ON THE ADDITIVE DILOGARITHM

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Abstract. Let k be a field of characteristic zero, and $k[\varepsilon]_n := k[\varepsilon]/(\varepsilon^n)$. We construct an additive dilogarithm $Li_{2,n} : B_2(k[\varepsilon]_n) \to k^{\oplus(n-1)}$, where B_2 is the Bloch group which is crucial in studying weight two motivic cohomology. We use this construction to show that the Bloch complex of $k[\varepsilon]_n$ has cohomology groups expressed in terms of the K-groups $K_.(k[\varepsilon]_n)$ as expected. Finally we compare this construction to the construction of the additive dilogarithm by Bloch and Esnault [5] defined on the complex $T_n\mathbb{Q}(2)(k)$.

1. introduction

1.1. For any scheme S one expects a category $\mathcal{M}_S$ of motivic (perverse) sheaves over S, which should be an abelian tensor category that satisfies all the formalism of mixed sheaf theory (5.10, [1]). The Tate sheaves $\mathbb{Z}_{\mathcal{M}}(n)$ should play a special role. Namely, letting

$$H^i(S, \mathbb{Z}_{\mathcal{M}}(n)) := \mathrm{Ext}^i_{\mathcal{M}_S}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n)),$$

the Chern character map

$$K_{2n-i}(S)^{(n)}_{\mathbb{Q}} \to H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \tag{1.1.1}$$

from the n-th graded piece of Quillen's K-theory tensored with $\mathbb{Q}$, defined as the k^n-eigenspace for the k-th Adams operator (*Remark* 3.1.2), to motivic cohomology of weight n should be an isomorphism when S is regular (5.10, [1]). Since $\mathcal{M}_S$ is to have realizations corresponding to various cohomology theories, the regulator map:

$$K_{2n-i}(S)^{(n)}_{\mathbb{Q}} \to H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \to H^i_*(S, \mathbb{Q}_*(n)),$$

where $*$ is the relevant realization, gives arithmetically important information.

The complexes $\underline{\mathrm{RHom}}_{Zar}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))$ of sheaves on the Zariski site should have the property that $H^i(S_{Zar}, \underline{\mathrm{RHom}}_{Zar}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))) = H^i(S, \mathbb{Z}_{\mathcal{M}}(n))$. Hence the motivic cohomology of S of weight n could be computed as the hypercohomology of a complex of sheaves on S_{Zar}.

Recently progress has been made in motivic cohomology by Voevodsky. If $S = \mathrm{Spec}(k)$, where k is a field of characteristic zero, Voevodsky constucts a triangulated category $DM^{eff,-}_{Nis}(k)$ (Ch. 14, [19]) and a complex of sheaves $\underline{\mathbb{Z}}(n)$ on the big Zariski site over k, which should be isomorphic to the hypothetical $\underline{\mathrm{RHom}}_{Zar}(\mathbb{Z}_{\mathcal{M}}(0), \mathbb{Z}_{\mathcal{M}}(n))$ above, such that for any smooth scheme X over k,

$$H^i(X_{Zar}, \underline{\mathbb{Z}}(n)) \simeq \mathrm{Ext}^i_{DM^{eff,-}_{Nis}}(M(X), \underline{\mathbb{Z}}(n))$$

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(14.16, [19]), where $M(X)$ is the motive of X (Definition 14.1, [19]). Since $\underline{\mathbb{Z}}(n)$ and Bloch's complex of algebraic cycles of codimension n are isomorphic (Ch. 19, [19]), the Bloch-Grothendieck-Riemann-Roch theorem [4] implies that the hypercohomology of $\underline{\mathbb{Q}}(n)$ on X_{Zar} is expressed in terms of the K-groups of X as above:

$$K_{2n-i}(X)^{(n)}_{\mathbb{Q}} \simeq H^i(X_{Zar}, \underline{\mathbb{Q}}(n)). \tag{1.1.2}$$

In order to study the motivic cohomology of S, it would be sufficient to restrict to a subcategory of $\mathcal{M}_S$. Let $\mathcal{MTM}_S$ denote the smallest full-subcategory of $\mathcal{M}_S$ that contains the Tate motives and is closed under extensions. Then $H^i(S, \mathbb{Q}_{\mathcal{M}}(n)) \simeq \mathrm{Ext}^i_{\mathcal{M}_S}(\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n)) = \mathrm{Ext}^i_{\mathcal{MTM}_S}((\mathbb{Q}_{\mathcal{M}}(0), \mathbb{Q}_{\mathcal{M}}(n))$. The category $\mathcal{MTM}_S$ would be simpler than $\mathcal{M}_S$. In fact for $S = \mathrm{Spec}(k)$, where k is a number field, a candidate for $\mathcal{MTM}_S$ has been constructed as a tannakian category in [8], using $DM^{eff,-}_{Nis}$.

It is natural to expect that $\mathcal{MTM}_S$ can be constructed by using only the relative cohomologies of hyperplane arrangements and in turn that motivic cohomology can be computed using complexes of *linear* algebraic objects [2], rather than all algebraic cycles. There are special degenerate configurations of hyperplanes, called the polylogarithmic configurations ([2], [10]) which act as building blocks for all configurations and thus play a special role in describing motivic cohomology.

Using the relations satisfied by the polylogarithmic configurations Goncharov defines a complex $\Gamma_k(n)_{\mathbb{Q}}$:

$$\mathcal{B}_n(k) \to \mathcal{B}_{n-1}(k) \otimes k^{\times}_{\mathbb{Q}} \to \mathcal{B}_{n-2}(k) \otimes \Lambda^2 k^{\times}_{\mathbb{Q}} \to \cdots \to \mathcal{B}_2(k) \otimes \Lambda^{n-2} k^{\times}_{\mathbb{Q}} \to \Lambda^n k^{\times}_{\mathbb{Q}}$$

which he conjectures to compute the motivic cohomology of weight n (Conjecture A, Conjecture 1.17, [10]).

If $k = \mathbb{C}$, integration over the polylogarithmic configurations can be used to define a map $\mathbb{Q}[\mathbb{P}^1(\mathbb{C})] \to \mathbb{R}$, the single valued real analytic version of the n-th polylogarithmic function (1.0, [10]), which factors through the projection $\mathbb{Q}[\mathbb{P}^1(\mathbb{C})] \to \mathcal{B}_n(\mathbb{C})$ (1.0, [10]) to give

$$\mathcal{L}_n : \mathcal{B}_n(\mathbb{C}) \to \mathbb{R}$$

the n-th polylogarithm that is expected to induce the regulator $K_{2n-1}(\mathbb{C})^{(n)}_{\mathbb{Q}} \simeq H^1(\mathrm{Spec}(\mathbb{C}), \mathbb{Q}_{\mathcal{M}}(n)) \to \mathcal{B}_n(\mathbb{C}) \to \mathbb{R}$ (p. 224, [10]).

