in k^\times$ and $r_1, \tilde{r}_2, \dots \in k$.

First we simplify the expression $\sum_{c \in |C|} \ell(\text{res}_c(\tilde{q}_c))$, where $\tilde{q}_c = (\tilde{r}, \tilde{g}_c, \tilde{h}_c)$. Note that $\text{res}_c(\tilde{q}_c) = -\tilde{r} \wedge \text{res}_c(\tilde{g}_c \wedge \tilde{h}_c)$. For $\alpha \in k[[t]]$, let $\alpha[t^n]$ denote the coefficient of t^n in α . Then we have,

$$\ell(\text{res}_c(\tilde{q}_c)) = -\ell(\tilde{r} \wedge \text{res}_c(\tilde{g}_c \wedge \tilde{h}_c)) = -\tilde{r}_1 \cdot \log^\circ(\text{res}_c(\tilde{g}_c \wedge \tilde{h}_c))[t] + r_1 \cdot \log^\circ(\text{res}_c(\tilde{g}_c \wedge \tilde{h}_c))[t^2].$$

We can rewrite $\log^\circ(\text{res}_c(\tilde{g}_c \wedge \tilde{h}_c))[t]$ in terms of residues as:

$$\log^\circ(\text{res}_c(\tilde{g}_c \wedge \tilde{h}_c))[t] = \text{res}_t \frac{1}{t} d\log(\text{res}_c(\tilde{g}_c \wedge \tilde{h}_c)) = \text{res}_t \frac{1}{t} (\text{res}_c(d\log(\tilde{g}_c) \wedge d\log(\tilde{h}_c))).$$

By the anti-commutativity of the residues, the last expression can be replaced with

$$-\text{res}_c(\text{res}_t(\frac{1}{t}d\log(\tilde{g}_c) \wedge d\log(\tilde{h}_c))).$$

Similar to the arguments in the preceding propositions, $\text{res}_t(\frac{1}{t}d\log(\tilde{g}_c) \wedge d\log(\tilde{h}_c))$ defines a rational 1-form on C , independent of c , and hence the sum of its residues on the curve C is equal to 0. Therefore,

$$(3.4.2) \quad \sum_{c \in |C|} \ell(\text{res}_c(\tilde{q}_c)) = r_1 \sum_{c \in |C|} \log^\circ(\text{res}_c(\tilde{g}_c \wedge \tilde{h}_c))[t^2].$$

Let α be a generic lifting map $\alpha : \tilde{\mathcal{A}}/(t^2) \xrightarrow{\sim} \mathcal{O}_{C_2, \eta}$; and γ_c , the local lifting maps $\gamma_c : \tilde{\mathcal{B}}_c^\circ/(t^2) \xrightarrow{\sim} \hat{\mathcal{O}}_{C_2, c}$, as in the definition (3.2.3). Let us denote a generic lifting of (r, g, h) by $\tilde{p}_\eta := (\tilde{r}, \tilde{g}_\eta, \tilde{h}_\eta)$. Note that we denoted the local liftings by $\tilde{q}_c := (\tilde{r}, \tilde{g}_c, \tilde{h}_c)$. Let $\psi : \tilde{\mathcal{A}} \rightarrow \tilde{\mathcal{A}}$ be a splitting and $\bar{\psi} : \tilde{\mathcal{A}}[[t]] \xrightarrow{\sim} \tilde{\mathcal{A}}$, the corresponding isomorphism.

Then, by the definition of ω , we have

$$(3.4.3) \quad \text{res}_c \omega(\tilde{p}_\eta, \tilde{q}_c, \tilde{\gamma}_{c, \eta}^{-1} \circ \alpha_c) = \text{res}_c(\Omega(\bar{\psi}^{-1}(\tilde{p}_\eta), \bar{\psi}^{-1}(\bar{\chi}_c^{-1}(\tilde{q}_c))),$$

where $\bar{\chi}_c$ is any lifting of the isomorphism $\tilde{\gamma}_{c,\eta}^{-1} \circ \alpha_c$. We will simplify the last expression. Let us put $\tilde{q} := \bar{\psi}^{-1}(\tilde{p}_\eta)$ and $\hat{q} := \bar{\psi}^{-1}(\bar{\chi}_c^{-1}(\tilde{q}_c))$. Using the notation in (2.3.1), we have $\tilde{q} := (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$, $\hat{q} := (\hat{y}_1, \hat{y}_2, \hat{y}_3)$, with $\tilde{y}_1 = \hat{y}_1 = \tilde{r}$. Since Ω is a 1-form relative to k , the terms involving $d \log(r_0)$ in the expression of Ω will vanish. Also since $\tilde{y}_1 = \hat{y}_1$, we see also that the terms involving \tilde{r}_2 vanish. Therefore, Ω contains only the terms that have the factor r_1 in them. More precisely,

$$\Omega(\tilde{q}, \hat{q}) = r_1 (d \log(\alpha_{02})(\tilde{\alpha}_{23} - \hat{\alpha}_{23}) - d \log(\alpha_{03})(\tilde{\alpha}_{22} - \hat{\alpha}_{22})).$$

Taking residues we have

$$(3.4.4) \quad \text{res}_c \Omega(\tilde{q}, \hat{q}) = r_1 \cdot \text{res}_c (d \log(\alpha_{02})\tilde{\alpha}_{23} - d \log(\alpha_{03})\tilde{\alpha}_{22}) - r_1 \cdot \log^\circ(\text{res}_c(\tilde{g}_c \wedge \tilde{h}_c))[t^2],$$

where we replaced the term $\text{res}_c(d \log(\alpha_{02})\hat{\alpha}_{23} - d \log(\alpha_{03})\hat{\alpha}_{22})$ with $\log^\circ(\text{res}_c(\tilde{g}_c \wedge \tilde{h}_c))[t^2]$ using the fact that \tilde{q}_c is a good lifting.

