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Additive polylogarithms and their functional equations

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Abstract Let $k[\varepsilon]_2 := k[\varepsilon]/(\varepsilon^2)$. The single valued real analytic n-polylogarithm $\mathcal{L}_n : \mathbb{C} \to \mathbb{R}$ is fundamental in the study of weight n motivic cohomology over a field k, of characteristic 0. In this paper, we extend the construction in Ünver (Algebra Number Theory 3:1–34, 2009) to define additive n-polylogarithms li_n:$k[\varepsilon]_2 \to k$ and prove that they satisfy functional equations analogous to those of $\mathcal{L}_n$. Under a mild hypothesis, we show that these functions descend to an analog of the nth Bloch group $B'_n(k[\varepsilon]_2)$ defined by Goncharov (Adv Math 114:197–318, 1995). We hope that these functions will be useful in the study of weight n motivic cohomology over $k[\varepsilon]_2$.

1 Introduction

1.1 Extensions in the conjectural category of mixed Tate motives

Let S be any scheme. One expects an abelian category $\mathcal{M}_S$, of mixed motivic perverse $\mathbb{Q}$-sheaves on S, together with Tate sheaves $\mathbb{Q}_{\mathcal{M}}(n)$ as objects in $\mathcal{M}_S$, for every $n \in \mathbb{Z}$ [1, 5.10]. The interest in such a category lies in the expectation that this category be rich enough that the K-theory of S, which holds arithmetic and geometric significance, can be expressed as extension groups in this category. Namely, that

$$\mathrm{Ext}^i_{\mathcal{M}_S}(\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n)) \simeq K_{2n-i}(S)^{(n)}_{\mathbb{Q}},$$

where the right hand side denotes the nth graded piece of the K-theory of S with respect to the gamma filtration tensored with $\mathbb{Q}$, [1, 5.10]). The left hand side of the last equation is called the ith motivic cohomology of S of weight n and is denoted

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by $\mathrm{H}^i_{\mathcal{M}}(S, \mathbb{Q}(n))$. For more information on motivic cohomology as it relates to the discussion below, see the introduction of [21] and the references therein.

It is expected that there is a much smaller full subcategory of $\mathcal{M}_S$, called the category of mixed Tate motives and denoted by $\mathcal{MTM}_S$, whose objects consist of iterated extensions of the Tate motives $\mathbb{Q}_{\mathcal{M}}(n)$, for various n. The extension groups should be the same [1, 5.10]:

$$\mathrm{Ext}^i_{\mathcal{M}_S}(\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n)) = \mathrm{Ext}^i_{\mathcal{MTM}_S}(\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n)).$$

In the case when $S := \operatorname{Spec} k$, where k is a number field, a candidate for the category $\mathcal{MTM}_S$ has been constructed as a tannakian category over $\mathbb{Q}$ by Deligne and Goncharov in [8], using Voevodsky's triangulated category of motives. The tannakian fundamental group of this category, at a canonical fiber functor ω, is a semi-direct product $\mathbb{G}_m \ltimes U_\omega$ of the multiplicative group and a unipotent group U_ω [8, §2]. Let $\mathcal{A}_\cdot(k)$ denote the graded Hopf algebra corresponding to the ring of regular functions on U_ω: the grading on $\mathcal{A}_\cdot(k)$ comes from the $\mathbb{G}_m$ action on U_ω [8, §2.1]. The general conjectures on $\mathcal{MTM}_S$ imply that it is a mixed Tate category in the sense of [12, §1.10]. These conjectures and the formalism in [8, §2], then gives a graded Hopf algebra $\mathcal{A}_\cdot(k)$ as above, such that a mixed Tate motive over k is the same as a graded vector space over $\mathbb{Q}$ with a comodule structure over $\mathcal{A}_\cdot(k)$. We will need this Hopf algebra *only* in order to describe where our construction stands in regards to the general theory and conjectures.

1.2 Volume map on Hodge–Tate structures

Let $\mathbb{E} \in \{\mathbb{Q}, \mathbb{R}\}$. A mixed Tate $\mathbb{E}$-Hodge structure is a mixed $\mathbb{E}$-Hodge structure such that for every $r \in \mathbb{Z}$, its graded piece of degree $-2r$ with respect to the weight filtration are direct sums of the Tate $\mathbb{E}$-Hodge structure $\mathbb{E}(r)$, of weight $-2r$; and its graded pieces of odd degree are equal to 0. Let $\mathcal{H}_{\mathbb{E}\cdot}$ denote the graded Hopf algebra associated to the tannakian category of mixed Tate $\mathbb{E}$-Hodge structures. The Hodge realization functor should give a morphism $\mathcal{A}_\cdot(\mathbb{C}) \to \mathcal{H}_{\mathbb{Q}\cdot}$ of graded Hopf algebras.

A construction of Beilinson and Deligne [§2.5, [2]; pp. 248–249, [12]] gives a map $p_{\mathcal{H},n} : \mathcal{H}_{\mathbb{R},n} \to \mathbb{R}$. Composing $p_{\mathcal{H},n}$ with the natural map $\mathcal{H}_{\mathbb{Q}\cdot} \to \mathcal{H}_{\mathbb{R}\cdot}$ gives a homomorphism:

$$\mathrm{vol}_n(\mathbb{R}) \colon \mathcal{A}_n(\mathbb{C}) \to \mathbb{R},$$

which we denote by $\mathrm{vol}_n(\mathbb{R})$, for reasons that are going to be explained below. In other words, if $\mathcal{A}_\cdot(\mathbb{C})$ is the graded Hopf algebra associated to mixed Tate motives, $\mathrm{vol}_n(\mathbb{R})$ is the composition of the $\mathbb{R}$-Hodge realization functor and $p_{\mathcal{H},n}$.

1.3 Volume map in the infinitesimal case

In general, for an arbitrary field k one cannot expect a nontrivial natural map $\mathcal{A}_n(k) \to k$, similar to $\mathrm{vol}_n(\mathbb{R})$ above, since the construction of $\mathrm{vol}_n(\mathbb{R})$ uses integration.

However, below we will try to explain why one can expect such a map in the infinitesimal case. More precisely, let k be a field of characteristic 0 and let $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$. Then if $\mathcal{A}.(k[\varepsilon]_2)$ denotes the Hopf algebra of the (as yet undefined) tannakian category of mixed Tate motives over $k[\varepsilon]_2$ then we expect a natural non-trivial map

$$\mathrm{vol}_n^{\circ}(k)\colon \mathcal{A}_n(k[\varepsilon]_2) \to k.$$

We will give more details and motivation about this question in Sect. 2, especially as it relates to the scissors congruence class groups and Hilbert's 3rd problem [11,13].

1.4 Goncharov's motivic complexes

Since the objects in $\mathcal{MTM}_S$ should be constructed from Tate objects by means of extensions, one expects $\mathcal{MTM}_S$ to have a *linear* algebraic description. In [3], a graded Hopf algebra $A.(k)$ was defined, using linear algebraic objects, such that one should have a natural map $A.(k) \to \mathcal{A}.(k)$.

Let $A.(k)$ denote the graded Hopf algebra of Aomoto polylogarithms over k defined in [3] (also [12, §1.16]). $A_n(k)$ is generated by pairs of simplices $(L; M)$ in $\mathbb{P}_k^n$ [12, §1.16]. There are certain configurations, called polylogarithmic configurations, in $A_n(k)$ that play an important role in understanding the motivic cohomology of k, since they act as building blocks for all configurations [3,12, §1.16 and Fig. 1.14]. Namely, for every $t \in k^\flat := \{x \in k^\times | 1 - x \in k^\times\}$, there is a special configuration $(L, M_t) \in A_n(k)$ [12, Fig. 1.14]. This defines a map $\mathbb{Z}[k^\flat] \to A_n(k)$, which induces:

$$l_n\colon \mathbb{Z}[k^\flat] \to A_n(k)/P_n(k),$$

where $P_n(k)$ denotes the subgroup of *prisms* [12, §1.16, p. 242]. Denote the image of l_n in $A_n(k)/P_n(k)$ by $B_n'(k)$. One expects the comultiplication on $A.(k)$ to induce a complex:

$$B_n'(k) \to B_{n-1}'(k) \otimes k^\times \to \cdots \to B_2'(k) \otimes \Lambda^{n-2}k^\times \to \Lambda^n k^\times$$

[12, §1.9, (1.25b)]; [12, §1.16, Conjecture 1.40], which would compute the motivic cohomology of k of weight n (§1.9, Conjecture A, [12]).

1.5 Volume maps and polylogarithms

The canonical mixed Hodge structure on $\mathrm{H}^n(\mathbb{P}_{\mathbb{C}}^n \backslash L, M \backslash L)$ is a mixed Tate Hodge structure. As above this gives a map $A.(\mathbb{C}) \to \mathcal{H}_{\mathbb{Q}}.$ and hence, after composing with the map $p_{\mathcal{H}}$ in Sect. 1.2, a map: $A_n(\mathbb{C}) \to \mathbb{R}$, which we also denote by $\mathrm{vol}_n(\mathbb{R})$ since it would not cause confusion. This map in fact descends to a map [12, p. 248]:

$$\mathrm{vol}_n(\mathbb{R})\colon A_n(\mathbb{C})/P_n(\mathbb{C}) \to \mathbb{R}.$$

In the complex case, we then have a map

$$\mathrm{vol}_n(\mathbb{R}) \circ l_n : \mathbb{Z}[\mathbb{C}^\flat] \to \mathbb{R},$$

which has the following description. Let $\ell i_n(z) := \sum_{1 \leq k} \frac{z^k}{k^n}$, denote the classical complex valued n-polylogarithm, convergent on the disc $|z| < 1$ and $\mathcal{L}_n$ its real single valued analytic continuation defined by Zagier [24, p. 202]; [12]:

$$\mathcal{L}_n(z) := \mathcal{R}_n \left(\sum_{j=0}^{n} \frac{2^j B_j}{j!} (\log |z|)^j \ell i_{n-j}(z) \right),$$

where B_n is the nth Bernoulli number; $\mathcal{R}_n$ is the real part if n is odd and the imaginary part if n is even; and $\ell i_0(z) := -1/2$. Then for $z \in \mathbb{C}^\flat$, $\mathrm{vol}_n(\mathbb{R}) \circ l_n(z) = \mathcal{L}_n(z)$ (§1.5, [2]; cf. Remark in §1.16, [12]).

1.6 Additive polylogarithms

Our principal aim in this note is to define an analog of the map $\mathcal{L}_n : \mathbb{C}^\flat \to \mathbb{R}$, in the infinitesimal case, where the base ring $\mathbb{C}$ is replaced with $k[\varepsilon]_2$.

Let k be a field of characteristic 0. The definitions of $A_n(k)$, $P_n(k)$, l_n and $B'_n(k)$ in [12, §1.16]] exactly carry over to the $k[\varepsilon]_2$ case to define the groups $A_n(k[\varepsilon]_2)$, $P_n(k[\varepsilon]_2)$, and $B'_n(k[\varepsilon]_2)$, and a map, $l_n : \mathbb{Z}[k[\varepsilon]_2^\flat] \to A_n(k[\varepsilon]_2)$. One would again expect a natural map $A_\cdot(k[\varepsilon]_2) \to \mathcal{A}_\cdot(k[\varepsilon]_2)$ so that $\mathrm{vol}_n^\circ(k)$ on $\mathcal{A}_n(k[\varepsilon]_2)$ would give a corresponding map

$$\mathrm{vol}_n^\circ(k) : A_n(k[\varepsilon]_2) \to k,$$

which we continue to denote with the same notation.