For a general field k, one cannot expect a polylogarithm on $\mathcal{B}_n(k)$. However through his interpretation of hyperbolic scissor congruence groups in terms of mixed Tate motives, Goncharov expected that there should be an infinitesimal polylogarithmic function which acts like a regulator map on $K_{2n-1}(k[\varepsilon]_2, (\varepsilon))^{(n)}$, for any field k of characteristic 0 (pp. 616-617, [9]; [11]), where $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$. In our notation, assuming the existence of mixed Tate motives and the complex Γ_n over the dual numbers, this is translated into the existence of a map

$$\mathcal{B}_n(k[\varepsilon]_2)/\mathcal{B}_n(k) \to k \tag{1.1.3}$$

such that when composed with $K_{2n-1}(k[\varepsilon]_2, (\varepsilon))^{(n)} \to \mathcal{B}_n(k[\varepsilon]_2)/\mathcal{B}_n(k)$, gives an isomorphism. The map (1.1.3) is to be an analog of both the volume map for euclidean scissor congruence groups and of polylogarithms.

In this paper we are interested in this question for weight two. Next we give details about this case.

1.2. Let A be an artinian local ring and I an ideal of A. In the rest of the paper, when we refer to weight two (rational) motivic cohomology of A relative to I, what we mean are the groups $K_3(A,I)^{(2)}_{\mathbb{Q}}$ and $K_2(A,I)^{(2)}_{\mathbb{Q}}$ and not to the Voevodsky motivic cohomology groups in §1.1, which were there only to motivate the main results of this paper. This common abuse of notions is partly justified by the expected Chern character isomorphism (1.1.1), which is known to be true when A is a field (1.1.2).

Let k be an algebraically closed field of characteristic 0, S the semi-local ring of rational functions on $\mathbb{A}^1_k$ that are regular on $\{0,1\}$, and J the Jacobson radical of S.

The first complex computing the weight two motivic cohomology is constructed by Bloch as follows. Localizing $\mathbb{A}^1_k$ away from 0 and 1 gives an exact sequence

$$0 \to K_3(k)^{(2)} \to K_2(S,J) \xrightarrow{\ \varphi\ } \bigoplus_{x\in k^\times\setminus\{1\}} k^\times \to K_2(k) \to 0,$$

(proof of 7.1, [18]; [3]), where φ is the tame symbol map. Let

$$B(k) := K_2(S,J)/im((1+J)\otimes k^\times),$$

the part of $K_2(S,J)$ that does not come from the products of weight 1 terms, then $(\bigoplus_{x\in k^\times\setminus\{1\}} k^\times)/\varphi((1+J)\otimes k^\times) = k^\times \otimes k^\times$ and the sequence

$$0 \to K_3(k)^{(2)}_{\mathbb{Q}} \to B(k)_{\mathbb{Q}} \to (k^\times\otimes k^\times)_{\mathbb{Q}} \to K_2(k)_{\mathbb{Q}} \to 0,$$

remains exact (proof of 7.1, [18]; [3]).

In complete analogy, Bloch and Esnault define a complex that computes the motivic cohomology of $k[t]_2$ relative to the ideal (t) as follows. Let R be the local ring of $\mathbb{A}^1_k$ at 0. Then localizing away from 0 on $\mathbb{A}^1$ gives the sequence

$$K_2(k[t],(t^2)) \to K_2(R,(t^2)) \xrightarrow{\ \varphi\ } \oplus_{x\in k^\times} k^\times \to K_1(k[t],(t^2)).$$

Letting $\mathcal{C}$ denote the subgroup generated by the symbols $\langle a,b\rangle \in K_2(R,(t^2))$ with $a\in(t^2)$ and $b\in k$, and $TB(k) := K_2(R,(t^2))/\mathcal{C}$, we have $k^\times\otimes k = (\oplus_{x\in k^\times} k^\times)/\varphi(\mathcal{C})$ and an exact sequence

$$0 \to K_2(k[t],(t^2))^{(2)} \to TB(k) \to k^\times\otimes k \to K_1(k[t],(t^2)) \to 0$$

(Prop. 2.1 and Cor. 2.5, [5]). Then we have $K_2(k[t],(t^2))^{(2)}_{\mathbb{Q}} \simeq K_3(k[t]_2,(t))^{(2)}_{\mathbb{Q}}$, and $K_1(k[t],(t^2)) \simeq K_2(k[t]_2,(t))$ (loc. cit.). Therefore the complex

$$TB(k) \to k^\times \otimes k$$

(tensored with $\mathbb{Q}$), really computes the motivic cohomology of $k[t]_2$ relative to (t). Moreover Bloch and Esnault define a dilogarithm map on $TB(k)$:

Theorem 1.2.1. *(Bloch-Esnault, Corollary 2.5,* [5] *) Let $\mathfrak{m}$ be the maximal ideal of R. There is a well-defined map $\rho : TB(k) \to \mathfrak{m}^3/\mathfrak{m}^4$ such that for $\langle a,b\rangle \in K_2(R,(t^2))$ with $a\in\mathfrak{m}^2, b\in R$,*

$$\rho(\langle a,b\rangle) = -a\cdot db,$$

and ρ induces an isomorphism $K_3(k[t],(t^2))^{(2)} \to \mathfrak{m}^3/\mathfrak{m}^4$ of abelian groups.

1.3. For k a field of characteristic zero there is another natural complex, which is of more geometric origin, and hence easier to relate to various definitions of categories of mixed Tate motives, that computes the weight two motivic cohomology groups of k.