By the definition (3.2.3) of ρ , we have

$$\rho(r \wedge g \wedge h) = \sum_{c \in |C|} (\ell(\text{res}_c(\tilde{q}_c)) + \text{res}_c \omega(\tilde{p}_\eta, \tilde{g}_c, \tilde{\gamma}_c^{-1} \circ \alpha_c)).$$

Using (3.4.2), (3.4.3), and (3.4.4), we can rewrite this as

$$r_1 \sum_{c \in |C|} \text{res}_c (d \log(\alpha_{02})\tilde{\alpha}_{23} - d \log(\alpha_{03})\tilde{\alpha}_{22}).$$

However, $d \log(\alpha_{02})\tilde{\alpha}_{23} - d \log(\alpha_{03})\tilde{\alpha}_{22}$ is a rational 1-form on C , and hence the sum of its residues is equal to 0. This finishes the proof. \square

In order to prove the infinitesimal version of the strong reciprocity conjecture, we then put together the results we have proved so far about ρ in this paper and about the Bloch group $B_2(k_2)$ in [11].

Theorem 3.4.4. (*Infinitesimal strong reciprocity conjecture*) *The infinitesimal part of the residue map*

$$\text{res}_{|C|} : \Gamma(k(C_2, \mathcal{P}_2), 3) \rightarrow \Gamma(k, 2)^\circ$$

is homotopic to 0. More precisely, there is a map $h : \Lambda^3 k(C_2, \mathcal{P}_2)^\times \rightarrow B_2(k_2)^\circ$ which makes the diagram

$$\begin{array}{ccc} B_2(k(C_2, \mathcal{P}_2)) \otimes k(C_2, \mathcal{P}_2)^\times & \xrightarrow{\Delta} & \Lambda^3 k(C_2, \mathcal{P}_2)^\times \\ \downarrow \text{res}_{|C|} & \swarrow h & \downarrow \text{res}_{|C|} \\ B_2(k_2)^\circ & \xleftarrow{\delta^\circ} & (\Lambda^2 k_2^\times)^\circ \end{array}$$

commute and has the property that $h(k_2^\times \wedge \Lambda^2 k(C_2, \mathcal{P}_2)^\times) = 0$.

Proof. By the theorem in [11], $B_2(k_2)^\circ \simeq \ker(\delta^\circ) \oplus \text{Im}(\delta^\circ)$, and $\ker(\delta^\circ) \xrightarrow{\sim} k$, such that the composition with the projection $B_2(k_2)^\circ \rightarrow \ker(\delta^\circ)$, is the same as the additive dilogarithm $li_2 : B_2(k_2)^\circ \rightarrow k$. Suppose that we have such an h as in the statement of the theorem. Let us denote the first component of h by h_1 and the second component by h_2 . Then $h_1 : \Lambda^3 k(C_2, \mathcal{P}_2)^\times \rightarrow \ker(\delta^\circ)$ and $h_2 : \Lambda^3 k(C_2, \mathcal{P}_2)^\times \rightarrow \text{Im}(\delta^\circ)$. By the commutativity of the lower triangle we must have that $h_2 = \text{res}_{|C|}$. In order to be able to choose $h_2 := \text{res}_{|C|}$, we need to know that $\text{Im}(\text{res}_{|C|}) \subseteq \text{Im}(\delta^\circ)$. Since the cokernel of δ° is $K_2^M(k_2)^\circ$, this is equivalent to Proposition 3.4.2.

We know that by Proposition 3.3.3, $li_{2,|C|} \circ \text{res} = \rho \circ \Delta$. Clearly, from the definitions, we also have $li_{2,|C|} \circ \text{res} = li_2 \circ \text{res}_{|C|}$. Therefore, we have $li_2 \circ \text{res}_{|C|} = \rho \circ \Delta$. By the commutativity of the upper triangle, we see then that we must have $li_2 \circ (h_1 \circ \Delta) = li_2 \circ \text{res}_{|C|} = \rho \circ \Delta$. Therefore, we choose $h_1 := li_2^{-1} \circ \rho$, with $li_2^{-1} : k \rightarrow \ker(\delta^\circ)$. These choices of h_1 and h_2 give us an h that makes the diagram commutative.

In order to show that $h(k_2^\times \wedge \Lambda^2 k(C_2, \mathcal{P}_2)^\times) = 0$, we need to show that $h_1(k_2^\times \wedge \Lambda^2 k(C_2, \mathcal{P}_2)^\times) = 0$, and $h_2(k_2^\times \wedge \Lambda^2 k(C_2, \mathcal{P}_2)^\times) = 0$. The first statement follows from $\rho(k_2^\times \wedge \Lambda^2 k(C_2, \mathcal{P}_2)^\times) = 0$, proved in Proposition 3.4.3. For the second statement we need to show that $\text{res}_{|C|}(k_2^\times \wedge$

$\Lambda^2 k(C_2, \mathcal{P}_2)^\times = 0$. If $r \in k_2^\times$ then $\text{res}_c(r \wedge f \wedge g) = -r \wedge \text{res}_c(f \wedge g)$. Therefore, it suffices to prove that $\text{res}_{|C_1|} : \Lambda^2 k(C_2, \mathcal{P}_2)^\times \rightarrow k_2^\times$ is 0. This is precisely Proposition 3.4.1. \square

3.5. Comparison with the additive dilogarithm. In this section, we no longer assume that k is algebraically closed. We will show that when the infinitesimal regulator ρ in this paper is restricted to the case of $\mathbb{P}_2^1 := \mathbb{P}_{k_2}^1$ and the rational functions are restricted to those of degree 1, then we obtain the additive dilogarithm li_2 of [11]. We first compute the regulator ρ for the triple of rational functions on \mathbb{P}_2^1 that is related to the Totaro cycle.

Lemma 3.5.1. *For $a \in k_2^\flat$, the triple of rational functions $(1-x, x, 1-\frac{a}{x})$ on the projective line \mathbb{P}_2^1 over k_2 , with the parameter x , are good with respect to a system of uniformizers \mathcal{P}_2 and the regulator value on the triple $(1-x) \wedge x \wedge (1-\frac{a}{x}) \in \Lambda^3 k(\mathbb{P}_2^1, \mathcal{P}_2)^\times$ is given by*

$$\rho((1-x) \wedge x \wedge (1-\frac{a}{x})) = \text{li}_2(\{a\}_2),$$

where $\{a\}_2$ is the element corresponding to $a \in k^\flat$ in the Bloch group $B_2(k_2)$.