Then the analog of $\mathcal{L}_n$ that we are seeking is the hypothetical map

$$li_n : \mathbb{Z}[k[\varepsilon]_2^\flat] \to k \tag{1.6.1}$$

that would be the composition $\mathrm{vol}_n^\circ(k) \circ l_n$.

1.7 Outline

The main aim of this note is to give an explicit construction of (1.6.1) that does not rely on any conjecture on mixed Tate motives. Such a construction was made by Bloch and Esnault [4] in the slightly different context of additive Chow groups in weight two. This construction was the main motivation for [21] where additive dilogarithms over $k[\varepsilon]_2$ was constructed. This note can be considered as a generalization of [21] to higher weights. We briefly sketch the contents of the paper.

In Sect. 2, we recall very briefly the basic constructions and the conjectures of [11,12] in the infinitesimal case. This is necessary to put our construction in perspective.

Section 3 is the main part of the construction where a lifting to $k[\varepsilon]_{n+1}$ argument is used as in [21] to define the polylogarithm. In Definition 1, a formula for li_n is given in terms of $\Delta_n^{(n+1)}$. If one had a tannakian category of mixed Tate motives over $k[\varepsilon]_r$ this formula would immediately give a construction of additive polylogarithms over the Hopf algebra corresponding to such a category, where $\Delta_n^{(n+1)}$ would be replaced with a map constructed out of the comultiplication map (see e.g. Sect. 3.2). In Corollary 1 and 2, li_n is made explicit. In Sects. 3.2–3.4, we need to assume the existence of a comultiplication map on $A.(R)$. This map is known on the subgroup of generic configurations in $A.(R)$ [3], but at this point this map is not known on the whole group. Assuming this, in Theorem 1 we prove that li_n descends to give a map $B'_n(k[\varepsilon]_2) \to k$. In Sects. 3.3 and 3.4, using the analogs of the standard conjectures of mixed Tate motives in the $k[\varepsilon]_2$ case, we show that the additive polylogarithm map that we define is injective when restricted to $\ker^{\circ}(\Delta_{n-1,1})$ (Proposition 2 and Remark 6). This is the analog of the injectivity of the regulator conjecture of Ramakrishnan.

In Sect. 4, we show that li_n satisfies the analogs of the functional equations for classical polylogarithms, such as the inversion formula, distribution formula, and Gangl's functional equations for the trilogarithm and the tetralogarithm. In Sect. 4.2, we use the context of functional equations of additive polylogarithms to prove that the additive dilogarithm and its higher modulus generalizations in [21] satisfy Wojtkowiak's functional equations.

There are many questions that are left unanswered in this paper. One of these is the question of additive polylogarithms over a field of characteristic p, this situation being radically different from the characteristic 0 case. Another one is to construct a volume-like map as in Sect. 1.6, at least for small weights, $n \leq 4$. We will address these questions in a future paper. Another question that is left out is to relate the cohomology of $\Gamma'_{k[\varepsilon]_2}(n)$ to motivic cohomology. We hope that our construction of the additive polylogarithm will be useful in understanding the infinitesimal part of motivic cohomology.

For infinitesimal motivic cohomology in the context of additive Chow groups we refer the reader to [5,18,16,19].

Notation For an abelian group A, we let $A_{\mathbb{Q}} = A \otimes \mathbb{Q}$. We let $R^{\flat} := \{r \in R| r(1-r) \in R^{\times}\}$ and $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$. For a set S, we let $\mathbb{Z}[S]$ denote the free abelian group generated by S. For $r \in R^{\flat}$, let $\{r\}_n$ denote the image of $r \in \mathbb{Z}[R^{\flat}]$ under the natural projection l_n:$\mathbb{Z}[R^{\flat}] \to B'_n(R)$.

If A is an object defined over $k[\varepsilon]_m$, we let A° denote its infinitesimal part. Since the map $k[\varepsilon]_m \to k$ has a canonical splitting, A° is naturally a direct summand of A. In general, we will not mention the natural maps $A \to A^{\circ}$ or $A^{\circ} \to A$, and take other liberties of this kind. For example, $k[\varepsilon]_m^{\circ} = \varepsilon k[\varepsilon]_m$, $A_n(k[\varepsilon]_m) = A_n^{\circ}(k[\varepsilon]_m) \oplus A_n(k)$ etc.

2 On the volume map

2.1 Expected cohomology in the inifinitesimal case

Continuing with the notation in Sects. 1.4 and 1.6, one expects a complex $\Gamma'_{k[\varepsilon]_2}(n)$, concentrated in degrees $[1, n]$:

$$B'_n(k[\varepsilon]_2) \to B'_{n-1}(k[\varepsilon]_2) \otimes k[\varepsilon]_2^\times \to \cdots \to B'_2(k[\varepsilon]_2) \otimes \Lambda^{n-2} k[\varepsilon]_2^\times \to \Lambda^n k[\varepsilon]_2^\times$$

induced by a comultiplication map on $A_\cdot(k[\varepsilon]_2)$ and such that $\{x\}_i \otimes y \in B'_i(k[\varepsilon]_2) \otimes \Lambda^{n-i} k[\varepsilon]_2^\times$ is mapped to:

$$\{x\}_{i-1} \otimes x \wedge y \in B'_{i-1}(k[\varepsilon]_2) \otimes \Lambda^{n-i+1} k[\varepsilon]_2^\times \tag{2.1.1}$$

if $i \geq 3$, and to

$$(1-x) \wedge x \wedge y \in \Lambda^n k[\varepsilon]_2^\times \tag{2.1.2}$$

if $i = 2$ (§1.9, (1.25b); §1.16, Conjecture 1.40, [12]). The conjectures in [12], extended to the case of the base ring $k[\varepsilon]_2$, imply that $\mathrm{H}^i(\Gamma'_{k[\varepsilon]_2}(n)_{\mathbb{Q}}) \simeq K_{2n-i}(k[\varepsilon]_2)^{(n)}_{\mathbb{Q}}$.

If k is a field of characteristic 0, Goodwillie's theorem [14] gives the isomorphism:

$$K_{2n-i}(k[\varepsilon]_2, (\varepsilon))^{(n)}_{\mathbb{Q}} \simeq \mathrm{HC}_{2n-i-1}(k[\varepsilon]_2, (\varepsilon))^{(n-1)},$$

where the right hand side denotes cyclic homology with respect to $\mathbb{Q}$. The fact that the graded pieces of the gamma filtrations on both sides of the last equation matches is proved in [6]. The relative cyclic homology of $(k[\varepsilon]_2, (\varepsilon))$ is given by [6, p. 594]:

$$\mathrm{HC}_n(k[\varepsilon]_2, (\varepsilon))^{(m)} \simeq \Omega^{2m-n}_{k/\mathbb{Q}},$$

for $\left[\frac{n+1}{2}\right] \leq m \leq n$, and is 0 otherwise. Moreover, for $\lambda \in k$, the automorphism ρ_λ of $k[\varepsilon]_2$ that sends ε to $\lambda\varepsilon$ induces multiplication by $\lambda^{2(n-m)+1}$ on $\Omega^{2m-n}_{k/\mathbb{Q}}$ (loc. cit.).

Combining all of these, the expectation for the infinitesimal part of the cohomology of $\Gamma'(n)$ is:

$$\mathrm{H}^i(\Gamma'_{k[\varepsilon]_2}(n)^\circ_{\mathbb{Q}}) \simeq \Omega^{i-1}_{k/\mathbb{Q}}, \tag{2.1.3}$$

for $1 \leq i \leq n$ and that ρ_λ induces multiplication by $\lambda^{2(n-i)+1}$ on $\Omega^{i-1}_{k/\mathbb{Q}}$.

Remark 1 One can readily generalize the statements above to the $k[\varepsilon]_m$ case, using the description of the relative cyclic homology of $(k[\varepsilon]_m, (\varepsilon))$ and the straightforward definition of the objects $A_n(k[\varepsilon]_m)$, $B'_n(k[\varepsilon]_m)$, etc. (cf. §1.6). In the weight two case, i.e. when $n = 2$, the computations in [3] (Corollary in § 3.1, Corollary 3.6.2 and Main Theorem 2 in §3.8) generalize with the same proof to the $k[\varepsilon]_m$ case to show that in our notation $B'_2(k[\varepsilon]_m)_{\mathbb{Q}} = B_2(k[\varepsilon]_m)_{\mathbb{Q}}$, where the right hand side is the Bloch group in [21, §1.3]. This implies that $\Gamma'_{k[\varepsilon]_m}(2)_{\mathbb{Q}}$ is isomorphic to $\gamma_{k[\varepsilon]_m}(2)_{\mathbb{Q}}$ in [21, §1.3]. Then Theorem 1.3.1 in [21] gives that:

$$\mathrm{H}^i(\Gamma'_{k[\varepsilon]_m}(2)^{\circ}_{\mathbb{Q}}) \simeq (\Omega^{i-1}_{k/\mathbb{Q}})^{\oplus(m-1)},$$

for $1 \leq i \leq 2$, and ρ_λ induces multiplication by $\lambda^{m(2-i)+j}$ on the jth component of $(\Omega^{i-1}_{k/\mathbb{Q}})^{\oplus(m-1)}$, for $1 \leq i \leq 2$ and $1 \leq j \leq m-1$. This shows that the generalizations of the above conjectures to the ring $k[\varepsilon]_m$ is true in the weight two case.

2.2 Relation to scissors congruence class groups

If $\mathcal{G}^n$ is one of the three n-dimensional classical geometries: $\mathcal{E}^n$, the euclidean; $\mathcal{H}^n$, the hyperbolic; or $\mathcal{S}^n$, the spherical, then let $\mathcal{P}(\mathcal{G}^n)$ denote the scissors congruence class group corresponding to $\mathcal{G}^n$ as defined in [11, §3.1]. The Dehn invariant map [11, p. 572]:

$$D_n^{\mathcal{G}} : \mathcal{P}(\mathcal{G}^n) \to \oplus_{i=1}^{n-2}\mathcal{P}(\mathcal{G}^i) \otimes \mathcal{P}(\mathcal{S}^{n-i-1})$$

endows $\oplus\mathcal{P}(\mathcal{S}^{\cdot})$ with the structure of a coalgebra and, $\oplus\mathcal{P}(\mathcal{H}^{\cdot})$ and $\oplus\mathcal{P}(\mathcal{E}^{\cdot})$ with structures of comodules over this coalgebra.

Goncharov defines a Hopf algebra $S_{\cdot}(k)$ [11, p. 591] similar to scissors congruence groups and expects a map $S_{\cdot}(k) \to \mathcal{A}_{\cdot}(k)$. This is known, if k is a number field [11, p. 612]. There is a map: $\mathcal{P}(\mathcal{H}^{2n-1}) \to S_n(\mathbb{C})$ [11, Theorem 5.2] and hence one expects a map: $\mathcal{P}(\mathcal{H}^{2n-1}) \to \mathcal{A}_n(\mathbb{C})$.