Let A be an artinian local ring with residue field k. The Bloch group $B_2(A)$ (denoted by $\mathfrak{p}(A)$ in [23]) is the free abelian group generated by the symbols $[x]$, such that $x(1-x) \in A^\times$, modulo the subgroup generated by elements of the form

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)],$$

for all $x, y \in A^\times$ such that $(1-x)(1-y)(1-x/y) \in A^\times$. The map that sends $[x]$ to $x \wedge (1-x) \in \Lambda^2_{\mathbb{Z}} A^\times$ induces the two term complex $\gamma_A(2)$ which sits in $[1,2]$:

(1.3.1) $$\delta_A : B_2(A) \to \Lambda^2_{\mathbb{Z}} A^\times.$$

$\gamma_k(2)$ can be thought of as a more explicit version of $\Gamma_k(2)$. In fact, there is a natural map $\gamma_k(2)_{\mathbb{Q}} \to \Gamma_k(2)_{\mathbb{Q}}$ which is expected to be an isomorphism (Conj. 1.20, [10]), and there is an exact sequence [23]:

$$0 \to K_3(k)^{(2)}_{\mathbb{Q}} \to B_2(k)_{\mathbb{Q}} \to (\Lambda^2 k^\times)_{\mathbb{Q}} \to K_2(k)_{\mathbb{Q}} \to 0.$$

For $n \geq 2$, we are interested in the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$, where $\delta_{k[\varepsilon]_n}$ will be denoted by δ_n. We show that it has the expected cohomology:

Theorem 1.3.1. *For k a field of characteristic 0, there is an exact sequence:*

$$0 \to K_3(k[\varepsilon]_n)^{(2)}_{\mathbb{Q}} \longrightarrow B_2(k[\varepsilon]_n)_{\mathbb{Q}} \xrightarrow{\delta_n} (\Lambda^2 k[\varepsilon]_n^\times)_{\mathbb{Q}} \longrightarrow K_2(k[\varepsilon]_n)_{\mathbb{Q}} \to 0.$$

For $n = 2$ this theorem gives an affirmative answer to Problem 2.3 in [11].

While proving the previous theorem we construct an additive dilogarithm map on $B_2(k[\varepsilon]_n)$:

Theorem 1.3.2. *For every $n \geq 2$, there is a natural map*

$$Li_{2,n} : B_2(k[\varepsilon]_n) \to k^{\oplus(n-1)}$$

such that when composed with $K_3(k[\varepsilon]_n, (\varepsilon))^{(2)} \hookrightarrow B_2(k[\varepsilon]_n)$ it induces an isomorphism $K_3(k[\varepsilon]_n, (\varepsilon))^{(2)} \simeq k^{\oplus(n-1)}$ of abelian groups.

The advantage of defining a dilogarithm map on $B_2(k[\varepsilon]_n)$ is that this group is closely related to the linear algebra-geometric complexes of mixed Tate motives. More precisely, $Li_{2,n}$ immediately gives an analog of the volume map for a pair of triangles over $k[\varepsilon]_n$, as in [2]: all one needs to do is to take the image of the pair of triangles in $B_2(k[\varepsilon]_n)$ under the map in Proposition 3.7, [2] and then apply $Li_{2,n}$. In this context Theorem 1.3.1 and 1.3.2 imply that the class of a pair of triangles in $A_2(k[\varepsilon_n])/A_2(k)$ ([2]) is determined by its image in $\Lambda^2 k[\varepsilon_n]^\times/\Lambda^2 k^\times$ and its image under $Li_{2,n}$. This is a precise analog of Sydler's theorem on Hilbert's 3rd problem that the scissors congruence class of a three dimensional polyhedron is determined by its volume and its Dehn invariant (§2.7, [9]). We do not, however, pursue this application in the paper.

1.4. In order to compare $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$ with the complex of Bloch and Esnault, we show that the argument of Bloch and Esnault extends to define a complex $T_n\mathbb{Q}(2)(k)$:

$$T_nB(k) \to k^\times \otimes (\varepsilon \cdot k[\varepsilon]_n)$$

(for $n = 2$ this is the complex in **1.2**). Letting $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}} = \gamma_k(2)_{\mathbb{Q}} \oplus \gamma_{k[\varepsilon]_n}(2)^\circ_{\mathbb{Q}}$, we note that the cohomology groups of $\gamma_{k[\varepsilon]_n}(2)^\circ_{\mathbb{Q}}$ and $T_n\mathbb{Q}(2)(k)$ coincide. We define a subcomplex $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ of $\gamma_{k[\varepsilon]_n}(2)^\circ_{\mathbb{Q}}$ that has the same cohomology groups and a direct consequence of Theorem 1.2.1, Theorem 1.3.1, and Theorem 1.3.2 is that

Corollary 1.4.1. *For k an algebraically closed field of characteristic 0, the complexes $T_n\mathbb{Q}(2)(k)$ and $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ are isomorphic.*

1.5. The paper is organized as follows. In §2, we construct the additive dilogarithm, $Li_{2,n} : B_2(k[\varepsilon]_n) \to k^{\oplus(n-1)}$. Two results in §2 are useful in studying $Li_{2,n}$. On the one hand, $Li_{2,n}$ is explicitly described in Proposition 2.2.3 and Definition 2.2.4. On the other hand, $Li_{2,n}$ has a conceptual description. Namely, the image of an element in $B_2(k[\varepsilon]_n)$ under $Li_{2,n}$ is obtained by lifting that element to an arbitrary element in $B_2(k[\varepsilon]_{2n-1})$ then taking its image in $\Lambda^2 k[\varepsilon]^\times_{2n-1}$ under the map in (1.3.1), and finally choosing certain algebraic combinations of its coordinates in $\Lambda^2 k[\varepsilon]^\times_{2n-1}$ as in Proposition 2.1.2, 2.2.1 and 2.2.2. It is this flexibility in the choice of the lifting that is used in the computations in §4.

In this paper, rather than working with K-theory we work with cyclic homology most of the time. This is possible since $K_*(k[\varepsilon]_n) = K_*(k[\varepsilon]_n, (\varepsilon)) \oplus K_*(k)$ and by the theorem of Goodwillie [12], $HC_{*-1}(k[\varepsilon]_n, (\varepsilon)) \simeq K_*(k[\varepsilon]_n, (\varepsilon))$, where HC denotes cyclic homology with respect to $\mathbb{Q}$. Note that since we are working with $\mathbb{Q}$-coefficients, K-theory is nothing other than the primitive part of the rational homology of GL (Corollary 11.2.12, [16]).

From §3.1 to §3.6 we make Goodwillie's theorem explicit, following [16], by giving the description of a map from $HC_2(k[\varepsilon]_n, (\varepsilon))$ to $H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q})$. Then in §3.7 and §3.8, Suslin-Guin's stability theorem and a construction of Bloch, Suslin, Goncharov is used to construct a map $H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q}) \to ker(\delta_n)$. More details about §3 are given in §3.1. This explicit description will be needed in §4.