Proof. The first statement follows by choosing the uniformizers $x, x-1, x-a$, and $1/x$ at the points $0, 1, a$, and ∞ . For the second statement, note that $\Delta(\{x\}_2 \otimes (1-\frac{a}{x})) = (1-x) \wedge x \wedge (1-\frac{a}{x})$ and by Proposition 3.3.3, $\rho(\Delta(\{x\}_2 \otimes (1-\frac{a}{x}))) = \text{li}_{2, |\mathbb{P}_1|}(\text{res}(\{x\}_2 \otimes (1-\frac{a}{x}))) = \text{li}_2(\{a\}_2)$. \square

For $B = k_m$ or A , we let $\mathbb{P}_B^n = \text{Proj}(B[z_0, \dots, z_n])$. For $0 \leq i \leq n$, let H_i denote the hyperplane defined by $z_i = 0$. Let $z_i^q(\mathbb{P}_B^n)$ denote the free abelian group generated by codimension q linear subspaces of \mathbb{P}_B^n , which are in general position with respect to the intersections $\bigcap_{i \in I} H_i$, for all $I \subseteq \{0, 1, \dots, n\}$. We do *not* assume that the linear subspaces under consideration come from linear subspaces of \mathbb{P}_k^n by pull-back. In other words, they are not necessarily constant on B . For $0 \leq i \leq n$, let $\partial_i : z_i^q(\mathbb{P}_B^n) \rightarrow z_i^q(\mathbb{P}_B^{n-1})$ denote the map induced by sending L to $L \cap H_i$. The map $\partial := \sum_{0 \leq i \leq n} (-1)^i \partial_i$ then defines a complex $z_i^q(\mathbb{P}_B^\bullet)$. We denote $\mathbb{P}_{k_2}^n$ by \mathbb{P}_2^n .

Definition 3.5.2. Restricting our ρ to linear objects we obtain a map

$$\rho_l : z_1^2(\mathbb{P}_2^3) \rightarrow k,$$

as follows. Given a line L_2 in \mathbb{P}_2^3 , in general position with respect to the intersections of the coordinate hyperplanes as above, then for a system of uniformizers \mathcal{P}_2 on L_2 , $\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0} \in \Lambda^3 k(L_2, \mathcal{P}_2)^\times$ and we define $\rho_l(L_2) := \rho(\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0})$.

We can give a more explicit description of the function ρ_l above. We first define $\tilde{\rho}_l : z_1^2(\mathbb{P}_A^3) \rightarrow k$ as follows. Suppose that \tilde{L} is a line in $z_1^2(\mathbb{P}_A^3)$, then there is a system of uniformizers \mathcal{P}_A on \tilde{L} such that $\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0} \in \Lambda^3 k(\tilde{L}, \mathcal{P}_A)^\times$. Let L denote the reduction of \tilde{L} modulo (t) . Then the value of $\tilde{\rho}_l$ on \tilde{L} is defined by

$$(3.5.1) \quad \tilde{\rho}_l(\tilde{L}) := \sum_{c \in |L|} \text{Tr}_k(\ell(\text{res}_c \frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0})) = \sum_{c \in |L| \cap |H|} \ell(\text{res}_c \frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0}),$$

where $|H| := \cup_{0 \leq i \leq 3} |H_i|$. The last equality follows because the singularities of the above triple of functions lie only on $|L| \cap |H|$ and that the intersections $\tilde{L} \cap H_i$, being the intersections of linear subspaces, are A -rational and hence the values of the function ℓ in the summands are already in k and there is no need to take trace.

Proposition 3.5.3. *The natural reduction mod (t^2) map from $z_1^2(\mathbb{P}_A^3)$ to $z_1^2(\mathbb{P}_2^3)$ is surjective and the map $\tilde{\rho}_l$ factors through this map to give ρ_l .*

Proof. Given a line in L_2 in \mathbb{P}_2^3 , which is in general position with respect to the intersections of the coordinate hyperplanes. Let \tilde{L} be a linear subspace in \mathbb{P}_A^3 which reduces to L modulo (t^2) . Then $\tilde{L} \in z_1^2(\mathbb{P}_A^3)$. This proves the first statement. For the second statement we need to compute $\rho_l(L) = \rho(\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0})$. But \tilde{L} together with the triple of rational functions $\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0}$ on \tilde{L} gives a good global lifting. Therefore when computing ρ on L_2 , we can use Corollary 3.2.3 with $i = 1$. This proves that $\rho_l(L_2) = \tilde{\rho}_l(\tilde{L})$ as desired. \square

Proposition 3.5.4. *The regulator $\rho_l : z_1^2(\mathbb{P}_2^3) \rightarrow k$ factors through the quotient $z_1^2(\mathbb{P}_2^3)/\partial(z_1^2(\mathbb{P}_2^4))$.*

Proof. By the previous proposition and the fact that the map $z_i^2(\mathbb{P}_A^4) \rightarrow z_i^2(\mathbb{P}_2^4)$ is a surjection, it suffices to show that $\tilde{\rho}_l$ factors through $z_i^2(\mathbb{P}_A^3)/\partial(z_i^2(\mathbb{P}_A^4))$. By the definition (3.5.1) of $\tilde{\rho}_l$, we see that

$$\tilde{\rho}_l(\tilde{L}) = \ell\left(\left(\frac{z_1}{z_0} \wedge \frac{z_2}{z_0}\right)\Big|_{\tilde{L} \cap H_3}\right) - \ell\left(\left(\frac{z_1}{z_0} \wedge \frac{z_3}{z_0}\right)\Big|_{\tilde{L} \cap H_2}\right) + \ell\left(\left(\frac{z_2}{z_0} \wedge \frac{z_3}{z_0}\right)\Big|_{\tilde{L} \cap H_1}\right) - \ell\left(\left(\frac{z_2}{z_1} \wedge \frac{z_3}{z_1}\right)\Big|_{\tilde{L} \cap H_0}\right).$$

For a scheme X/A , let $X(A)$ denote the set of A -valued points of X . Note that we can express $\tilde{\rho}_l$ in terms of the map

$$f : \mathcal{H} := \cup_{0 \leq i \leq 3} H_i(A) \setminus \cup_{0 \leq i < j \leq 3} (H_i \cap H_j)(A) \rightarrow k$$

defined by $f([a_0, \dots, 0, \dots, a_2]) := \ell(\frac{a_1}{a_0} \wedge \frac{a_2}{a_0})$, as follows. First, we extend f linearly to a map from the free abelian group $\mathbb{Z}[\mathcal{H}]$ on \mathcal{H} . Then we note that $\partial(\tilde{L}) \in \mathbb{Z}[\mathcal{H}]$ and

$$\tilde{\rho}_l(\tilde{L}) = -f(\partial(\tilde{L})).$$

The statement that $\tilde{\rho}_l(\partial(\tilde{P})) = 0$, then follows from $\partial^2 = 0$. \square