If one considers the Cayley spherical model for the hyperbolic geometry then as the sphere gets bigger the hyperbolic geometry approaches to the euclidean geometry [11, p. 616]. Therefore in the limit case one should have a natural map $\mathcal{P}(\mathcal{E}^{2n-1}) \to \mathcal{A}_n(\mathbb{C}[\varepsilon]_2)^{\circ}$.

The euclidean scissors congruence class group $\mathcal{P}(\mathcal{E}^n_k)$ can be defined for any field k [13, §3.2], where this is denoted by $\mathcal{E}_n(k)$), and, as above, one expects a map

$$\mathcal{P}(\mathcal{E}_k^{2n-1}) \to \mathcal{A}_n(k[\varepsilon]_2)^{\circ}.$$

The euclidean Dehn invariant endows $\oplus\mathcal{P}(\mathcal{E}^{\cdot}_k)$ with the structure of a comodule over $S_{\cdot}(k)$ [13, §3.4]. If one has a map $S_{\cdot}(k) \to \mathcal{A}_{\cdot}(k)$ then this gives an $\mathcal{A}_{\cdot}(k)$-comodule structure on $\oplus\mathcal{P}(\mathcal{E}^{\cdot}_k)$, and a cobar complex as in [13, §3.4]; if we let $\mathrm{H}^i(\oplus_{2n-1}\mathcal{P}(\mathcal{E}^{\cdot}_k))$ denote the ith cohomology of the $(2n-1)$th graded part of this complex then we would have

$$\mathrm{H}^i(\oplus_{2n-1}\mathcal{P}(\mathcal{E}^{\cdot}_k)) \simeq \Omega^{i-1}_{k/\mathbb{Q}},$$

for k a field of characteristic 0 [11,13, Question 6.4]], and $1 \leq i \leq n$.

Let $\mathrm{H}^i(\mathcal{A}_{\cdot}(k)(n))$ denote the ith cohomology of the nth graded piece of the cobar complex of $\mathcal{A}_{\cdot}(k)$ [3, §3.16]. The general conjectures implying that this cohomology group is isomorphic to the ith cohomology of $\Gamma'(n)_{\mathbb{Q}}$ (Conjecture A′ in §1.11 and Conjecture 1.40 in [12, §1.16]) together with (2.1.3) give that

$$\mathrm{H}^i(\mathcal{A}_{\cdot}(k[\varepsilon]_2)(n)^{\circ}) \simeq \Omega^{i-1}_{k/\mathbb{Q}},$$

for $1 \le i \le n$.

These suggest a close similarity between the structures of $\mathcal{A}_n(k[\varepsilon]_2)^{\circ}$ and $\mathcal{P}(\mathcal{E}_k^{2n-1})$. The euclidean scissors congruence class group has a volume map

$$\mathcal{P}(\mathcal{E}_k^{2n-1}) \to k$$

which is conjectured to induce an isomorphism from $\mathrm{H}^1(\oplus_{2n-1}\mathcal{P}(\mathcal{E}_k^{\cdot}))$, the kernel in $\mathcal{P}(\mathcal{E}_k^{2n-1})$ of the Dehn invariant map, to k. For $n = 2$ and $k = \mathbb{R}$, this is Sydler's theorem [20]. In analogy, we expect a map:

$$\mathrm{vol}_n^{\circ}(k) : \mathcal{A}_n(k[\varepsilon]_2)^{\circ} \to k,$$

which induces an isomorphism from $\mathrm{H}^1(\mathcal{A}_{\cdot}(k[\varepsilon]_2)(n)^{\circ})$ to k. Moreover, we should have the identity $\mathrm{vol}_n^{\circ}(k) \circ \rho_\lambda = \lambda^{2n-1}\mathrm{vol}_n^{\circ}(k)$, for $\lambda \in k$.

Remark 2 For $n = 2$ and $\mathcal{A}_{\cdot}$ replaced with $A_{\cdot}$, the above conjecture and its generalization to higher modulus is the main content of Theorem 1.3.2 in [21]. Namely, if one lets $\mathrm{vol}_2^{\circ}(k)_r$ denote the compositions of the inclusion $A_2(k[\varepsilon]_r)^{\circ} \to A_2(k[\varepsilon]_r)$, the map $A_2(k[\varepsilon]_r) \to B_2(k[\varepsilon_r])$ in §3.3, [3]; and $\mathrm{Li}_{2,r}$:$B_2(k[\varepsilon]_r) \to k^{\oplus(r-1)}$ in [21], then Theorem 3.1.2 of [21] and Corollary 3.15.4 of [3] imply that $\mathrm{vol}_2^{\circ}(k)_r$ induces an isomorphism between the kernel in $A_2(k[\varepsilon]_r)^{\circ}$ of the comultiplication map and $k^{\oplus(r-1)}$.

3 Additive polylogarithms

3.1 Main construction

For an R-module M let $M^{\otimes_R n}$ denote the tensor product of M with itself n-times. We drop the R from the notation if $R = \mathbb{Z}$.

Consider the map

$$\delta_n^{(m)} : k[\varepsilon]_m^{\flat} \to (k[\varepsilon]_m^{\times})^{\otimes n},$$

that sends a to $(1 - a) \otimes a \otimes \cdots \otimes a$.

Let

$$\log^{\otimes n} : (k[\varepsilon]_m^{\times})^{\otimes n} \to (k[\varepsilon]_m^{\circ})^{\otimes n}$$

denote the map induced by the composition of the natural projection $k[\varepsilon]_m^{\times} \to k[\varepsilon]_m^{\times\circ}$ and the logarithm map, $\log : k[\varepsilon]_m^{\times\circ} \to k[\varepsilon]_m^{\circ}$.

Let us denote the composition of $\log^{\otimes n} \circ \delta_n^{(m)}$ and the natural projection

$$(k[\varepsilon]_m^\circ)^{\otimes n} \to (k[\varepsilon]_m^\circ \wedge_k k[\varepsilon]_m^\circ) \otimes_k (k[\varepsilon]_m^\circ)^{\otimes_k (n-2)},$$

by

$$\Delta_n^{(m)} : k[\varepsilon]_m^\flat \to (k[\varepsilon]_m^\circ \wedge_k k[\varepsilon]_m^\circ) \otimes_k (k[\varepsilon]_m^\circ)^{\otimes_k (n-2)}.$$

For $1 \le j_2 < j_1$ and $1 \le j_3, \ldots, j_n$ let

$$\pi^{(m)}(j_1, j_2, \ldots, j_n) : (k[\varepsilon]_m^\circ \wedge_k k[\varepsilon]_m^\circ) \otimes_k (k[\varepsilon]_m^\circ)^{\otimes_k (n-2)} \to k$$

be defined by the identiy

$$\alpha = \sum_{\substack{1 \le j_2 < j_1 \le m-1 \\ 1 \le j_3, \cdots, j_n \le m-1}} \pi^{(m)}(j_1, j_2, \cdots, j_n)(\alpha)(\varepsilon^{j_1} \wedge \varepsilon^{j_2}) \otimes \varepsilon^{j_3} \otimes \cdots \otimes \varepsilon^{j_n}.$$

Let $I_n := \{(i_1, i_2, \cdots, i_{n-1}) \in \mathbb{N}^{n-1} | i_1 \ge 2, i_2 \ge i_3 \ge \cdots \ge i_{n-1} \ge 1, i_1 + i_2 + \cdots + i_{n-1} = 2n-2\}$.

We are going to define the additive polylogarithm by first finding a function

$$c : I_n \to \mathbb{Q}$$

such that the sum

$$\sum_{(i_1, i_2, \cdots, i_{n-1}) \in I_n} c(i_1, i_2, \cdots, i_{n-1}) \pi^{(n+1)}(i_1, 1, i_2, \cdots, i_{n-1}) \circ \Delta_n^{(n+1)}, \tag{3.1.1}$$

which is a map from $k[\varepsilon]_{n+1}^\flat$ to k, factors through the canonical projection $k[\varepsilon]_{n+1}^\flat \to k[\varepsilon]_2^\flat$. This then gives a map from $k[\varepsilon]_2^\flat$ to k, which will be defined to be the additive polylogartihm of weight n.

The following proposition is crucial in what follows:

Proposition 1 *There is, up to multiplication by a scalar, a unique $c : I_n \to \mathbb{Q}$ that makes* (3.1.1) factor through the canonical projection $k[\varepsilon]_{n+1}^\flat \to k[\varepsilon]_2^\flat$.

Proof Let us first prove the existence of such a map.

Note that if

$$\log\left(1 + \frac{1}{s}(a_1\varepsilon + \cdots a_n\varepsilon^n)\right) = b_1\varepsilon + \cdots + b_n\varepsilon^n =: u(\varepsilon)$$

then

$$\begin{aligned}\Delta_n^{(n+1)}(s + a_1\varepsilon + \cdots a_n\varepsilon^n) &= \log\left(1 - \frac{s}{1-s}(e^{u(\varepsilon)} - 1)\right) \wedge u(\varepsilon) \otimes (u(\varepsilon))^{\otimes(n-2)} \\ &= -\ell i_1(t(e^{u(\varepsilon)} - 1)) \wedge u(\varepsilon) \otimes (u(\varepsilon))^{\otimes(n-2)},\end{aligned}$$

where $t = s/(1-s)$ and $\ell i_1(z) := \sum_{1\leq n} \frac{z^n}{n}$. Letting

$$u(\varepsilon) := \sum_{1\leq n} b_1\varepsilon^n \in (k[b_1, b_2, \cdots])[[\varepsilon]],$$
$$\psi_t(u) := \ell i_1(t(e^u - 1)) \in (k[t])[[u]],$$

and

$$\psi_t(u(\varepsilon)) = \sum_{1\leq n} \beta_n(t)\varepsilon^n,$$

where $\beta_n(t) \in k[t, b_1, b_2, \cdots]$, finding c as above is equivalent to finding c such that the sum

$$\sum_{(i_1,\cdots,i_{n-1})\in I_n} c(i_1, \cdots, i_{n-1})(\beta_{i_1}(t)b_1 - \beta_1(t)b_{i_1})b_{i_2}\cdots b_{i_{n-1}}$$

depends only on t and b_1. Since $\beta_1(t) = b_1 t$ this is equivalent to requiring that

$$\sum_{(i_1,\cdots,i_{n-1})\in I_n} c(i_1, \cdots, i_{n-1})\overline{\beta}_{i_1}(t)b_{i_2}\cdots b_{i_{n-1}} \tag{3.1.2}$$

depend only on t and b_1, where $\overline{\psi}_t(u) = \psi_t(u) - tu$ and

$$\overline{\psi}_t(u(\varepsilon)) = \sum_{2\leq n} \overline{\beta}_n(t)\varepsilon^n = \sum_{1\leq n}(\beta_n(t) - b_n t)\varepsilon^n.$$

From now on for $(i_1, i_2, \cdots, i_{n-1}) \in I_n$, let

$$c(i_1, i_2, \ldots, i_{n-1}) := (-1)^{n-(l_2+\cdots+l_{n-1})} i_1 \frac{(n-1+(l_2+\cdots+l_{n-1}))!}{(n-1)!(l_2)!\cdots(l_{n-1})!}, \tag{3.1.3}$$

where l_k is the number of k's occuring among $(i_2, \ldots, i_{n-1})$, for $2 \leq k \leq n-1$.