In §4, we prove Theorem 1.3.2. This is done by first using the description of $HC_2(k[\varepsilon]_n, (\varepsilon))$ given in [7] in §4.1.1, then constructing certain elements

$$\alpha_w \in HC_2(k[\varepsilon]_n, \varepsilon)^{(1)} \text{ for } n+1 \leq w \leq 2n-1,$$

and chasing the images of these elements under the maps described in §2 and §3. The proof also shows that $\{\alpha_w\}_{n+1 \leq w \leq 2n-1}$ form a basis for $HC_2(k[\varepsilon]_n, \varepsilon)^{(1)}$.

In §5, using [23] and [14], and §4 we show that the infinitesimal part of $ker(\delta)$ is canonically isomorphic to $HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$. From this Theorem 1.3.1 follows.

In §6, we first define a subcomplex $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ of $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$. Then we extend the construction of Bloch and Esnault to higher moduli and finally prove Corollary 1.4.1 which compares the two constructions.

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Remarks. (i) We would like to mention the work of J. Park [20] where he gives an additive Chow theoretic description of the additive dilogarithm of Bloch and Esnault and the work of K. Rülling [21] where he proves that the complex of additive Chow groups with modulus (not necessarily of 2) has the expected cohomology groups on the level of zero cycles.

(ii) There are many problems that are left unanswered in this note. The most important of these is the construction of additive polylogarithms for higher weights. We have made this construction, but proving that the complex has the right cohomology groups is still unanswered at the time of writing. The question of what happens in characteristic p; and the question of comparing our construction to the work of Park and Rülling will be addressed in another paper.

2. ADDITIVE DILOGARITHM

Notation 2.0.1. Let k be a field of characteristic zero. An abelian group A endowed with a group homomorphism $k^\times \to \mathrm{Aut}_{ab}(A)$ is said to be a $k^\times$-abelian group; we denote the action of $\lambda \in k^\times$ on $a \in A$ by $\lambda \times a$. If $f : A \to k$ is an additive map that satisfies $f(\lambda \times a) = \lambda^w \cdot f(a)$ for all $\lambda \in k^\times$ and $a \in A$ then we say that f is of $k^\times$-weight w.

If V is a k-module with a $k^\times$-action that is k-linear, i.e. defined by a homomorphism $k^\times \to \mathrm{Aut}_{k-mod}(V)$ then we let $V_{\langle w \rangle} := \{v \in V | \lambda \times v = \lambda^w \cdot v, \text{ for all } \lambda \in k^\times\}$, be the subspace of elements of V of weight w.

Let $k[\varepsilon]_m := k[\varepsilon]/(\varepsilon^m)$, $V_m := k[\varepsilon]_m^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, and $B_2(k[\varepsilon]_m)$ as in §1.3.

For an object A defined over $k[\varepsilon]_m$ we denote by A° its infinitesimal part, e.g. $B_2(k[\varepsilon]_m) = B_2(k) \oplus B_2(k[\varepsilon]_m)^\circ$, $k[\varepsilon]_m^\circ = \varepsilon \cdot k[\varepsilon]_m$, $V_m^\circ = 1 + \varepsilon \cdot k[\varepsilon]_m$ etc. When the context requires it we, for example, write $K_*(k[\varepsilon]_m)^\circ$ instead of $K_*(k[\varepsilon]_m, (\varepsilon))$. Finally, since in what follows the infinitesimal part A° of an object A is canonically a direct summand of A, we never mention the natural maps $A^\circ \to A$ and $A \to A^\circ$, and take other liberties of this kind.

Note that the exponential map gives an isomorphism $k[\varepsilon]_m^\circ \simeq V_m^\circ$, which endows V_m° with a k-space structure. For $\lambda \in k^\times$, the k-algebra map that sends ε to $\lambda \cdot \varepsilon$ defines an action of $k^\times$ on $k[\varepsilon]_m$ and V_m°. Denote the weight i subspace of V_m° under this action by $V_{m,\langle i \rangle}$, i.e. $V_{m,\langle i \rangle} = \{v \in V_m^\circ | \lambda \times v = \lambda^i \cdot v, \text{ for all } \lambda \in k^\times\} = \{\exp(a \cdot \varepsilon^i) | a \in k\}$. Then $V_m^\circ = \oplus_{1 \le i \le m-1} V_{m,\langle i \rangle}$. Since it simplifies the notation we also put $V_{m,\langle 0 \rangle} := k^\times \otimes \mathbb{Q}$.

Let $k[\varepsilon]_m^{\times\times} \subseteq k[\varepsilon]_m$ denote the set of exceptional units, i.e. those $a \in k[\varepsilon]_m^\times$ such that $1 - a \in k[\varepsilon]^\times$.

Let $\delta : \mathbb{Q}[k[\varepsilon]_m^{\times\times}] \to \Lambda^2 V_m$ be the map that sends $x \in k[\varepsilon]_m^{\times\times}$ to $x \wedge (1-x)$. If we want to emphasize that we are working over $k[\varepsilon]_m$, we will use the notation δ_m instead of δ. The map on $B_2(k[\varepsilon]_m)$ induced by δ_m is denoted by the same letter (cf. (1.3.1)).

2.1. Construction of li_2. In this section we collect the combinatorial arguments in the construction of the additive dilogarithm over $k[\varepsilon]_n$. The crucial step is the statement that $S_k(m,n)_{\langle w \rangle}$ is one dimensional in Proposition 2.1.2. This implies that if one thinks that the additive dilogarithm on $k[\varepsilon]_n$ should be constructed by first lifting to $k[\varepsilon]_{2n-1}$ then using δ then there is essentially one way to define it. That this is the right definition is justified in the next section.

Definition 2.1.1. Let $n, m \in \mathbb{N}$ such that $2 \leq n \leq m$. Let

$$\alpha_{m,n} : \mathbb{Q}[k[\varepsilon]_m^{\times\times}] \to \Lambda^2 V_m$$

denote the map that sends $\gamma \in k[\varepsilon]_m^{\times\times}$ to $\delta(\gamma) - \delta(\gamma|n)$, where $\gamma|n$ is the truncation of γ to the sum of first n powers of ε, i.e. if $\gamma = a_0 + a_1 \cdot \varepsilon + \cdots + a_{m-1} \cdot \varepsilon^{m-1}$ then $\gamma|n = a_0 + a_1 \cdot \varepsilon + \cdots + a_{n-1} \cdot \varepsilon^{n-1}$.