Let $\tilde{C}_m(\mathbb{P}_B^1)$ denote the free abelian group generated by an m -tuple of distinct points in $\mathbb{P}_B^1(B)$, and $C_m(\mathbb{P}_B^1)$ be the co-invariants of this group with respect to the natural action of $PGL(2, B)$. Let $d_i : \tilde{C}_m(\mathbb{P}_B^1) \rightarrow \tilde{C}_{m-1}(\mathbb{P}_B^1)$, be the map that sends $(c_0, \dots, c_i, \dots, c_{m-1})$ to $(c_0, \dots, \hat{c}_i, \dots, c_{m-1})$, and $d := \sum_{0 \leq i \leq m-1} (-1)^i d_i$. This defines a complex $(\tilde{C}_\bullet(\mathbb{P}_B^1), d)$, which descends to a complex $(C_\bullet(\mathbb{P}_B^1), d)$.

There is a commutative diagram

$$\begin{array}{ccc} z_l^2(\mathbb{P}_B^4) & \xrightarrow{\partial} & z_l^2(\mathbb{P}_B^3) \\ f_5 \downarrow & & f_4 \downarrow \\ C_5(\mathbb{P}_B^1) & \xrightarrow{d} & C_4(\mathbb{P}_B^1) \end{array}$$

with surjective vertical maps f_4 and f_5 . The map f_4 is given by sending \tilde{L} to the 4-tuple of points $(\gamma(\tilde{L} \cap H_0), \gamma(\tilde{L} \cap H_1), \gamma(\tilde{L} \cap H_2), \gamma(\tilde{L} \cap H_4))$ on \mathbb{P}_B^1 , where $\gamma : \tilde{L} \xrightarrow{\sim} \mathbb{P}_B^1$ is any isomorphism of \tilde{L} with \mathbb{P}_B^1 . Since we are taking coinvariants with respect to the action of the projective general linear group, this is independent of γ . Similarly, f_5 is defined by sending \tilde{P} to $(\gamma(\tilde{L} \cap H_0), \gamma(\tilde{L} \cap H_1), \gamma(\tilde{L} \cap H_2), \gamma(\tilde{L} \cap H_4), \gamma(\tilde{L} \cap H_4))$, where \tilde{L} is any line in \tilde{P} in general position with respect to the lines $\{\tilde{P} \cap H_i\}_{0 \leq i \leq 4}$ in \tilde{P} and $\gamma : \tilde{L} \xrightarrow{\sim} \mathbb{P}_B^1$ is any isomorphism of \tilde{L} with \mathbb{P}_B^1 . The above commutative diagram induces the following surjective map

$$(3.5.2) \quad g : z_l^2(\mathbb{P}_2^3)/\partial(z_l^2(\mathbb{P}_2^4)) \rightarrow C_4(\mathbb{P}_2^1)/d(C_5(\mathbb{P}_2^1)).$$

Proposition 3.5.5. *The linear regulator map $\rho_l : z_l^2(\mathbb{P}_2^3)/\partial(z_l^2(\mathbb{P}_2^4)) \rightarrow k$, factors through the surjection g in (3.5.2) and induces a map $\rho_c : C_4(\mathbb{P}_2^1)/d(C_5(\mathbb{P}_2^1)) \rightarrow k$.*

Proof. In order to prove the statement we need to prove that if \tilde{L} and \tilde{L}' are in $z_l^2(\mathbb{P}_A^3)$ with the property that $f_4(\tilde{L}) = f_4(\tilde{L}')$ then $\rho_l(\tilde{L}_2) = \rho_l(\tilde{L}'_2)$, where $\tilde{L}_2 := \tilde{L}|_{t^2}$ and $\tilde{L}'_2 = \tilde{L}'|_{t^2}$. The assumption $f_4(\tilde{L}) = f_4(\tilde{L}')$ implies that there is an isomorphism $\alpha : \tilde{L} \xrightarrow{\sim} \tilde{L}'$ such that $\alpha(\tilde{L} \cap H_i) = \tilde{L}' \cap H_i$, for all $0 \leq i \leq 3$. By the definition of ρ_l , we need to prove that

$$(3.5.3) \quad \rho\left(\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0}\Big|_{\tilde{L}_2}\right) = \rho\left(\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0}\Big|_{\tilde{L}'_2}\right),$$

where our notation is for emphasizing that on the left hand side inside the parenthesis we consider triples of rational functions on \tilde{L}_2 whereas on the right hand side it is on \tilde{L}'_2 .

By the functoriality of ρ we have $\rho(\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0}\Big|_{\tilde{L}'_2}) = \rho(\alpha^*(\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0}\Big|_{\tilde{L}_2}))$. On the other hand, for each $1 \leq i \leq 3$, $\alpha^*(\frac{z_i}{z_0}\Big|_{\tilde{L}'_2})$ has the same zeroes and poles as $\frac{z_i}{z_0}\Big|_{\tilde{L}_2}$. This implies that there exist $\lambda_i \in A^\times$ such that $\alpha^*(\frac{z_i}{z_0}\Big|_{\tilde{L}'_2}) = \lambda_i \cdot \frac{z_i}{z_0}\Big|_{\tilde{L}_2}$, for all $1 \leq i \leq 3$. Combining with the above, this implies that

$$\rho\left(\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0}\Big|_{\tilde{L}'_2}\right) = \rho\left(\left(\lambda_1 \frac{z_1}{z_0} \wedge \lambda_2 \frac{z_2}{z_0} \wedge \lambda_3 \frac{z_3}{z_0}\right)\Big|_{\tilde{L}_2}\right).$$

By Proposition 3.4.3, we see that the right hand side is equal to $\rho((\frac{z_1}{z_0} \wedge \frac{z_2}{z_0} \wedge \frac{z_3}{z_0})|_{\bar{L}_2})$. This implies (3.5.3) and hence finishes the proof of the proposition. \square

By Remark 3.8.2 in [11], there is a map from $B_2(k_2)$ to $C_4(\mathbb{P}_2^1)/d(C_5(\mathbb{P}_2^1))$ which sends $\{a\}_2$ to $(0, a, 1, \infty)$ and this map is an isomorphism, when the source and the target are tensored with \mathbb{Q} . Let us continue to denote the map from $B_2(k_2) \rightarrow k$, which corresponds to ρ_c via this isomorphism, by the same symbol. Then the following theorem proves that ρ restricted to the case of the projective line and linear fractional transformations is essentially the same as the additive dilogarithm of [11].