The following lemma, due to D. Zagier, then proves the existence part of Proposition 1:

Lemma 1 (Zagier) *With $c : I_n \to \mathbb{Q}$ defined as in (3.1.3), the sum*

$$\sum_{(i_1,\ldots,i_{n-1})\in I_n} c(i_1, \ldots, i_{n-1})\overline{\beta}_{i_1}(t)b_{i_2}\ldots b_{i_{n-1}}$$

depends only on b_1 and t. In fact, is equal to

$$(-1)^n b_1^{2n-2}\frac{\ell i_1(t(e^u - 1))^{(n)}(0)}{(n-1)!}.$$

Proof The sum in the statement above is equal to

$$\sum_{0\le l\le n-2}(n-l)\overline{\beta}_{n-l}(t)S_{n,l}(b_1,\ldots,b_{n-1})$$

where

$$\begin{aligned}&S_{n,l}(b_1,\ldots,b_{n-1})\\&\quad=\sum_{\substack{l_2,\ldots,l_{n-1}\ge 0\\ l_2+2l_3+\cdots+(n-2)l_{n-1}=l}}\frac{((n-1)+(l_2+\cdots+l_{n-1}))!}{(n-1)!(l_2)!\cdots(l_{n-1})!}\\&\qquad\times(-b_1)^{(n-2)-(l_2+\cdots+l_{n-1})}b_2^{l_2}\ldots b_{n-1}^{l_{n-1}}.\end{aligned}$$

Now notice that for $0\le l\le n-2$,

$$\begin{aligned}&S_{n,l}(b_1,\ldots,b_{n-1})\\&\quad=\text{Coefficient of }\varepsilon^l\text{ in}\sum_{0\le j}\binom{n-1+j}{j}(-b_1)^{n-2-j}(b_2\varepsilon+\cdots+b_{n-1}\varepsilon^{n-2})^j\\&\quad=\text{Coefficient of }\varepsilon^l\text{ in }(-1)^n b_1^{2n-2}(b_1+b_2\varepsilon+\cdots+b_{n-1}\varepsilon^{n-2})^{-n}\\&\quad=(-1)^n b_1^{2n-2}\mathrm{Res}_{\varepsilon=0}(\frac{\varepsilon^{n-l-1}}{u(\varepsilon)^n}d\varepsilon).\end{aligned}$$

This implies that the expression in the statement of the lemma is equal to

$$\begin{aligned}&(-1)^n b_1^{2n-2}\mathrm{Res}_{\varepsilon=0}\left(\frac{d\overline{\psi}_t(u(\varepsilon))/d\varepsilon}{u(\varepsilon)^n}d\varepsilon\right)\\&\quad=(-1)^n b_1^{2n-2}\mathrm{Res}_{u=0}(\frac{d\overline{\psi}_t(u)}{u^n})\\&\quad=(-1)^n b_1^{2n-2}n\frac{\overline{\psi}_t^{(n)}(0)}{n!}\\&\quad=(-1)^n b_1^{2n-2}\frac{\ell i_1(t(e^u-1))^{(n)}(0)}{(n-1)!}.\end{aligned}$$

In order to prove the uniqueness, we only need to show that if $c: I_n\to\mathbb{Q}$ is a function such that (3.1.2) depends only on t and b_1 and that $c(n,1,\ldots,1)=0$ then $c=0$. Assume that $c\ne 0$ and let n_0 be the greatest integer such that there is an $(n-1)$-tuple $(n_0,k_2,\ldots,k_{n-1})\in I_n$ such that $c(n_0,k_2,\ldots,k_{n-1})\ne 0$. Note that $2\le n_0<n$.

By looking at the coefficient of t^{n_0} in

$$\overline{\psi}_t(u(\varepsilon)) = \sum_{2\le n} \overline{\beta}_n(t)\varepsilon^n = -tu(\varepsilon) + t(u(\varepsilon) + u(\varepsilon)^2/2! + \cdots) + \frac{1}{2}(t(u(\varepsilon) + u(\varepsilon)^2/2! + \cdots)^2 + \cdots$$

we see that the coefficient of t^{n_0} in (3.1.2) is

$$\frac{b_1^{n_0}}{n_0} \sum_{(n_0,i_2,\ldots,i_{n-1})\in I_n} c(n_0, i_2, \ldots, i_{n-1}) b_{i_2} \ldots b_{i_{n-1}}. \tag{3.1.4}$$

Since $n_0 < n$, if $(n_0, i_2, \ldots, i_{n-1}) \in I_n$ then at least one of the $i_2, \ldots, i_{n-1}$ is greater than 1. This implies that the sum (3.1.4) is 0. But since $c(n_0, k_2, \ldots, k_{n-1}) \neq 0$, this contradicts the algebraic independence of $b_1, b_2, \ldots$.

Definition 1 Let li_n denote

$$\frac{1}{n} \sum_{(i_1,i_2,\cdots,i_{n-1})\in I_n} c(i_1, i_2, \ldots, i_{n-1}) \pi^{(n+1)}(i_1, 1, i_2, \ldots, i_{n-1}) \circ \Delta_n^{(n+1)}$$

where c is defined as in (3.1.3). Then by Lemma 1, li_n descends to a map

$$li_n : k[\varepsilon]_2^\flat \to k$$

which we call the additive polylogarithm of weight n.

Corollary 1 *For $s + a\varepsilon \in k[\varepsilon]_2^\flat$,*

$$li_n(s + a\varepsilon) = \frac{(-1)^{n+1}}{n!}(a/s)^{2n-1} \ell i_1 \left(\frac{s(e^u - 1)}{1 - s}\right)^{(n)}(0),$$

where the derivative is with respect to u.

Proof Follows directly from the definition of li_n and the proof of Proposition 1.

Remark 3 The referee noted that some of the values of li_n can be expressed in terms of the values of the Riemann zeta function at negative integers. Namely, after making the substitution $t = e^u$, Corollary 1 gives

$$li_n(s + a\varepsilon) = \frac{(-1)^{n+1}}{n!}(a/s)^{2n-1}(td/dt)^{n-1} \left(\frac{st}{1 - st}\right)\Bigg|_{t=1}.$$

Then using Euler's formula ([15, p. 27]):

$$(1 - 2^n)\zeta(1 - n) = (td/dt)^{n-1} \left(\frac{t}{1 + t}\right)\Bigg|_{t=1}$$

for $n \geq 2$, we get

$$li_n(-1+a\varepsilon) = \frac{(-1)^{n+1}}{n!} a^{2n-1}(1-2^n)\zeta(1-n).$$

Corollary 2 *For $s + a\varepsilon \in k[\varepsilon]_2^\flat$, we have*

$$li_n(s+a\varepsilon) = \frac{(-1)^{n+1}}{n!}(a/s)^{2n-1} \sum_{1\leq i\leq m\leq n} \frac{(-1)^{m-i}}{m}\left(\frac{s}{1-s}\right)^m \binom{m}{i} i^n$$

Proof Note that the coefficient of u^n in $\ell i_1(\frac{s(e^u-1)}{1-s})$ is

$$\frac{1}{n!}\sum_{1\leq i\leq m\leq n}(-1)^{m-i}\frac{(s/(1-s))^m}{m}\binom{m}{i} i^n.$$

The corollary follows.

Remark 4 As the referee pointed out to us, there is an analogy between our construction and that of Dupont in [9]. Namely, let $\underline{\mathbb{C}}^\times$ denote $\mathbb{C}^\times$ modulo torsion. Consider the map

$$\underline{L}_n : \mathbb{C}^\flat \to \underline{\mathbb{C}}^\times \otimes \underline{\mathbb{C}}^\times \otimes \cdots \otimes \underline{\mathbb{C}}^\times \quad (n \text{ copies})$$

that sends z to $(1-z)\otimes z \otimes \cdots \otimes z$. Let

$$\omega_i := \frac{1}{2\pi i}\frac{dz}{z-i},$$

and let t_{01} denote the unit tangential basepoint at 0 pointing towards 1 as in [7]. If $p : \tilde{\mathbb{C}}^\flat \to \mathbb{C}^\flat$ denotes the projection map from the universal covering space of $\mathbb{C}^\flat$ and

$$e : \mathbb{C}\otimes\mathbb{C}\otimes\cdots\otimes\mathbb{C} \to \underline{\mathbb{C}}^\times \otimes \underline{\mathbb{C}}^\times \otimes \cdots \otimes \underline{\mathbb{C}}^\times$$

denotes the nth tensor power of the map that sends z to $\exp(2\pi i z)$ then we have a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{C}}^\flat & \longrightarrow & \mathbb{C}\otimes\mathbb{C}\otimes\cdots\otimes\mathbb{C} \\ {\scriptstyle p}\downarrow & & {\scriptstyle e}\downarrow \\ \mathbb{C}^\flat & \xrightarrow{\underline{L}_n} & \underline{\mathbb{C}}^\times \otimes \underline{\mathbb{C}}^\times \otimes \cdots \otimes \underline{\mathbb{C}}^\times \end{array}$$

where the upper horizontal map is the one that sends a path γ starting at t_{01} to

$$\int_\gamma \omega_1 \otimes \int_\gamma \omega_0 \otimes \cdots \otimes \int_\gamma \omega_0.$$

Since the integrals depend on the path, this map does not descend to $\mathbb{C}^\flat$. But Dupont shows [9, §4] that by adding the correct linear combination of the tensor product of iterated integrals $\int_\gamma \omega_{i_1} \circ \cdots \circ \omega_{i_k}$, where i_j is 0 or 1, to the upper horizontal map, one defines a map

$$L_n : \tilde{\mathbb{C}}^\flat \to \mathbb{C} \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}$$

such that the above diagram still commutes, and moreover L_n descends to give a map

$$L_n : \mathbb{C}^\flat \to \mathbb{C} \otimes \mathbb{C} \otimes \cdots \otimes \mathbb{C}.$$

Composing this map with maps of the form $\mathbb{C} \otimes \cdots \otimes \mathbb{C} \to \mathbb{R}$, obtained by taking linear combinations of products of the real or imaginary parts of the factors, one gets real valued polylogarithmic functions. For example, when $n = 2$, we have

$$(m \circ (\mathrm{Re} \otimes \mathrm{Im}) \circ L_2)(z) = \frac{-1}{4\pi^2} D(z),$$

where $m{:}\mathbb{R} \otimes \mathbb{R} \to \mathbb{R}$ is the multiplication and $D(z)$ is the single valued real analytic Bloch–Wigner dilogarithm [12, p. 201].