Let $V(m,n)$ denote

$$\bigoplus_{\substack{0<i\leq n-1 \\ n\leq j\leq m-1}} \Big(V_{m,\langle i\rangle} \otimes V_{m,\langle j\rangle} \Big) \subseteq \Lambda^2 V_m,$$

which we also consider as a quotient of $\Lambda^2 V_m$ via the direct sum decomposition

$$\Lambda^2 V_m = \oplus_{0\leq i<j<m} \Big(V_{m,\langle i\rangle} \otimes V_{m,\langle j\rangle} \Big) \oplus \Big(\oplus_{0\leq i<m} \Lambda^2 V_{m,\langle i\rangle} \Big). \tag{2.1.1}$$

Finally let $V_k(m,n)$ denote the quotient

$$\bigoplus_{\substack{0<i\leq n-1 \\ n\leq j\leq m-1}} \Big(V_{m,\langle i\rangle} \otimes_k V_{m,\langle j\rangle} \Big)$$

of $V(m,n)$, $p(m,n) : \Lambda^2 V_m \to V_k(m,n)$ the canonical projection, $S_k(m,n)$, the $k^\times$-abelian group $V_k(m,n)/im(p(m,n) \circ \alpha_{m,n})$ and $S_k(m,n)_{\langle i\rangle}$ the image of $V_k(m,n)_{\langle i\rangle}$ in $S_k(m,n)$. This notation is justified by noting that $S_k(m,n)$ has a natural k-module structure induced from that of $V_k(m,n)$ such that its weight i subspace is equal to $S_k(m,n)_{\langle i\rangle}$ and $S_k(m,n) = \oplus_{0<i} S_k(m,n)_{\langle i\rangle}$.

For $0 < i < j < m$ let $p_{i,j} : \Lambda^2 V_m \to V_{m,\langle i\rangle} \otimes V_{m,\langle j\rangle}$ denote the projection determined by the decomposition (2.1.1). Then

$$l_{i,j} : \Lambda^2 V_m \to k$$

is defined by letting for any $\alpha \in \Lambda^2 V_m$, $(\log \otimes \log)(p_{i,j}(\alpha)) = l_{i,j}(\alpha) \cdot (\varepsilon^i \otimes \varepsilon^j)$ in $k[\varepsilon]_m \otimes_k k[\varepsilon]_m$.

Proposition 2.1.2. *Let $n, m, w \in \mathbb{N}$ such that $2 \leq n < w \leq min(2n-1, m)$. Then $S_k(m,n)_{\langle w\rangle}$ is a one dimensional k-module. The unique linear functional*

$$li_{2,(m,n),w} : S_k(m,n)_{\langle w\rangle} \to k$$

such that $li_{2,(m,n),w}(\exp(\varepsilon) \otimes \exp(\varepsilon^{w-1})) = 1$ is given by

$$li_{2,(m,n),w} = \sum_{1\leq j\leq w-n} j \cdot l_{j,w-j}.$$

Proof. Let $li_{2,(m,n),w}$ denote the map from $\Lambda^2 V_m$ to k given by the formula

$$li_{2,(m,n),w} = \sum_{1\leq j\leq w-n} j \cdot l_{j,w-j}.$$

We would like to see that $li_{2,(m,n),w} \circ \alpha_{m,n} = 0$. Fix $x := s + s(1-s)a_1\varepsilon + \cdots + s(1-s)a_{m-1}\varepsilon^{m-1} \in k[\varepsilon]_m^{\times\times}$.

Let $A_m := \{1, \cdots, m-1\}$ and let $(A_m)^{\times\alpha}$ denote the cartesian product of A_m with itself α-times. Define $\mathfrak{p} : (A_m)^{\times\alpha} \to k$ by $\mathfrak{p}(i_1, \cdots, i_\alpha) := a_{i_1} \cdot a_{i_2} \cdots a_{i_\alpha}$, and $\mathfrak{w} : (A_m)^{\times\alpha} \to \mathbb{N}$ by $\mathfrak{w}(i_1, \cdots, i_\alpha) := i_1 + i_2 + \cdots + i_\alpha$. Note that even though $\mathfrak{p}$ depends on x, we suppress it from the notation. In order to simplify the notation let $A(\alpha) := (A_m)^{\times\alpha}$ and $B(\alpha) := (A_m)^{\times\alpha} \setminus (A_n)^{\times\alpha}$.

If $1 \leq \alpha, \beta \leq w$ let

$$C(\alpha, \beta) := \{(a, b) | a \in A(\alpha),\ b \in B(\beta),\ \text{ such that } \mathfrak{w}(a) + \mathfrak{w}(b) = w\}.$$

Let the permutation group on $\alpha + \beta$ letters $S_{\alpha+\beta}$ act on $A(\alpha) \times A(\beta)$ by permuting the coordinates. On $C(\alpha, \beta) \subseteq A(\alpha) \times A(\beta)$ consider the following equivalence relation. If $(a, b), (c, d) \in C(\alpha, \beta)$ then (a, b) and (c, d) are equivalent if there exists a permutation $\sigma \in S_{\alpha+\beta}$ such that $(a, b)^\sigma = (c, d)$. Denote the equivalence class of (a, b) by $[(a, b)]$ and the set of all equivalence classes by $\mathcal{S}(\alpha, \beta)$. Let $\mathfrak{p}([a, b]) = \mathfrak{p}(a) \cdot \mathfrak{p}(b)$.

Assume from now on that $\alpha + \beta \leq w$. Note that, since $w \leq 2n - 1$ any element $(a, b) \in C(\alpha, \beta)$ has a unique coordinate which is greater than or equal to n. Denote this coordinate by $e(a, b)$. Denote by $(a, b)_0$ the element of $C(\alpha, \beta)$ obtained by interchanging the last coordinate of (a, b) with the coordinate containing $e(a, b)$.