Theorem 3.5.6. *The map $\rho_c : B_2(k_2) \rightarrow k$ which is induced by restriction of the regulator ρ to the projective line, is given by*

$$\rho_c(\{a\}_2) = -\text{li}_2(\{a\}_2),$$

where $\text{li}_2(\{a_0 + a_1 t\}_2) = -\frac{a_1^3}{2a_0^2(1-a_0)^2}$.

Proof. By the definition of ρ_c on $B_2(k_2)$, the left hand side of the expression is $\rho_c(0, a, 1, \infty)$. On the other hand, again by the definition of ρ_c we can express it as the regulator of a triple of functions of \mathbb{P}_2^1 :

$$\rho_c(0, a, 1, \infty) = \rho((1 - \frac{a}{x}) \wedge (1 - \frac{1}{x}) \wedge \frac{1}{x}) = -\rho((1-x) \wedge x \wedge (1 - \frac{a}{x})) = -\text{li}_2(\{a\}_2),$$

where the last equality follows from the Totaro cycle computation in Lemma 3.5.1. \square

4. INFINITESIMAL REGULATOR ON ALGEBRAIC CYCLES

In this section, we will construct a regulator map for cycles over S , which has the properties that it vanishes on boundaries and assumes the same value if the cycles are equivalent modulo (t^2) , for an appropriately defined equivalence relation.

4.1. Bloch's cubical higher Chow groups. We first recall Bloch's definition [1] of cubical higher Chow groups in the case of a smooth k -scheme X/k . Let $\square_k := \mathbb{P}_k^1 \setminus \{1\}$ and \square_k^n the n -fold product of \square_k with itself over k , with the coordinate functions y_1, \dots, y_n . For a smooth k -scheme X , we let $\square_X^n := X \times_k \square_k^n$. A codimension 1 face of \square_X^n is a divisor F_i^a of the form $y_i = a$, for $1 \leq i \leq n$, and $a \in \{0, \infty\}$. A face of \square_X^n is either the whole scheme \square_X^n or an arbitrary intersection of codimension 1 faces.

Let $\underline{z}^q(X, n)$ be the free abelian group on the set of codimension q , integral, closed subschemes $Z \subseteq \square_X^n$ which are *admissable*, i.e. which intersect each face properly on \square_X^n . For each codimension one face F_i^a , and irreducible $Z \in \underline{z}^q(X, n)$, we let $\partial_i^a(Z)$ be the cycle associated to the scheme $Z \cap F_i^a$. We let $\partial := \sum_{i=1}^n (-1)^n (\partial_i^\infty - \partial_i^0)$ on $\underline{z}^q(X, n)$. One checks immediately that $\partial^2 = 0$. We therefore obtain a complex $(\underline{z}^q(X, \cdot), \partial)$.

Let $\underline{z}^q(X, n)_{\text{degn}}$ denote the subgroup of degenerate cycles, i.e. sums of those obtained by pulling back via one of the standard projections $p_i : \square_X^n \rightarrow \square_X^{n-1}$, for $0 \leq i \leq n$, which omits the i -th coordinate on \square_X^n and $z^q(X, \cdot) := \underline{z}^q(X, \cdot) / \underline{z}^q(X, \cdot)_{\text{degn}}$ the corresponding non-degenerate complex. The complex $(z^q(X, \cdot), \partial)$ is called the *higher Chow complex* of X and its homology $\text{CH}^q(X, n) := H_n(z^q(X, \cdot))$ is the higher Chow group of X . It is a theorem of Voevodsky that the higher Chow groups $\text{CH}^q(X, n)$ compute the motivic cohomology $H_{\mathcal{M}}^{2q-n}(X, \mathbb{Z}(q))$, for smooth varieties X/k .

4.2. Cycles over S . In the following, we need to work with cycles over S which have finite reduction in a certain sense. We define these groups as follows. Let $\bar{\square}_k := \mathbb{P}_k^1$, $\bar{\square}_k^n$, the n -fold product of $\bar{\square}_k$ with itself over k , and $\bar{\square}_S^n := \bar{\square}_k^n \times_k S$. We define a subcomplex $\underline{z}_f^q(S, \cdot) \subseteq \underline{z}^q(S, \cdot)$, as the subgroup generated by integral, closed subschemes $Z \subseteq \bar{\square}_S^n$ which are admissable in the above sense and have *finite reduction*, i.e. \bar{Z} intersects each $s \times \bar{F}$ properly on $\bar{\square}_S^n$, for every face F of $\bar{\square}_k^n$. Modding out by degenerate cycles, we have a complex $z_f^q(S, \cdot)$.

We want to emphasize that the more familiar notion of s -admissability is not enough for our purposes. This condition would require that the cycle $Z \subseteq \bar{\square}_S^n$ intersect each $S \times F$ and $s \times F$ properly on $\bar{\square}_S^n$, for every face F of $\bar{\square}_k^n$. For example, the cycle $(1+t, t)$ in $\underline{z}^2(S, 2)$ is s -admissable,

but does not have finite reduction because its closure contains the point $(1, 0)$ over the special fiber. Our regulator will not be defined on this cycle.

4.3. Definition of the regulator. An irreducible cycle p in $z_f^2(S, 2)$ is given by a closed point p_η of \square_η^2 whose closure \bar{p} in \square_S^2 does not meet $(\{0, \infty\} \times \bar{\square}_S) \cup (\bar{\square}_S \times \{0, \infty\})$. Let \tilde{p} denote the normalisation of \bar{p} and $\{s_1, \dots, s_m\}$ the closed fiber of \tilde{p} . We have surjections $\hat{\mathcal{O}}_{\tilde{p}, s_i} \rightarrow k(s_i)$. Since $k(s_i)/k$ is finite étale there is a unique splitting $\sigma_{\tilde{p}, s_i} : k(s_i) \rightarrow \hat{\mathcal{O}}_{\tilde{p}, s_i}$. We define $\log_{\tilde{p}, s_i}^\circ : \hat{\mathcal{O}}_{\tilde{p}, s_i}^\times \rightarrow \hat{\mathcal{O}}_{\tilde{p}, s_i}$, by

$$\log_{\tilde{p}, s_i}^\circ(y) = \log\left(\frac{y}{\sigma_{\tilde{p}, s_i}(y(s_i))}\right).$$

Let

$$(4.3.1) \quad l(p) := \sum_{1 \leq i \leq m} \text{Tr}_k \left(\text{res}_{\tilde{p}, s_i} \left(\frac{1}{t^3} \left(\log_{\tilde{p}, s_i}^\circ(y_1) \cdot d \log(y_2) - \log_{\tilde{p}, s_i}^\circ(y_2) \cdot d \log(y_1) \right) \right) \right).$$

The following lemma shows that the value of l on the graph of a function assumes a familiar form.