Our construction of additive polylogarithms is similar to this. We have a commutative diagram

$$\begin{array}{ccc} k[\varepsilon]_{n+1}^\flat & \longrightarrow & k[\varepsilon]_{n+1}^\circ \otimes k[\varepsilon]_{n+1}^\circ \otimes \cdots \otimes k[\varepsilon]_{n+1}^\circ \\ \downarrow & & \downarrow \\ k[\varepsilon]_2^\flat & \longrightarrow & k[\varepsilon]_2^{\times\circ} \otimes k[\varepsilon]_2^{\times\circ} \otimes \cdots \otimes k[\varepsilon]_2^{\times\circ} \end{array}$$

where the upper and lower horizontal maps are the ones that send z to $\log(1-z) \otimes \log(z) \otimes \cdots \otimes \log(z)$ and to $(1-z) \otimes z \otimes \cdots \otimes z$, respectively, and the vertical maps are the reduction map and the n-the tensor power of the exponential map composed with the reduction map. Then composing the upper horizontal map with a map from $k[\varepsilon]_{n+1}^\circ \otimes k[\varepsilon]_{n+1}^\circ \otimes \cdots \otimes k[\varepsilon]_{n+1}^\circ$ to k, that is constructed by choosing the right algebraic combinations of the coefficients in $k[\varepsilon]_{n+1}$, we get a map from $k[\varepsilon]_{n+1}^\flat$ to k that descends to give the additive n-polylogarithm $li_n : k[\varepsilon]_2^\flat \to k$.

3.2 Aomoto polylogarithms in the infinitesimal case

A *simplex* in $\mathbb{P}^n_{k[\varepsilon]_r}$ is an ordered $(n+1)$-tuple $H := (H_0, \ldots, H_n)$ of hyperplanes in $\mathbb{P}^n_{k[\varepsilon]_r}$. We denote the reduction of H to a simplex in $\mathbb{P}^n_k$ by $\underline{H}$. H is said to be *non-degenerate*, if $\cap_{0\le i\le n}\underline{H}_i = \emptyset$. A *face* of H is an intersection $\cap_{i\in I} H_i$, for some $I \subset \{0, 1, \ldots n\}$. A pair of simplices (L, M) is said to be *admissible* if $\underline{L}$ and $\underline{M}$ do not have a common face, and *generic* if all the faces of $\underline{L}$ and $\underline{M}$ are in general position.

The groups $A_\cdot(k)$ were first defined in [3, §2.1]. We let $A_0(k[\varepsilon]_r) = \mathbb{Z}$. For $n > 0$, let $A_n(k[\varepsilon]_r)$ be the abelian group generated by pairs of *admissible* simplices (L, M) modulo the following relations:

(i) $(L, M) = 0$, if L or M is degenerate

(ii) For $\sigma \in Sym(n)$, the group of permutations of $\{0, \ldots, n\}$, let

$$\sigma(L_0, \ldots, L_n) := (L_{\sigma(0)}, \ldots, L_{\sigma(n)}).$$

Then for every $\sigma \in Sym(n)$,

$$(\sigma(L), M) = (L, \sigma(M)) = \mathrm{sgn}(\sigma)(L, M).$$

(iii) If $L_0, \ldots, L_{n+1}$ are $n+2$ hyperplanes in $\mathbb{P}^n_{k[\varepsilon]_r}$ such that for all $0 \le i \le n+1$,

$$((L_0, \ldots, \hat{L}_i, \ldots, L_{n+1}), M)$$

is admissible then we have the additivity relation for the first component:

$$\sum_{0 \le i \le n+1} (-1)^j ((L_0, \ldots, \hat{L}_i, \ldots, L_{n+1}), M) = 0.$$

We have the analogous additivity relation for the second component.

(iv) For every $\alpha \in \mathrm{PGL}_{n+1}(k[\varepsilon]_r)$,

$$(\alpha L, \alpha M) = (L, M).$$

There is a product from $A_{n'}(k[\varepsilon]_r) \otimes A_{n''}(k[\varepsilon]_r)$ to $A_{n'+n''}(k[\varepsilon]_r)$ that can be described as follows [3, p. 148]. Let $L' \subseteq \mathbb{P}^{n'}_{k[\varepsilon]_r}$, $L'' \subseteq \mathbb{P}^{n''}_{k[\varepsilon]_r}$ and $L \subseteq \mathbb{P}^{n'+n''}_{k[\varepsilon]_r}$ be the standard coordinate simplices: $L' = (z_0 = 0, \ldots, z_{n'} = 0)$, etc. By (iv), we can always assume this. Let M' and M'' be non-degenerate simplices such that (L', M') and (L'', M'') are admissible. Identifying $\mathbb{A}^n_{k[\varepsilon]_r}$ with $\mathbb{P}^n_{k[\varepsilon]_r} \backslash \{z_0 = 0\}$, we have $(M' \cap \mathbb{A}^{n'}_{k[\varepsilon]_r}) \times (M'' \cap \mathbb{A}^{n''}_{k[\varepsilon]_r}) \subseteq \mathbb{A}^{n'+n''}_{k[\varepsilon]_r}$. Let P be the closure of this in $\mathbb{P}^{n'+n''}_{k[\varepsilon]_r}$. In general, P is not a simplex but can be subdivided into simplices $\sum_i M_i$. Then the product of (L', M') and (L'', M'') is $\sum_i (L, M_i)$. By (iii), this is independent of the subdivision.

A *prism* in $A_n(k[\varepsilon]_r)$ is a product $(L', M') \cdot (L'', M'')$ with (L', M') and (L'', M'') simplices in $\mathbb{P}^{n'}_{k[\varepsilon]_r}$ and $\mathbb{P}^{n''}_{k[\varepsilon]_r}$ and $n', n'' \ge 1$, $n = n' + n''$. The group of prisms $P_n(k[\varepsilon]_r)$ is the subgroup generated by the prisms in $A_n(k[\varepsilon]_r)$.

For $x \in k[\varepsilon]_r^\flat$, let (L, M_x) be the configuration in $\mathbb{P}^n_{k[\varepsilon]_r}$ such that L is the standard coordinate simplex and M_x is the simplex defined as

$$(z_1 - z_0 = 0, z_1 + z_2 = z_0, z_3 - z_2 = 0, \ldots, z_n - z_{n-1} = 0, z_n - x z_0 = 0).$$

This then defines a map $l_n : \mathbb{Z}[k[\varepsilon]_r^\flat] \to A_n(k[\varepsilon]_r)$, by letting $l_n(x) := (L, M_x)$. We denote the induced map from $\mathbb{Z}[k[\varepsilon]_r^\flat]$ to $A_n(k[\varepsilon]_r)/P_n(k[\varepsilon]_r)$ by the same letter.

We let $B'_n(k[\varepsilon]_r)$ denote the image of l_n in $A_n(k[\varepsilon]_r)/P_n(k[\varepsilon]_r)$, and the image of $l_n(x)$ in $B'_n(k[\varepsilon]_r)$ by $\{x\}_n$.

One expects that there is a natural coproduct on the graded ring

$$A.(k[\varepsilon]_r) := \sum_{0\leq n} A_n(k[\varepsilon]_r)$$

that will make it a graded Hopf algebra over $\mathbb{Z}$ [3, §2.16]. Such a coproduct was defined by Zhao on the subgroup of *generic* configurations in [25] and in the case of all admissable configurations for $n \leq 3$ in [26]. In Theorem 1, we will assume that there is a coproduct on $A.(k[\varepsilon]_r)$ that makes it a Hopf algebra and that this coproduct behaves as expected on the elements $\{x\}_n$ (c.f. Proposition, [3, §2.10]). Next we proceed to make this more precise.

A coproduct that makes $A.(k[\varepsilon]_r)$ a Hopf algebra, induces a Lie coalgebra structure on $A.(k[\varepsilon]_r)/P.(k[\varepsilon]_r)$ [3, §2.16]:

$$A.(k[\varepsilon]_r)/P.(k[\varepsilon]_r) \to A.(k[\varepsilon]_r)/P.(k[\varepsilon]_r) \wedge A.(k[\varepsilon]_r)/P.(k[\varepsilon]_r).$$

We obtain, by restriction, a map from $B'_n(k[\varepsilon]_r)$ to

$$(A.(k[\varepsilon]_r)/P.(k[\varepsilon]_r) \wedge A.(k[\varepsilon]_r)/P.(k[\varepsilon]_r))_n.$$

The projection of this to the $(n-1, 1)$ component gives a map to

$$\begin{aligned} A_{n-1}(k[\varepsilon]_r)/P_{n-1}(k[\varepsilon]_r) \otimes A_1(k[\varepsilon]_r) &= A_{n-1}(k[\varepsilon]_r)/P_{n-1}(k[\varepsilon]_r) \otimes k[\varepsilon]_r^\times : \\ \Delta_{n-1,1} : B'_n(k[\varepsilon]_r) &\to A_{n-1}(k[\varepsilon]_r)/P_{n-1}(k[\varepsilon]_r) \otimes k[\varepsilon]_r^\times, \end{aligned}$$

if $n \geq 3$, and

$$\Delta_{1,1} : B'_2(k[\varepsilon]_r) \to \Lambda^2 k[\varepsilon]_r^\times.$$

If $\Delta_{n-1,1}(\{x\}_n) = \{x\}_{n-1} \otimes x$ then the additive polylogarithm defines a map from $B'_n(k[\varepsilon]_2)$ to k:

Theorem 1 *Suppose that $A.(k[\varepsilon]_r)$ has a natural coproduct such that*

$$\Delta_{n-1,1}(\{x\}_n) = \{x\}_{n-1} \otimes x$$

for $n \geq 3$ and $\Delta_{1,1}(\{x\}_2) = (1-x) \wedge x$. Then the additive polylogarithm $li_n : \mathbb{Z}[k[\varepsilon]_2^\flat] \to k$ factors through the projection l_n:$\mathbb{Z}[k[\varepsilon]_2^\flat] \to B'_n(k[\varepsilon]_2)$ to give a map $B'_n(k[\varepsilon]_2) \to k$.

Proof Note that any prism in $A_n(k[\varepsilon]_2)$ can obviously be lifted to one in $A_n(k[\varepsilon]_{n+1})$. Then by the definition of li_n and $B'_n(k[\varepsilon]_2)$, it suffices to show that $\Delta_n^{(n+1)}(l_n^{-1}(P_n^{(n+1)})) = 0$, where $P_n^{(n+1)}$ is the group of prisms in $A_n(k[\varepsilon]_{n+1})$.

Let $id^{\otimes m}$ denote the identity function on $(k[\varepsilon]_{n+1}^\times)^{\otimes m}$. By the hypothesis on the value of $\Delta_{n-1,1}$ on $\{x\}_n$, it is easy to see that, $\Delta_n^{(n+1)}$ factors through

$$(\Delta_{1,1}\otimes id^{\otimes(n-2)})\circ\cdots\circ(\Delta_{n-3,1}\otimes id^{\otimes 2})\circ(\Delta_{n-2,1}\otimes id^{\otimes 1})\circ\Delta_{n-1,1}\circ l_n.$$

The claim then follows from that $\Delta_{n-1,1}(P_n^{(n+1)})=0$.

Corollary 3 *The additive polylogarithm* $li_n:\mathbb{Z}[k[\varepsilon]_2^\flat]\to k$ *induces a map* $B_n'(k[\varepsilon]_2)\to k$, *for* $n\le 3$.

Proof The hypotheses of the previous theorem are satisfied in case $n\le 3$. Namely, Zhao constructs such a coproduct [26, Example 6.4].

3.3 Nontriviality of the additive polylogarithm

We will need the following lemma in order to show the non-triviality of li_n. From now on let $\omega:=\frac{1+\sqrt{-3}}{2}$. We assume in this section that $\omega\in k$.