Then we define a map

$$\iota : C(\alpha, \beta) \to C(\beta, \alpha)$$

as follows. Let $(a, b) \in C(\alpha, \beta)$ then $\iota(a, b) \in C(\beta, \alpha)$ is the element $(a, b)_0$, where we think of both $C(\alpha, \beta)$ and $C(\beta, \alpha)$ as subsets of $A(\alpha) \times A(\beta) \simeq A(\alpha + \beta) \simeq A(\beta) \times A(\alpha)$. This passes to equivalence classes and gives a map $\mathcal{S}(\alpha, \beta) \to \mathcal{S}(\beta, \alpha)$ that we continue to denote by ι. Note that $\iota^2 = 1$, and if $G \in \mathcal{S}(\alpha, \beta)$, then $\mathfrak{p}(\iota(G)) = \mathfrak{p}(G)$, and

$$\sum_{(a,b) \in G} \mathfrak{w}(a) = \sum_{(c,d) \in \iota(G)} \mathfrak{w}(c).$$

Letting

$$z = a_1 \varepsilon + a_2 \varepsilon^2 + \cdots + a_{m-1} \varepsilon^{m-1}$$

we have

$$x = s(1 + (1 - s)z) \ \text{ and } \ 1 - x = (1 - s)(1 - sz).$$

Computing in $k[\varepsilon]_m$, this gives

$$\log(x/s) = -\sum_{\ell=1}^{m-1} \frac{(s-1)^\ell z^\ell}{\ell} \ \text{ and } \ \log((1-x)/(1-s)) = -\sum_{\ell=1}^{m-1} \frac{s^\ell z^\ell}{\ell}.$$

Since

$$z^\alpha = \sum_{u \in A(\alpha)} \mathfrak{p}(u) \varepsilon^{\mathfrak{w}(u)},$$

we have

$$\log(x/s) = -\sum_{\ell=1}^{m-1} \frac{(s-1)^\ell}{\ell} \sum_{u \in A(\ell)} \mathfrak{p}(u) \varepsilon^{\mathfrak{w}(u)}$$

and

$$\log((1-x)/(1-s)) = -\sum_{\ell=1}^{m-1} \frac{s^\ell}{\ell} \sum_{u \in A(\ell)} \mathfrak{p}(u) \varepsilon^{\mathfrak{w}(u)}.$$

Then $li_{2,(m,n),w}(\alpha_{m,n}(x))$

$$= \sum_{\substack{1 \leq \alpha \leq w \\ 1 \leq \beta \leq w}} \sum_{\substack{a \in A(\alpha) \\ b \in B(\beta) \\ \mathfrak{w}(a) + \mathfrak{w}(b) = w}} \frac{\mathfrak{w}(a) \cdot \mathfrak{p}(a) \cdot \mathfrak{p}(b)}{\alpha \cdot \beta} \cdot ((s-1)^\alpha \cdot s^\beta - s^\alpha \cdot (s-1)^\beta)$$

$$= \sum_{\substack{1\le\alpha\le w\\1\le\beta\le w}} ((s-1)^\alpha\cdot s^\beta - s^\alpha\cdot(s-1)^\beta) \sum_{G\in\mathcal{S}(\alpha,\beta)} \binom{\mathfrak{p}(G)}{\alpha\cdot\beta} \sum_{(a,b)\in G}\mathfrak{w}(a)$$

On the other hand $\sum_{G\in\mathcal{S}(\alpha,\beta)}\binom{\mathfrak{p}(G)}{\alpha\cdot\beta}\sum_{(a,b)\in G}\mathfrak{w}(a)$

$$= \sum_{G\in\mathcal{S}(\alpha,\beta)}\binom{\mathfrak{p}(\iota(G))}{\alpha\cdot\beta}\sum_{(c,d)\in\iota(G)}\mathfrak{w}(c) = \sum_{G\in\mathcal{S}(\beta,\alpha)}\binom{\mathfrak{p}(G)}{\beta\cdot\alpha}\sum_{(a,b)\in G}\mathfrak{w}(a).$$

Therefore $li_{2,(m,n),w}(\alpha_{m,n}(x)) = 0$, and we have a linear functional

$$li_{2,(m,n),w} : S_k(m,n)_{\langle w\rangle} \to k.$$

By the definition of $li_{2,(m,n),w}$ it is clear that

$$li_{2,(m,n),w}(\exp(\varepsilon)\otimes\exp(\varepsilon^{w-1})) = 1.$$

In order to finish the proof we only need to show that the space of linear functionals on $S_k(m,n)_{\langle w\rangle}$ is generated by $li_{2,(m,n),w}$. Or equivalently that if l is a linear combination of

$$\{l_{2,w-2}, l_{3,w-3}\cdots, l_{w-n,n}\}$$

such that $l(\alpha_{m,n}(x)) = 0$, for all $x\in k[\varepsilon]_m^{\times\times}$ then l is zero. So let

$$l = \sum_{2\le i\le w-n} c_i\cdot l_{i,w-i}$$

satisfy $l(\alpha_{m,n}(x)) = 0$, for all $x\in k[\varepsilon]_m^{\times\times}$. Assume that $l\neq 0$ and let i_0 be the smallest integer i such that $c_i\neq 0$. For all $s\in k^{\times\times}$ and $a_1, a_{i_0-1}, a_{w-i_0}\in k$ we have

$$l(\alpha_{m,n}(s + s(1-s)\cdot a_1\cdot\varepsilon + s(1-s)\cdot a_{i_0-1}\cdot\varepsilon^{i_0-1} + s(1-s)\cdot a_{w-i_0}\cdot\varepsilon^{w-i_0})) = 0.$$

If we denote the left hand side of the above equation by $l(s, a_1, a_{i_0-1}, a_{w-i_0})$, then we have

$$c_{i_0}\cdot\frac{(s-1)^2s - s^2(s-1)}{2}\cdot(a_1\cdot a_{i_0-1}\cdot a_{w-i_0}) = l(s,a_1,a_{i_0-1},a_{w-i_0}) - l(s,a_1,0,a_{w-i_0})$$

$= 0$. Therefore $c_{i_0} = 0$ contradicting the assumption. □

2.2. Construction of Li. Using the construction in the previous section we show that $li_{2,(2n-1,n),w}$ descends to a function on $B_2(k[\varepsilon]_n)$, as defined in §1.3.

Proposition 2.2.1. *For $n+1\le w\le 2n-1$, the map*

$$li_{2,(2n-1,n),w}\circ\delta : \mathbb{Q}[k[\varepsilon]_{2n-1}^{\times\times}] \to k$$

factors through the canonical projection $\mathbb{Q}[k[\varepsilon]_{2n-1}^{\times\times}]\to\mathbb{Q}[k[\varepsilon]_n^{\times\times}]$. We denote the induced map from $\mathbb{Q}[k[\varepsilon]_n^{\times\times}]$ to k by $Li_{2,n,w}$.

Proof. This follows from the fact that $li_{2,(2n-1,n),w}\circ\alpha_{2n-1,n} = 0$ by the construction in Proposition 2.1.2.