Lemma 4.3.1. *Suppose that $f_1(t), f_2(t) \in k[[t]]^\times$. Then $p := (f_1(t), f_2(t))$ is an irreducible 0-cycle in $z_f^2(S, 2)$ and*

$$l(p) = \log^\circ(f_1)[t] \cdot \log^\circ(f_2)[t^2] - \log^\circ(f_2)[t] \cdot \log^\circ(f_1)[t^2].$$

Proof. Note that with the above notation $\tilde{p} = \bar{p} = p$, and $m = 1$ with $s_1 = (f_1(0), f_2(0))$. Let us write $\log^\circ(f_1) = a_1 t + a_2 t^2 + \dots$ and $\log^\circ(f_2) = b_1 t + b_2 t^2 + \dots$. We have $\log_{\tilde{p}, s_1}^\circ(y_i) = \log^\circ(f_i)$, and $d \log(y_i) = d \log^\circ(f_i)$. The expression (4.3.1) then takes the form

$$\text{res}_t \frac{1}{t^3} \left((a_1 t + a_2 t^2 + \dots)(b_1 + 2b_2 t + \dots) - (b_1 t + b_2 t^2 + \dots)(a_1 + 2a_2 t + \dots) \right) dt.$$

This expression is equal to $a_1 b_2 - a_2 b_1$, which is exactly the expression in the statement of the lemma. \square

Definition 4.3.2. We will define a regulator $\rho_f : z_f^2(S, 3) \rightarrow k$ as the composition $l \circ \partial$, where $l : z_f^2(S, 2) \rightarrow k$ is the map defined in (4.3.1).

4.4. Properties of the regulator. In this subsection, we prove that the regulator has the expected properties.

4.4.1. *Vanishing on the boundaries.* By the construction of the regulator, it is easy to see that it is 0 on the boundaries.

Proposition 4.4.1. *The composition $\rho_f \circ \partial : z_f^2(S, 4) \rightarrow k$ is 0.*

Proof. This follows from the fact that $\rho = l \circ \partial$ and that $\partial \circ \partial = 0$. \square

4.4.2. *Anti-symmetry.* Let G_n be the semi-direct product of S_n with $(\mathbb{Z}/2)^n$, with S_n acting on $(\mathbb{Z}/2)^n$ by permuting the factors. Let $\chi : G_n \rightarrow \mathbb{Z}/2 \simeq \{-1, 1\}$ be the homomorphism which restricts to identity on each $\mathbb{Z}/2$ factor and to the sign character on S_n . There is a natural action of G_n on $z^q(S, n)$, where S_n permutes the coordinates in \square_S^n and $\mathbb{Z}/2$ in the i -th coordinate in $(\mathbb{Z}/2)^n$ acts by switching 0 and ∞ in the i -th coordinate in \square_S^n . Then the regulator has the following anti-symmetry property.

Proposition 4.4.2. *For $\sigma \in G_3$ and $Z \in z_f^2(S, 3)$, $\rho_f(\sigma(Z)) = \chi(\sigma)\rho_f(Z)$.*

Proof. Note that by the description (4.3.1), $l : z_f^2(S, 2) \rightarrow k$ is anti-symmetric with respect to the action of G_2 . If $\sigma_i \in (\mathbb{Z}/2)^3$ is the element which is non-trivial only in the i -th coordinate then $(\partial_i^0 - \partial_i^\infty)\sigma_i(Z) = -(\partial_i^0 - \partial_i^\infty)(Z)$. For $j \neq i$, $l((\partial_j^0 - \partial_j^\infty)\sigma_i(Z)) = l(\alpha_j(\sigma_i)((\partial_j^0 - \partial_j^\infty)(Z)))$, where $\alpha_j : (\mathbb{Z}/2)^3 \rightarrow (\mathbb{Z}/2)^2$ is the homomorphism which omits the i -th term. Then the anti-symmetry of l gives the desired equality for σ_i . To complete the proof we need to show anti-symmetry for $\sigma \in S_3$, and hence only for $\sigma = (12)$ or (23) , since they generate S_3 . This is then seen by a direct computation. \square

4.4.3. *Modulus property.* Suppose that Z_i for $i = 1, 2$ are two irreducible cycles in $\underline{z}_f^2(S, 3)$. We say that Z_1 and Z_2 are equivalent modulo t^m if the following condition (M_m) holds:

- (i) \overline{Z}_i/S are smooth with $(\overline{Z}_i)_s \cup (\cup_{j,a} \partial_j^a Z_i)$ a strict normal crossings divisor on \overline{Z}_i . and more importantly
- (ii) $\overline{Z}_1|_{t^m} = \overline{Z}_2|_{t^m}$.

The main result of this section is the following:

Theorem 4.4.3. *If $Z_i \in \underline{z}_f^2(S, 3)$, for $i = 1, 2$, satisfy the condition (M_2) then they have the same infinitesimal regulator value:*

$$\rho_f(Z_1) = \rho_f(Z_2).$$

Proof. By assumption (i) in (M_2) , we see that the divisors F_j^a define a system of uniformizers $\mathcal{P}_S(i)$ on \overline{Z}_i such that the triple of functions $(y_1 \wedge y_2 \wedge y_3)|_{\overline{Z}_i}$ are $\mathcal{P}_S(i)$ -good. Therefore, if $\mathcal{P}_2(i)$ denote the reduction mod (t^2) of this system of uniformizers to $\overline{Z}_i|_{t^2}$ then $(y_1 \wedge y_2 \wedge y_3)|_{\overline{Z}_i|_{t^2}} \in \Lambda^3 k(\overline{Z}_i|_{t^2}, \mathcal{P}_2(i))^\times$ and $(y_1 \wedge y_2 \wedge y_3)|_{\overline{Z}_i}$ can be used as a global good lifting to compute $\rho((y_1 \wedge y_2 \wedge y_3)|_{\overline{Z}_i|_{t^2}})$ using Corollary 3.2.3 with $r = 1$. More precisely,

$$(4.4.1) \quad \rho((y_1 \wedge y_2 \wedge y_3)|_{\overline{Z}_i|_{t^2}}) = \sum_{z \in |(\overline{Z}_i)_s|} \text{Tr}_k \ell(\text{res}_{\tilde{\pi}_z(i)}((y_1 \wedge y_2 \wedge y_3)|_{\overline{Z}_i})),$$

where $\{\tilde{\pi}_z(i) | z \in |(\overline{Z}_i)_s|\}$ is the system of uniformizers $\mathcal{P}_S(i)$.