Lemma 2 *For* $n\ge 2$,

$$\ell i_1\left(\frac{\omega(e^u-1)}{1-\omega}\right)^{(n)}(0)\neq 0.$$

Proof Let $s_m:=\frac{(-1)^m}{m}\sum_{1\le i\le m}(-1)^i\binom{m}{i}i^n$. Since the coefficient of u^n in $(e^u-1)^m$ is 0 if $m>n$, $s_m=0$, for $m>n$.

First note that $\ell i_1(\frac{\omega(e^u-1)}{1-\omega})^{(n)}(0)$ is equal to

$$\omega^2\left(\sum_{\substack{1\le m\le n\\ m\equiv 1(\mathrm{mod}\ 3)}}s_m-\sum_{\substack{1\le m\le n\\ m\equiv 2(\mathrm{mod}\ 3)}}s_m\right)+\left(\sum_{\substack{1\le m\le n\\ m\equiv 0(\mathrm{mod}\ 3)}}s_m-\sum_{\substack{1\le m\le n\\ m\equiv 2(\mathrm{mod}\ 3)}}s_m\right),$$

since $\omega/(1-\omega)=\omega^2$, and $\omega^4=-\omega^2-1$. Therefore if the expression in the statement of the lemma is 0 then we have

$$\sum_{\substack{1\le m\le n\\ m\equiv 0(\mathrm{mod}\ 3)}}s_m=\sum_{\substack{1\le m\le n\\ m\equiv 1(\mathrm{mod}\ 3)}}s_m=\sum_{\substack{1\le m\le n\\ m\equiv 2(\mathrm{mod}\ 3)}}s_m. \tag{3.3.1}$$

We will see that this is not possible by looking at the 2-adic valuations of these terms.

Claim For $m\ge 3$, $2|s_m$.

Proof of the claim When m is odd, it is enough to remark that

$$\sum_{1\le i\le m}(-1)^i\binom{m}{i}i^n\equiv\sum_{\substack{1\le i\le m\\ \text{i, odd}}}\binom{m}{i}\equiv\sum_{0\le i\le m-1}\binom{m-1}{i}\equiv 0\ (\mathrm{mod}\ 2).$$

Assume, then, that m is even. If $\nu_2(a)$ denotes the 2-adic valuation of a number then

$$\nu_2\left(\binom{m}{i}\right) = \left(\left[\frac{m}{2}\right] - \left[\frac{i}{2}\right] - \left[\frac{m-i}{2}\right]\right) + \left(\left[\frac{m}{4}\right] - \left[\frac{i}{4}\right] - \left[\frac{m-i}{4}\right]\right) + \cdots \tag{3.3.2}$$

Note that each of the terms $\left[\frac{m}{2^n}\right] - \left[\frac{i}{2^n}\right] - \left[\frac{m-i}{2^n}\right]$ is either 0 or 1, and is 1 if i is not divisible by 2^k and m is. Therefore $\nu_2(\binom{m}{i}) \geq \nu_2(m) - \nu_2(\gcd(m, i))$, and hence if i is even $\nu_2(\binom{m}{i} i^n) \geq \nu_2(m) + 1$. So it is enough to show that

$$\nu_2\left(\sum_{\substack{1 \leq i \leq m \\ i,\ \text{odd}}} \binom{m}{i} i^n\right) \geq \nu_2(m) + 1.$$

First assume that $4|m$. Then $\sum_{\substack{1 \leq i \leq m \\ i,\ \text{odd}}} \binom{m}{i} i^n = \sum_{\substack{1 \leq i \leq m/2-1 \\ i,\ \text{odd}}} \binom{m}{i}(i^n + (m-i)^n)$, and each of the terms in the last sum have valuation at least $\nu_2(m) + 1$.

If, on the other hand, $\nu_2(m) = 1$ then

$$\sum_{\substack{1 \leq i \leq m \\ i,\ \text{odd}}} \binom{m}{i} i^n = \sum_{\substack{1 \leq i \leq m/2-2 \\ i,\ \text{odd}}} \binom{m}{i}(i^n + (m-i)^n) + \binom{m}{m/2}\left(\frac{m}{2}\right)^n.$$

The first sum on the right hand side of the equation has valuation at least $\nu_2(m)+1 = 2$, as above. We need to see that $\nu_2\binom{m}{m/2} \geq 2$. But this follows immediately from (3.3.2), since if $2^k < m/2 < 2^{k+1}$ then $[\frac{m}{2}] - 2[\frac{m}{4}]$ and $[\frac{m}{2^{k+1}}] - 2[\frac{m}{2^{k+2}}]$ are both equal to 1.

Since s_1 and s_2 are odd, the last claim implies that the first two sums in the expression (3.3.1) are not divisible by 2, whereas the last one is.

3.4 Injectivity of the regulator

For the rest of this section we assume the existence of the comultiplication map Δ as in Sect. 3.2, whose restriction to B'_n is given as in Theorem 1. We would like to find an element in the kernel of $\Delta_{n-1,1}$ in $B'_n(k[\varepsilon]_2)^{\circ}_{\mathbb{Q}}$ on which li_n does not vanish. In order to do this, we will look at the the action of ρ_λ, for $\lambda \in \mathbb{Q}$.

Remark 5 Let $\ker^{\circ}(\Delta_{n-1,1})$ denote the kernel of $\Delta_{n-1,1}$ in $B'_n(k[\varepsilon]_2)^{\circ}_{\mathbb{Q}}$. Then the $k[\varepsilon]_2$ analog of Goncharov's conjectures imply that $\ker^{\circ}(\Delta_{n-1,1}) \simeq k$ ((2.1.3) in Sect. 2.1), and that the automorphisms ρ_λ of $k[\varepsilon_2]$ act on $\ker^{\circ}(\Delta_{n-1,1})$ as multiplication by λ^{2n-1}, for $\lambda \in \mathbb{Q}$. Therefore the hypothesis of the next lemma is in consistent with these conjectures.

Lemma 3 *Assume that the automorphisms ρ_λ act on* $\ker^{\circ}(\Delta_{n-1,1})$ *as multiplication by λ^{2n-1}, for all $n \geq 2$ and $\lambda \in \mathbb{Q}$. Then there are integers $\alpha_n(a, b) \in \mathbb{Z}$, for $0 \leq a, b$,* *with* $\alpha_n(a,b) = 0$ *if* $a + b(2n-1) \geq (n-1)(2n-1)$, *such that for any* $\lambda \in \mathbb{Q}$ *and* $q \in k$,

$$\{\omega + \lambda^{n-1} q\varepsilon\}_n + \sum_{0 \leq a,b} \alpha_n(a,b)\lambda^a \{\omega + \lambda^b q\varepsilon\}_n$$

is in $\ker^\circ(\Delta_{n-1,1})$.

Proof Start with $\{\omega + q\varepsilon\}_n \in B'_n(k[\varepsilon]_2)^\circ_{\mathbb{Q}}$. This is mapped to

$$\{\omega + q\varepsilon\}_{n-1} \otimes (\omega + q\varepsilon) = \{\omega + q\varepsilon\}_{n-1} \otimes \left(1 + \frac{q}{\omega}\varepsilon\right) \in (B'_{n-1}(k[\varepsilon]_2) \otimes k[\varepsilon]_2^\times)^\circ_{\mathbb{Q}},$$

for $n \geq 3$ and to

$$(1 - (\omega + q\varepsilon)) \wedge (\omega + q\varepsilon) = \left(1 - \frac{q}{1-\omega}\varepsilon\right) \wedge \left(1 + \frac{q}{\omega}\varepsilon\right) \in (k[\varepsilon]_2^\times \wedge k[\varepsilon]_2^\times)^\circ_{\mathbb{Q}}$$

if $n = 2$, since ω and $1 - \omega$ are both roots of unity.

For $n = 2$ by the above computation we see that

$$\{\omega + \lambda q\varepsilon\}_2 - \lambda^2\{\omega + q\varepsilon\}_2$$

is in the kernel of $\Delta_{1,1}$.

Suppose that we know that the statement of the lemma is true for $n = k$. Let us look at the following expression

$$\{\omega + \lambda^{k-1} q\varepsilon\}_{k+1} + \sum_{0 \leq a,b} \alpha_k(a,b)\lambda^{a+(k-1-b)}\{\omega + \lambda^b q\varepsilon\}_{k+1}.$$

Note that the sum is only over a, b with $a + (2k-1)b < (k-1)(2k-1)$.

This expression maps to

$$\lambda^{k-1}(\{\omega + \lambda^{k-1} q\varepsilon\}_k + \sum_{0 \leq a,b} \alpha_k(a,b)\lambda^a\{\omega + \lambda^b q\varepsilon\}_k) \otimes \left(1 + \frac{q}{\omega}\varepsilon\right)$$

Since by assumption the expression on the left in the tensor product lies in the kernel of $\Delta_{k-1,1}$, ρ_λ acts on it by multiplication with λ^{2k-1}. Therefore ρ_λ acts on the whole expression by multiplication by λ^{2k}. This implies that the expression

$$\begin{aligned} &\{\omega + \lambda^k q\varepsilon\}_{k+1} + \sum_{0 \leq a,b} \alpha_k(a,b)\lambda^{a+(k-1-b)}\{\omega + \lambda^{b+1} q\varepsilon\}_{k+1} \\ &\quad -\lambda^{2k}\{\omega + \lambda^{k-1} q\varepsilon\}_{k+1} - \sum_{0 \leq a,b} \alpha_k(a,b)\lambda^{a+(3k-1-b)}\{\omega + \lambda^b q\varepsilon\}_{k+1} \end{aligned}$$

lies in the kernel of $\Delta_{k,1}$. Rearranging the terms, defining the $\alpha_{k+1}(\cdot,\cdot)$ appropriately and noting that if $a + (2k-1)b < (k-1)(2k-1)$ then $(a+k-1-b) + (2k+1)(b+1)$,

$2k+(2k+1)(k-1)$ and $(a+3k-1-b)+(2k+1)b$ are all less than $k(2k+1)$ proves the lemma.

Proposition 2 *Under the assumptions of* Lemma 3, $li_n : \ker^\circ(\Delta_{n-1,1}) \to k$ *is non-zero.*

Proof The image of an element

$$\{\omega+\lambda^{n-1}q\varepsilon\}_n + \sum_{0\le a,b} \alpha_n(a,b)\lambda^a\{\omega+\lambda^b q\varepsilon\}_n \in \ker^\circ(\Delta_{n-1,1}) \qquad (3.4.1)$$

under li_n is

$$(\lambda^{(n-1)(2n-1)} + \sum_{0\le a,b} \alpha_n(a,b)\lambda^{a+(2n-1)b})li_n(\omega+q\varepsilon). \qquad (3.4.2)$$

The polynomial in λ in the last expression is of degree $(n-1)(2n-1)$, since $\alpha_n(a,b)=0$ for $a+(2n-1)b \ge (n-1)(2n-1)$, and so is, in particular, non-zero. So choosing $\lambda \in \mathbb{Q}$ such that it is not a root of this polynomial and $q \in k^\times$ we see that (3.4.2) is non-zero by Corollary 1 and Lemma 2.