□

Proposition 2.2.2. *The map $Li_{2,n,w} : \mathbb{Q}[k[\varepsilon]_n^{\times\times}]\to k$ factors through the canonical projection $\mathbb{Q}[k[\varepsilon]_n^{\times\times}]\to B_2(k[\varepsilon]_n)$, we continue to denote the induced map by $Li_{2,n,w}$.*

Proof. We need to show that for $x, y \in k[\varepsilon]_n^{\times\times}$, such that $x/y \in k[\varepsilon]_n^{\times\times}$,

$$Li_{2,n,w}([x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)]) = 0.$$

If $\tilde{x}$ and $\tilde{y}$ are arbitrary liftings of x and y to $k[\varepsilon]_{2n-1}^{\times\times}$ then Proposition 2.2.1 implies that the left hand side of the last equation is equal to

$$(li_{2,(2n-1),w} \circ \delta)([\tilde{x}] - [\tilde{y}] + [\tilde{y}/\tilde{x}] - [(1-\tilde{x}^{-1})/(1-\tilde{y}^{-1})] + [(1-\tilde{x})/(1-\tilde{y})]).$$

The proposition then follows from that

$$\delta([\tilde{x}] - [\tilde{y}] + [\tilde{y}/\tilde{x}] - [(1-\tilde{x}^{-1})/(1-\tilde{y}^{-1})] + [(1-\tilde{x})/(1-\tilde{y})]) = 0.$$

□

If $\underline{c} = (c_1, \cdots, c_r) \in \mathbb{N}^r$ and $x = s + s(1-s)a_1\varepsilon + \cdots s(1-s)a_{n-1}\varepsilon^{n-1} \in k[\varepsilon]_n^{\times\times}$ then

$$\mathfrak{p}(x; \underline{c}) := a_{c_1} \cdot a_{c_2} \cdots a_{c_r}$$

and

$$\mathfrak{w}(\underline{c}) := c_1 + \cdots c_r.$$

Let

$$C(\alpha) := \{1, 2, \cdots, n-1\}^{\times\alpha}.$$

Proposition 2.2.3. *For $n+1 \leq w \leq 2n-1$, we have*

$$Li_{2,n,w}([x]) = \sum_{1 \leq \alpha, \beta \leq w} \frac{(s-1)^\alpha \cdot s^\beta - s^\alpha \cdot (s-1)^\beta}{\alpha \cdot \beta} \sum_{\substack{(a,b) \in C(\alpha) \times C(\beta) \\ \mathfrak{w}(a) \leq w-n \\ \mathfrak{w}(a,b) = w}} \mathfrak{w}(a) \cdot \mathfrak{p}(x; (a,b)).$$

Proof. Direct computation. □

Definition 2.2.4. The additive dilogarithm $Li_{2,n} : B_2(k[\varepsilon]_n) \to k^{\oplus(n-1)}$ is defined by

$$Li_{2,n} := \oplus_{n+1 \leq w \leq 2n-1} Li_{2,n,w}.$$

3. The map from cyclic homology to the Bloch group

3.1. This section is based on Goodwillie's theorem [12] and the construction of Bloch [3], Suslin [23] and Goncharov [10] of a map from the K_3 of a field to its Bloch group. Our main reference for cyclic homology and Goodwillie's theorem is [16]. In the following all cyclic homology groups are relative to $\mathbb{Q}$.

We will need the following to pass from cyclic homology to the rational homology of GL:

Theorem 3.1.1. *(Goodwillie,* [12]*; Theorem 11.3.1* [16]*) If A is a $\mathbb{Q}$-algebra and I is a nilpotent ideal in A then there is a canonical isomorphism*

$$HC_{n-1}(A, I) \simeq K_n(A, I)_{\mathbb{Q}},$$

for $n \geq 1$.

Remark 3.1.2. This isomorphism is compatible with the λ-structures on both sides by Theorem 1 in [7]. Hence, if $HC_*(A, I)^{(i-1)}$ and $K_*(A, I)_{\mathbb{Q}}^{(i)}$ denote the k^i-eigenspace for the k-th Adams operator (for any k) then the above isomorphism induces an isomorphism $HC_{*-1}(A, I)^{(i-1)} \simeq K_*(A, I)_{\mathbb{Q}}^{(i)}$ by the corollary in §1.3 [7].

For a ring A, the Hurewicz map induces an isomorphism from $\oplus_{n>0}K_n(A)_{\mathbb{Q}}$ to the primitive part $\mathrm{Prim}H_*(\mathrm{GL}(A),\mathbb{Q})$, of the homology of GL (11.2.12 Corollary, [16]). The map in Theorem 3.1.1 is constructed as the composition of a map from cyclic homology to the primitive part of the homology of GL and then using the inverse of the Hurewicz map. Since we will only need the map from cyclic homology to the homology of GL, we next describe the steps in its construction, following [16].

In §3.2, cyclic homology of A is computed as the homology of the Connes' complex. The natural map from the Connes' complex to the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{gl}$ is also described in §3.2. This map induces an isomorphism from cyclic homology to the primitive homology of $\mathfrak{gl}$. In §3.3 homology of $\mathfrak{gl}$ is replaced with the sum of the homology of its nilpotent parts $\mathfrak{t}_\sigma(A,I)$. In §3.4, homology of $\mathfrak{t}_\sigma(A,I)$ is replaced with that of the completion of its universal enveloping algebra and in §3.5 the latter is replaced with the homology of the group algebra of $T_\sigma(A,I)$, via Malcev theory. We reach the group homology of $\mathrm{GL}(A)$ in §3.6.

This construction combined with Suslin-Guin's stability theorem ([24], [14]) induces a map

$$HC_{n-1}(A,I)\to H_n(\mathrm{GL}_n(A),\mathbb{Q}), \tag{3.1.1}$$

when A is an artinian local algebra over $\mathbb{Q}$ and I is a proper ideal of A, in §3.7. We will use this map for $n=3$.

Finally we use a slight variation of the construction of Suslin-Goncharov in §3.8 to get a map $H_3(\mathrm{GL}_3(A),\mathbb{Q})\to ker(\delta)$.

The details can be found in §11.3 of [16] and the references therein. The main result of this section is Proposition 3.8.9.