The only contribution to the sum in (4.4.1) comes from when $z \in (\partial_j^a Z_i)_s$. The sum of the contributions coming from $z \in (\partial_j^a Z_i)_s$ is $l(\partial_j^a Z_i)$ by Lemma 4.3.1. Summing over all the faces we see that $\rho((y_1 \wedge y_2 \wedge y_3)|_{\overline{Z}_i|_{t^2}}) = l \circ \partial(Z_i) = \rho_f(Z_i)$. Since by the condition (ii) in (M_2) , $\overline{Z}_1|_{t^2} = \overline{Z}_2|_{t^2}$, the left hand sides of the last expression is the same for $i = 1$ and $i = 2$ and therefore the right hand sides are the same. This gives the desired equality. \square

Remark 4.4.4. By the same argument, one can prove a somewhat stronger statement than the above. Namely, that we do not need to assume that \overline{Z}_i/S are smooth with the the given divisors, being normal crossings divisors on them. But we assume that this holds after a common imbedded resolution of singularities for \overline{Z}_i in $\overline{\square}_S^3$ and these new smooth cycles are congruent modulo t^2 . Our main aim in doing this would be to define the correct higher Chow groups over k_2 , by defining the cycle complex to be cycles over S , and setting two cycles to be the same if they have the same reduction modulo t^2 after an appropriate resolution of singularities. We will pursue this approach in future work, the Milnor case of which is done in [9].

4.4.4. *Vanishing on the products.* Suppose that Z is an irreducible cycle in $\underline{z}_f^2(S, 3)$ which satisfies condition (i) in §4.4.3.

Proposition 4.4.5. *If there is $1 \leq i \leq 3$ such that y_i restricted to $Z_2 := Z|_{t^2}$ is in k_2^\times then $\rho_f(Z) = 0$.*

Proof. Without loss of generality assume that $i = 1$, and y_1 restricted to $Z|_2$ is $\alpha \in k_2^\times$. By exactly as in the proof of Theorem 4.4.3 we see that

$$\rho_f(Z) = \rho((y_1 \wedge y_2 \wedge y_3)|_{Z_2}) = \rho(\alpha \wedge (y_2 \wedge y_3)|_{Z_2}).$$

On the other hand, the right hand side is 0 by Proposition 3.4.3. \square

4.4.5. *Comparison with Park's regulator.* In this section, we will compare our construction to Park's construction in [8]. We first recall Park's construction in the cases that relate to our discussion. Let $\diamond_n := \mathbb{A}_k^1 \times_k \square_k^n$ and $\overline{\diamond}_n := \mathbb{A}_k^1 \times_k \overline{\square}_k^n$, with t being the coordinate on \mathbb{A}_k^1 . The codimension one face F_i^a , for $1 \leq i \leq n$, and $a \in \{0, \infty\}$ is given by $y_i = a$. For a cycle $Z \subseteq \diamond^n$, its face $\partial_i^a(Z)$ is defined as the cycle associated to $Z \cap F_i^a$. As usual $\partial := \sum_{1 \leq i \leq n} (\partial_i^0 - \partial_i^\infty)$. We will consider two different versions of the additive Chow groups: one with the strong sup modulus condition, denoted by the subscript "ss," and the other with the sup modulus condition, denoted by the subscript "s." In [8], only the one with the sup modulus condition is considered.

The groups $c_{p,s}(\diamond_n; 2)$ (resp. $c_{p,ss}(\diamond_n; 2)$) are defined inductively:

(a) $c_{0,s}(\diamond_n; 2) = c_{0,ss}(\diamond_n; 2)$ is defined to be the free abelian group on the set of closed points on $\diamond_n \setminus (\cup_{i,a} F_i^a \cup \{t = 0\})$.

(b) $c_{p,s}(\diamond_n; 2)$ (resp. $c_{p,ss}(\diamond_n; 2)$) is defined to be the free abelian group on the set of irreducible p -dimensional closed subvarieties W of \diamond_n which satisfy the following properties:

(i) W intersects all the faces properly.

(ii) For $1 \leq i \leq n$ and $a \in \{0, \infty\}$, $\partial_i^a(W) \in c_{p-1,s}(\diamond_{n-1}; 2)$ (resp. $\partial_i^a(W) \in c_{p-1,ss}(\diamond_{n-1}; 2)$).

Let $f : \overline{W} \rightarrow \overline{\diamond_n}$ be the normalisation of the closure of W in $\overline{\diamond_n}$.

Then for the sup modulus condition:

(iii)_s The divisor $\sup_{1 \leq i \leq n} (f^* \{y_i = 1\}) - 2 \cdot f^* \{t = 0\}$ is an effective divisor on \overline{W} . Here for a finite set of Weil divisors $\{D_i\}_{1 \leq i \leq n}$ on a normal variety X , $\sup_{1 \leq i \leq n} D_i$ is the divisor D such that for any prime divisor E on X , order of D along E is the maximum of the orders of D_i along E .

For the strong sup modulus condition:

(iii)_{ss} There exists an $i \in \{1, \dots, n\}$ such that $f^* \{y_i = 1\} - 2 \cdot f^* \{t = 0\}$ is an effective divisor on \overline{W} .

Dividing $c_{p,s}(\diamond_n; 2)$ (resp. $c_{p,ss}(\diamond_n; 2)$) by the subgroup of degenerate cycles one obtains the groups $\mathcal{Z}_{p,s}(\diamond_n; 2)$ (resp. $\mathcal{Z}_{p,ss}(\diamond_n; 2)$). Letting $q = n + 1 - p$, one denotes the same groups by $\mathcal{Z}_s^q(\diamond_n; 2)$ (resp. $\mathcal{Z}_{ss}^q(\diamond_n; 2)$).