Remark 6 Proposition 2 might be thought of as the infinitesimal version of the conjecture of Ramakrishnan [17, 7.1.2] on the injectivity modulo torsion of the Beilinson regulator. The general conjectures on mixed Tate motives in fact imply that $\ker^\circ(\Delta_{n-1,1})$ should be canonically isomorphic to $\mathrm{H}^1_{\mathcal{M}}(k[\varepsilon]_2, \mathbb{Q}(n))^\circ$ and hence that the map in Proposition 2 should be an isomorphism, and the elements (3.4.1) should generate $\mathrm{H}^1_{\mathcal{M}}(k[\varepsilon]_2, \mathbb{Q}(n))^\circ$. This and its generalizations to higher modulus was proved in [21] for $n=2$.

4 Functional equations

4.1 Functional equations for the additive polylogarithm

Our basic references for the functional equations of classical polylogarithms are [10, 22, 23].

The analogs of the inversion and distribution formula are valid for the additive polylogarithms. The situation in the case of additive polylogarithms is even better than the classical case since there are no lower order terms appearing in the functional equations. Below we list some of the functional equations that are satisfied by the additive polylogarithms. In fact we checked that any functional equation for the polylogarithm is also satisfied by the corresponding additive polylogarithm. Rather than listing all of these equations, below we state some of them in order to give the reader some idea about the statements and to give indications of the proofs.

Theorem 2 (Inversion formula) *For* $s+a\varepsilon \in k[\varepsilon]_2^\flat$ *and* $n \ge 2$, *we have*

$$li_n((s+a\varepsilon)^{-1}) = (-1)^{n-1} li_n(s+a\varepsilon).$$

Proof Denoting the lifting $s + a\varepsilon \in k[\varepsilon]^{\flat}_{n+1}$ of $s + a\varepsilon \in k[\varepsilon]^{\flat}_{2}$ by $\underline{s}$, we have the equality

$$\left(1 - \frac{1}{\underline{s}}\right) \wedge \frac{1}{\underline{s}} \otimes \frac{1}{\underline{s}} \otimes \cdots \otimes \frac{1}{\underline{s}} = (-1)^{n-1}(\underline{s} - 1) \wedge \underline{s} \otimes \underline{s} \otimes \cdots \otimes \underline{s}$$

in $(k[\varepsilon]^{\times}_{n+1} \wedge k[\varepsilon]^{\times}_{n+1}) \otimes k[\varepsilon]^{\times}_{n+1} \otimes \cdots \otimes k[\varepsilon]^{\times}_{n+1}$. Projecting this to $(k[\varepsilon]^{\circ}_{n+1} \wedge_k k[\varepsilon]^{\circ}_{n+1}) \otimes_k k[\varepsilon]^{\circ}_{n+1} \otimes_k \cdots \otimes_k k[\varepsilon]^{\circ}_{n+1}$ using $\Lambda^2 \log \otimes \log^{\otimes(n-2)}$ and then applying

$$\frac{1}{n} \sum_{(i_1, i_2, \ldots, i_{n-1}) \in I_n} c(i_1, i_2, \ldots, i_{n-1}) \pi^{(n+1)}(i_1, 1, i_2, \ldots, i_{n-1}),$$

to both sides of the equation gives the result.

Theorem 3 (Distribution formula) *Assume that k contains all the mth roots of unity. Then for $s + a\varepsilon \in k[\varepsilon]^{\flat}_{2}$ and $n \geq 2$, we have*

$$li_n((s + a\varepsilon)^m) = m^{n-1} \sum_{\zeta^m = 1} li_n(\zeta(s + a\varepsilon)).$$

Proof Following the notation in the proof of Theorem 2, we note that,

$$(1 - \underline{s}^m) \wedge \underline{s}^m \otimes \underline{s}^m \otimes \cdots \otimes \underline{s}^m = m^{n-1} \sum_{\zeta^m = 1} (1 - \zeta \underline{s}) \wedge \zeta \underline{s} \otimes \zeta \underline{s} \otimes \cdots \otimes \zeta \underline{s},$$

in $(k[\varepsilon]^{\times}_{n+1} \wedge k[\varepsilon]^{\times}_{n+1}) \otimes k[\varepsilon]^{\times}_{n+1} \otimes \cdots \otimes k[\varepsilon]^{\times}_{n+1}$ tensored with $\mathbb{Q}$. Since $1 - \underline{z}^m = (\prod_{\zeta^m = 1}(1 - \zeta z)$. Continuing as in the proof of Theorem 2 gives the result.

Theorem 4 (9-term relation for the additive trilogarithm) *For $x, y \in k[\varepsilon]^{\flat}_{2}$ such that all the terms below make sense, we have*

$$2li_3(x) + 2li_3(y) + 2li_3\left(\frac{x(1-y)}{x-1}\right) + 2li_3\left(\frac{y(1-x)}{y-1}\right) + 2li_3\left(\frac{1-x}{1-y}\right)$$
$$+2li_3\left(\frac{x(1-y)}{y(1-x)}\right) - li_3(xy) - li_3\left(\frac{x}{y}\right) - li_3\left(\frac{x(1-y)^2}{y(1-x)^2}\right) = 0.$$

Proof Let $\underline{x}, \underline{y} \in k[\varepsilon]_4^\flat$ be any two liftings of $x, y \in k[\varepsilon]_2^\flat$. Then we have the identity

$$2(1-\underline{x}) \wedge \underline{x} \otimes \underline{x} + 2(1-\underline{y}) \wedge \underline{y} \otimes \underline{y} + 2\frac{1-\underline{x}\underline{y}}{\underline{x}-1} \wedge \frac{\underline{x}(1-\underline{y})}{\underline{x}-1} \otimes \frac{\underline{x}(1-\underline{y})}{\underline{x}-1}$$
$$+2\frac{1-\underline{x}\underline{y}}{\underline{y}-1} \wedge \frac{\underline{y}(1-\underline{x})}{\underline{y}-1} \otimes \frac{\underline{y}(1-\underline{x})}{\underline{y}-1} + 2\frac{\underline{x}-\underline{y}}{1-\underline{y}} \wedge \frac{1-\underline{x}}{1-\underline{y}} \otimes \frac{1-\underline{x}}{1-\underline{y}}$$
$$+2\frac{\underline{y}-\underline{x}}{\underline{y}(1-\underline{x})} \wedge \frac{\underline{x}(1-\underline{y})}{\underline{y}(1-\underline{x})} \otimes \frac{\underline{x}(1-\underline{y})}{\underline{y}(1-\underline{x})} - (1-\underline{x}\underline{y}) \wedge \underline{x}\underline{y} \otimes \underline{x}\underline{y}$$
$$-\frac{\underline{y}-\underline{x}}{\underline{y}} \wedge \frac{\underline{x}}{\underline{y}} \otimes \frac{\underline{x}}{\underline{y}} - \frac{(\underline{y}-\underline{x})(1-\underline{x}\underline{y})}{\underline{y}(1-\underline{x})^2} \wedge \frac{\underline{x}(1-\underline{y})^2}{\underline{y}(1-\underline{x})^2} \otimes \frac{\underline{x}(1-\underline{y})^2}{\underline{y}(1-\underline{x})^2} = 0$$

in $(k[\varepsilon]_4^\times \wedge k[\varepsilon]_4^\times) \otimes k[\varepsilon]_4^\times$, tensored with $\mathbb{Q}$, which can be seen by direct computation.

Now note that if $f(u, v) \in k(u, v)$ is any rational function then $f(\underline{x}, \underline{y}) \in k[\varepsilon]_4$ is a lifting of $f(x, y) \in k[\varepsilon]_2$ and therefore

$$\begin{aligned} & li_3(f(x,y)) \\ & \quad = \frac{1}{3} \sum_{(i_1,i_2)\in I_3} c(i_1,i_2)(\pi^{(4)}(i_1,1,i_2) \circ (\Lambda^2 \log \otimes \log)) \\ & \qquad \times((1-f(\underline{x},\underline{y})) \wedge f(\underline{x},\underline{y}) \otimes f(\underline{x},\underline{y})) \end{aligned}$$

and the statement follows.

The following is the additive analog of Gangl's 9-term functional equation for the tetralogarithm [23, Example 4, p. 396].

Theorem 5 (9-term relation for the additive tetralogarithm) *For* $x \in k[\varepsilon]_2^\flat$ *such that* $1-x+x^2 \in k[\varepsilon]_2^\times$, *we have*

$$\begin{aligned} & 2li_4(x(1-x)) + 2li_4\left(\frac{-x}{(x-1)^2}\right) + 2li_4\left(\frac{x-1}{x^2}\right) \\ & -3li_4\left(\frac{1}{1-x+x^2}\right) - 3li_4\left(\frac{(1-x)^2}{1-x+x^2}\right) \\ & -3li_4\left(\frac{x^2}{1-x+x^2}\right) - 6li_4\left(\frac{1-x+x^2}{x(x-1)}\right) \\ & -6li_4\left(\frac{1-x+x^2}{x}\right) - 6li_4\left(\frac{1-x+x^2}{1-x}\right) = 0. \end{aligned}$$

Proof Exactly as in the proof of Theorem 4, one chooses a lifting $\underline{x}$ of x to $k[\varepsilon]_5$. Then the statement follows from the corresponding indentity for $\underline{x}$ in $(\Lambda^2 k[\varepsilon]_5^\times \otimes (k[\varepsilon]_5^\times)^{\otimes 2})_{\mathbb{Q}}$ and Definition 1.

4.2 Functional equations for the additive dilogarithm of higher modulus

4.2.1 Abel's functional equation for additive dilogarithm

Let us first recall the definition of the additive dilogarithm from [21]. Fix a modulus $2 \leq r$ and a weight $r+1 \leq w \leq 2r-1$. Then the additive dilogarithm of modulus r and weight w is a function

$$li_{2,r,w} : k[\varepsilon]_r^\flat \to k,$$

for which an explicit formula is given in Proposition 2.2.3 in [21]. Below we will need the following description of $li_{2,r,w}$, given in Proposition 2.1.2 in [21].

Following the notation in the beginning of Sect. 2, the map

$$\sum_{1 \leq j \leq w-r} j\pi^{(w)}(w-j, j) \circ \Delta_2^{(w)} : k[\varepsilon]_w^\flat \to k$$

descends to give $li_{2,r,w} : k[\varepsilon]_r^\flat \to k$. This definition is the same as that of the additive polylogarithm defined above for $n=2$, when $r=2$ and $w=3$.

In the proof of Proposition 2.2.2 in [21], it was shown that the $li_{2,r,w}$ satisfy Abel's 5-term functional equation: for $x, y \in k[\varepsilon]_r^\flat$ such that $x/y \in k[\varepsilon]_r^\flat$,

$$li_{2,r,w}(x) - li_{2,r,w}(y) + li_{2,r,w}\left(\frac{y}{x}\right) - li_{2,r,w}\left(\frac{1-x^{-1}}{1-y^{-1}}\right) + li_{2,r,w}\left(\frac{1-x}{1-y}\right) = 0.$$

4.2.2 Wojtkowiak's functional equation for additive dilogarithm

In this section we assume that k is an algebraically closed field of characteristic 0. We denote the reduction mod (ε) of an element $a \in k[\varepsilon]_r$ by $\overline{a}$.