Remark 4.4.6. If one would like to define a cycle complex over k_2 computing its motivic cohomology, one might try to do so by considering $z_f^q(S, n)/\mathcal{Z}^q(\diamond_n; 2)$, where $\cdot = s$ or ss . However, we are doubtful that this would give the correct answer. In some sense, this approach only would only make cycles which are close to infinity equal to zero, whereas we think that two cycles which are close to each other in the sense of Remark 4.4.4 should be made the same in this quotient group. An opposite reason for our doubt in this definition with the sup modulus condition is that such a definition would imply that the regulator should be 0 on $\mathcal{Z}_s^q(\diamond_n; 2)$. However, the example below shows that this is not the case.

Example. Consider the cycle $Z \in z_f^2(S, 3)$ given by the parametric equations

$$xy = t, \quad y_1 = 1 - x^3, \quad y_2 = \frac{1 + 2y^2}{1 + y^2}, \quad y_3 = \frac{1 + 2y}{1 + y}.$$

For simplicity, assume that k is algebraically closed. We would like to compute $\rho_f(Z)$. Let us look at the contributions from each of the faces. Since $\partial_1^\infty \overline{Z} = (1, 1)$ and $l(1, 1) = 0$, this term does not contribute to the regulator. For $i = 2, 3$, and $a = 0, \infty$, $\partial_i^a \overline{Z} = (1 - \alpha t^3, \cdot)$, for some $\alpha \in k$. Since $l(1 - \alpha t^3, \cdot) = 0$, these terms do not contribute to the regulator either. Therefore, $\rho_f(Z) = l(\partial_1^0 Z) = \sum_{\omega^3=1} l(\frac{1+2\omega t^2}{1+\omega t^2}, \frac{1+2\omega^2 t}{1+\omega^2 t}) = -3$.

This cycle given by the same formula above satisfies the sup modulus condition with modulus 2 and hence gives an element in $Z \in \mathcal{Z}_s^2(\diamond_3; 2)$, but does not satisfy the strong sup modulus condition with modulus 2 hence $Z \notin \mathcal{Z}_{ss}^2(\diamond_3; 2)$. In terms of our viewpoint, it is not surprising that the above cycle does not have 0 regulator value, since there is not a single function y_i which is constant on all of $Z|_{t^2}$: on one component y_1 is constant, on the component y_2 is constant.

Note that there is a natural map $\iota : \mathcal{Z}^q(\diamond_n; 2) \rightarrow z_f^q(S, n)$, where ι maps a cycle Z in $\mathbb{A}_k^1 \times_k \square_k^n$ to its completion $\iota(Z)$ along 0 in \mathbb{A}_k^1 . However, in principle $z_f^q(S, n)$ contains much more cycles since they need not satisfy any modulus condition.

Park defines a map $R : \mathcal{Z}_s^2(\diamond_2; 2) \rightarrow k$. We have the following relation between R and l .

Lemma 4.4.7. *We have the equality $l \circ \iota = R$ of the functions from $\mathcal{Z}_s^2(\diamond_2; 2)$ to k .*

Proof. Direct computation of both sides. □

Since we defined ρ_f as the composition $l \circ \partial$ and since $\partial \circ \iota = \iota \circ \partial$, we have the equality of the following functions on $\mathcal{Z}_s^2(\diamond_3; 2)$:

$$(4.4.2) \quad \rho_f \circ \iota = R \circ \partial.$$

The above example shows that $\rho_f \circ \iota$ is not necessarily 0 on $\mathcal{Z}_s^2(\diamond_3; 2)$. On the other hand, if Z is a cycle satisfying the strong modulus condition, in other words Z is in $\mathcal{Z}_{ss}^2(\diamond_3; 2)$, such that

$\iota(Z)$ is smooth then $\rho_f \circ \iota$ vanishes on Z [8, Theorem 3.1]. This can also be seen by applying Proposition 4.4.5 to $\iota(Z)$ since if Z satisfies the modulus condition $(iii)_{ss}$ then y_i restricted to $\iota(Z)|_{t^2}$ is equal to 1.

References

- [1] S. Bloch. *Algebraic cycles and higher K-theory*, Adv. Math., 61, (1986), no. 3, 267–304.
- [2] S. Bloch, H. Esnault. *The additive dilogarithm*, Doc. Math., Extra Vol., (2003), 131–155.
- [3] A. Goncharov. *Geometry of configurations, polylogarithms, and motivic cohomology*. Advances in Math. 114 (1995), 197-318.
- [4] A. Goncharov. *Volumes of hyperbolic manifolds and mixed Tate motives*. J. Amer. Math. Soc. 12 (1999), no. 2, 569-618.
- [5] A. Goncharov. *Polylogarithms, regulators, and Arakelov motivic complexes*. J. Amer. Math. Soc., 18 (2005), no. 1, 1-60.
- [6] M. Green, P. Griffiths. *On the tangent space to the space of algebraic cycles on a smooth algebraic variety*. Annals of Mathematics Studies, 157. Princeton University Press, Princeton, NJ, (2005). vi+200 pp.
- [7] A. Grothendieck. *Revêtements étales et groupe fondamental*, Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1). Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud. Lecture Notes in Mathematics, Vol. 224. Springer-Verlag, Berlin-New York, (1971), xxii+447 pp.
- [8] J. Park. *Regulators on additive higher Chow groups*, Amer. J. Math., 131, (2009), no. 1, 257–276.
- [9] J. Park, S. Ünver. *Motivic cohomology of fat points in Milnor range*. Preprint.
- [10] A. A. Suslin. *Reciprocity laws and the stable rank of rings of polynomials*. Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 6, 1394-1429.
- [11] S. Ünver. *On the additive dilogarithm*, Algebra Number Theory, 3, (2009), no. 1, 1–34.

KOÇ UNIVERSITY, MATHEMATICS DEPARTMENT. RUMELIFENERI YOLU, 34450, ISTANBUL, TURKEY

FREIE UNIVERSITÄT BERLIN, MATHEMATICS DEPARTMENT, ARNIMALLEE 3, 14195, BERLIN, GERMANY
E-mail address: sunver@ku.edu.tr