Suppose

$$f(z) = \lambda \prod_{1 \leq i \leq q} (z - a_i) / \prod_{1 \leq j \leq t} (z - b_j) \tag{4.2.1}$$

with $a_i, b_j \in k[\varepsilon]_r$ such that $\overline{a}_i \neq \overline{b}_j$ for $1 \leq i \leq q$ and $1 \leq j \leq t$ and $\lambda \in k^\times$ and

$$1 - f(z) = \mu \prod_{1 \leq k \leq l} (z - c_k) / \prod_{1 \leq j \leq t} (z - b_j). \tag{4.2.2}$$

with $\overline{c}_k \neq \overline{b}_j$ $1 \leq k \leq l$ and $1 \leq j \leq t$, and $\mu \in k^\times$. And moreover, suppose that the decompositions (4.2.1) and (4.2.2) can be lifted to similar decompositions over $k[\varepsilon]_w$: there are $\tilde{a}_i, \tilde{b}_j, \tilde{c}_k \in k[\varepsilon]_w$ lifting a_i, b_j, c_k, for $1 \leq i \leq q, 1 \leq j \leq t, 1 \leq k \leq l$ such that

$$\lambda \prod_{1 \leq i \leq q} (z - \tilde{a}_i) / \prod_{1 \leq j \leq t} (z - \tilde{b}_j) + \mu \prod_{1 \leq k \leq l} (z - \tilde{c}_k) / \prod_{1 \leq j \leq t} (z - \tilde{b}_j) = 1.$$

In this case we let $\tilde{f}(z)$ and $1-\tilde{f}(z)$ denote the first and second sum on the left in the last expression. Note that such decompositions, if they exist, need not be unique.

Remark 7 Note that such decompositions exist, and are unique if for example $f(z) \in k(z)^{\times}$ or if $f(z) \in k[\varepsilon]_r[z]$, with leading coefficient in k and such that $\overline{f}(z)$ and $1-\overline{f}(z)$ have no multiple roots in k. The existence and uniqueness of the decomposition for the latter follows immediately from Hensel's lemma.

The following is the exact additive analog of Wojtkowiak's Theorem A in [22]. Note that in our case the terms coming from the product of lower weight terms disappear.

Theorem 6 *With the assumptions as above, we have the functional equation*

$$\begin{aligned} & li_{2,r,w}(f(\alpha)) \\ & = \sum_{i,k} li_{2,r,w}\left(\frac{\alpha-a_i}{c_k-a_i}\right) - \sum_{j,k} li_{2,r,w}\left(\frac{\alpha-b_j}{c_k-b_j}\right) - \sum_{i,j} li_{2,r,w}\left(\frac{\alpha-a_i}{b_j-a_i}\right), \end{aligned}$$

for any $\alpha \in k[\varepsilon]_r$ sufficiently generic that all the terms above make sense.

Proof In the abelian group $\Lambda^2 k[\varepsilon]_w^{\times\circ}$ we have the following identity:

$$\begin{aligned} & \sum_{i,k}\left(1-\frac{\tilde{\alpha}-\tilde{a}_i}{\tilde{c}_k-\tilde{a}_i}\right)\wedge\frac{\tilde{\alpha}-\tilde{a}_i}{\tilde{c}_k-\tilde{a}_i} - \sum_{j,k}\left(1-\frac{\tilde{\alpha}-\tilde{b}_j}{\tilde{c}_k-\tilde{b}_j}\right)\wedge\frac{\tilde{\alpha}-\tilde{b}_j}{\tilde{c}_k-\tilde{b}_j} \\ & \quad - \sum_{i,j}\left(1-\frac{\tilde{\alpha}-\tilde{a}_i}{\tilde{b}_j-\tilde{a}_i}\right)\wedge\frac{\tilde{\alpha}-\tilde{a}_i}{\tilde{b}_j-\tilde{a}_i} \\ & = \sum_{i,k}((\tilde{\alpha}-\tilde{c}_k)\wedge(\tilde{\alpha}-\tilde{a}_i) - (\tilde{c}_k-\tilde{a}_i)\wedge(\tilde{\alpha}-\tilde{a}_i) + (\tilde{c}_k-\tilde{a}_i)\wedge(\tilde{\alpha}-\tilde{c}_k)) \\ & \quad - \sum_{j,k}((\tilde{\alpha}-\tilde{c}_k)\wedge(\tilde{\alpha}-\tilde{b}_j) - (\tilde{c}_k-\tilde{b}_j)\wedge(\tilde{\alpha}-\tilde{b}_j) + (\tilde{c}_k-\tilde{b}_j)\wedge(\tilde{\alpha}-\tilde{c}_k)) \\ & \quad - \sum_{i,j}((\tilde{\alpha}-\tilde{b}_j)\wedge(\tilde{\alpha}-\tilde{a}_i) - (\tilde{b}_j-\tilde{a}_i)\wedge(\tilde{\alpha}-\tilde{a}_i) + (\tilde{b}_j-\tilde{a}_i)\wedge(\tilde{\alpha}-\tilde{b}_j)) \\ & = \frac{\prod_k(\tilde{\alpha}-\tilde{c}_k)}{\prod_j(\tilde{\alpha}-\tilde{b}_j)}\wedge\frac{\prod_i(\tilde{\alpha}-\tilde{a}_i)}{\prod_j(\tilde{\alpha}-\tilde{b}_j)} - \sum_i \frac{\prod_k(\tilde{a}_i-\tilde{c}_k)}{\prod_j(\tilde{a}_i-\tilde{b}_j)}\wedge(\tilde{\alpha}-\tilde{a}_i) \\ & \quad + \sum_k \frac{\prod_i(\tilde{c}_k-\tilde{a}_i)}{\prod_j(\tilde{c}_k-\tilde{b}_j)}\wedge(\tilde{\alpha}-\tilde{c}_k) + \sum_j \frac{\prod_k(\tilde{b}_j-\tilde{c}_k)}{\prod_i(\tilde{b}_j-\tilde{a}_i)}\wedge(\tilde{\alpha}-\tilde{b}_j) \\ & = (1-\tilde{f}(\tilde{\alpha}))\wedge\tilde{f}(\tilde{\alpha}) - \sum_i(1-\tilde{f}(\tilde{a}_i))\wedge(\tilde{\alpha}-\tilde{a}_i) \\ & \quad + \sum_k \tilde{f}(\tilde{c}_k)\wedge(\tilde{\alpha}-\tilde{c}_k) + \sum_j(-\lambda/\mu)\wedge(\tilde{\alpha}-\tilde{b}_j) = (1-\tilde{f}(\tilde{\alpha}))\wedge\tilde{f}(\tilde{\alpha}) \end{aligned}$$

since $\tilde{f}(\tilde{a}_i) = 0,\ \tilde{f}(\tilde{c}_k) = 1$ and

$$\prod_j (z - \tilde{b}_j) = \mu \prod_k (z - \tilde{c}_k) + \lambda \prod_i (z - \tilde{a}_i).$$

Applying the map

$$\sum_{1 \le j \le w-r} j\pi^{(w)}(w-j, j) \circ \Lambda^2 \log$$

to both sides of the equation gives the statement.

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References

1. Beilinson, A.: Height pairing between algebraic cycles. Contemp. Math. **67**, 1–24 (1987)
2. Beilinson, A., Deligne, P.: Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs. In: Jannsen, U., Kleiman, S., Serre, J.-P. (eds.) Motives, AMS, Proceeding of Symposium in Pure Math., vol. 55, Part 2 (1994)
3. Beilinson, A., Goncharov, A., Schechtman, V., Varchenko, A.: Aomoto dilogarithms, mixed Hodge structures and motivic cohomology of pairs of triangles on the plane. Groth. Festschrift, vol. 1, pp. 135–172. Prog. Math. (1990)
4. Bloch, S., Esnault, H.: The additive dilogarithm. Doc. Math. Extra volume in honor of Kazuya Kato's fiftieth birthday, pp. 131–155
5. Bloch, S., Esnault, H.: An additive version of higher Chow groups. Ann. Sci. École Norm. Sup. **36**(3), 463–477 (2003)
6. Cathelineau, J.-L.: λ-structures in algebraic K-theory and cyclic homology. K-Theory **4**, 591–606 (1990/1991)
7. Deligne, P.: Le groupe fondamental de la droite projective moins trois points. Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), pp. 79–297, Math. Sci. Res. Inst. Publ., 16. Springer, New York (1989)
8. Deligne, P., Goncharov, A.: Groupes fondamentaux motiviques de Tate mixte. Ann. Sci. École Norm. Sup. **38**(1), 1–56 (2005)
9. Dupont, J.: On polylogarithms. Nagoya Math. J. **114**, 1–20 (1989)
10. Gangl, H.: Functional equations of higher logarithms. Selecta Math. **9**, 361–379 (2003)
11. Goncharov, A.: Volumes of hyperbolic manifolds and mixed Tate motives. J. Am. Math. Soc. **12**(2), 569–618 (1999)
12. Goncharov, A.: Geometry of configurations, polylogarithms, and motivic cohomology. Adv. Math. **114**, 197–318 (1995)
13. Goncharov, A.: Euclidean scissor congruence groups and mixed Tate motives over dual numbers. Math. Res. Lett. **11**, 771–784 (2004)
14. Goodwillie, T.G.: Relative algebraic K-theory and cyclic homology. Ann. Math. **124**(2), 347–402 (1986)
15. Hida, H.: Elementary theory of L-functions and Eisenstein Series. London Math. Soc. Student Texts 26. Cambridge University Press, Cambridge (1993)
16. Krishna, A., Levine, M.: Additive higher Chow groups of schemes. J. Reine Angew. Math. **619**, 75–140 (2008)

17. Ramakrishnan, D.: Regulators, algebraic cycles and values of L-functions. Contemp. Math. **83**, 183–310
18. Park, J.: Algebraic cycles and additive dilogarithm. Int. Math. Res. Not., 18 (2007)
19. Rülling, K.: The generalized de Rham–Witt complex over a field is a complex of zero cycles. J. Alg. Geom. **16**, 109–169 (2007)
20. Sydler, J.P.: Conditions nécessaires et suffisantes pour l'équivalence des polyèdres de l'espace euclidien à trois dimensions. Comment. Math. Helv. **40**, 43–80 (1965)
21. Ünver, S.: On the additive dilogarithm. Algebra Number Theory **3**(1), 1–34 (2009)
22. Wojtkowiak, Z.: The basic structure of polylogarithmic functional equations. In: Lewin, L. (ed.) Structural properties of polylogarithms (Mathematical surveys and monographs, AMS, vol. 37), pp. 205–231 (1991)
23. Zagier, D.: Special values and functional equations of polylogarithms (Appendix A). In: Lewin, L. (ed.) Structural properties of polylogarithms (Mathematical surveys and monographs, AMS, vol. 37), pp. 377–400 (1991)
24. Zagier, D.: Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields. Arithmetic Algebraic Geometry. In: van der Geer, G., Oort, F., Steenbrick, J. (eds.) Prog. Math., vol. 89, pp. 391–430. Birkhauser, Boston (1991)
25. Zhao, J.: Remarks on a Hopf algebra for defining motivic cohomology. Duke Math. J. **103**(3), 445–458 (2000)
26. Zhao, J.: Motivic cohomology of pairs of simplices. Proc. Lond. Math. Soc. **88**(3), 313–354 (2004)