3.2. Map from cyclic homology to Lie algebra homology.

3.2.1. For A any associative $\mathbb{Q}$-algebra the Connes' complex $C^\lambda_*(A)$ is defined as follows. Let $\mathbb{Z}/n\mathbb{Z}$ act on $A^{\otimes n}$ by

$$1\times(a_1\otimes a_2\otimes\cdots\otimes a_n)=(-1)^{n-1}a_2\otimes a_3\otimes\cdots\otimes a_n\otimes a_1,$$

and let $C^\lambda_{n-1}(A)$ denote the co-invariants of $A^{\otimes n}$ under this action. Let $b: C^\lambda_n(A)\to C^\lambda_{n-1}(A)$ be defined by

$$b(a_0,a_1,\cdots,a_n)=\sum_{0\le i\le n-1}(-1)^i(a_0,\cdots,a_i\cdot a_{i+1},\cdots,a_n)+(-1)^n(a_n\cdot a_0,a_1,\cdots,a_{n-1}).$$

Then $C^\lambda_*(A)$ is the complex:

$$\cdots\xrightarrow{b}C^\lambda_{n+1}(A)\xrightarrow{b}C^\lambda_n(A)\xrightarrow{b}\cdots\longrightarrow C^\lambda_0(A)\longrightarrow 0,$$

and $HC_*(A)=H_*(C^\lambda_*(A))$ (Theorem 2.1.5, [16]): the cyclic homology relative to $\mathbb{Q}$ can be computed as the homology of the Connes' complex.

3.2.2. For $\mathfrak{g}$ a Lie algebra over $\mathbb{Q}$, the Chevalley-Eilenberg complex $C_*(\mathfrak{g},\mathbb{Q})$ of $\mathfrak{g}$ with coefficients in $\mathbb{Q}$ is defined by:

$$\cdots\xrightarrow{d}\Lambda^n\mathfrak{g}\xrightarrow{d}\Lambda^{n-1}\mathfrak{g}\longrightarrow\cdots\xrightarrow{d}\mathfrak{g}\xrightarrow{d}\mathbb{Q}\longrightarrow 0,$$

where $d : \Lambda^n \mathfrak{g} \to \Lambda^{n-1} \mathfrak{g}$ is given by

$$d(a_1 \wedge a_2 \wedge \cdots \wedge a_n) = \sum_{1 \le i < j \le n} (-1)^{i+j-1} [a_i, a_j] \wedge a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge \hat{a}_j \wedge \cdots \wedge a_n.$$

The Lie algebra homology $H_*(\mathfrak{g}, \mathbb{Q})$ of $\mathfrak{g}$ with coefficients in $\mathbb{Q}$ is the homology of the complex $C_*(\mathfrak{g}, \mathbb{Q})$. The diagonal map $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ induces a map

$$\Delta : C_*(\mathfrak{g}, \mathbb{Q}) \to C_*(\mathfrak{g}, \mathbb{Q}) \otimes C_*(\mathfrak{g}, \mathbb{Q})$$

which makes $(C_*(\mathfrak{g}, \mathbb{Q}), d)$ a DG-coalgebra. Let $\mathrm{Prim} H_*(\mathfrak{g}, \mathbb{Q})$ denote the primitive elements in $H_*(\mathfrak{g}, \mathbb{Q})$, i.e. those α such that $\Delta(\alpha) = 1 \otimes \alpha + \alpha \otimes 1$.

Let $\mathfrak{gl}_n(A)$ denote the Lie algebra of $n \times n$ matrices and $\mathfrak{gl}(A)$ denote the direct limit $\lim_{n \to \infty} \mathfrak{gl}_n(A)$ with respect to the natural inclusions $\mathfrak{gl}_n(A) \subseteq \mathfrak{gl}_m(A)$, for $n \le m$. Then $\mathfrak{gl}(\mathbb{Q})$ acts on $C_*(\mathfrak{gl}(A), \mathbb{Q})$ by

$$[h, g_1 \wedge \cdots \wedge g_n] := \sum_{1 \le i \le n} g_1 \wedge \cdots \wedge [h, g_i] \wedge \cdots \wedge g_n.$$

Let $C_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ denote the complex of co-invariants with respect to this action and let $H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ and $\mathrm{Prim} H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$ denote respectively the homology and the primitive part of the homology of the complex $C_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{g}l(\mathbb{Q})}$. Then the theorem of Loday-Quillen-Tsygan says that:

Theorem 3.2.1. *(Loday-Quillen-Tsygan, Theorem 10.2.4,* [16]*) If A is an algebra over $\mathbb{Q}$ then there is a natural isomorphism*

$$HC_{*-1}(A) \simeq \mathrm{Prim} H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})} \simeq \mathrm{Prim} H_*(\mathfrak{g}l(A), \mathbb{Q}).$$

Explicitly the isomorphism

$$HC_{*-1}(A) \simeq \mathrm{Prim} H_*(\mathfrak{gl}(A), \mathbb{Q})_{\mathrm{gl}(\mathbb{Q})}$$

above is induced by the chain map that sends the class of $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ in $C^\lambda_{n-1}(A)$ to the class of $a_1 e_{12} \wedge a_2 e_{23} \wedge \cdots \wedge a_n e_{n1}$ in $C_n(\mathfrak{gl}(A), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})}$. Here e_{ij} denotes the matrix all of whose entries are zero except the one in the i-th row and the j-th column which is 1.

3.3. Volodin's construction in the Lie algebra case. Assume that I is a nilpotent ideal of A, and let $HC_*(A, I)$ denote the cyclic homology of A relative to I, the homology of the complex $C^\lambda_*(A, I)$ which is the kernel of the natural surjection $C^\lambda_*(A) \to C^\lambda_*(A/I)$.

For any permutation $\sigma \in S_n$ let $\mathfrak{t}_\sigma(A, I)$ denote the Lie subalgebra of $\mathfrak{gl}(A)$ given by $\mathfrak{t}_\sigma(A, I) := \{(a_{ij}) \in \mathfrak{gl}(A) : a_{ij} \in I \text{ if } \sigma(j) \le \sigma(i)\}$. Let $x(A, I) := \sum_\sigma C_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q})$ denote the sum of the subcomplexes

$$C_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) \subseteq C_*(\mathfrak{gl}(A), \mathbb{Q}),$$

over all n and $\sigma \in S_n$ and let $H_*(\mathfrak{gl}(A, I), \mathbb{Q})$ denote the homology of $x(A, I)$. Then the map in Theorem 3.2.1 induces an isomorphism:

$$HC_{*-1}(A, I) \simeq \mathrm{Prim}\ H_*(\mathfrak{g}l(A, I), \mathbb{Q}) \simeq \sum_\sigma \mathrm{Prim}\ H_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}), \tag{3.3.1}$$

(Proposition 11.3.12, [16]).

3.4. From the Lie algebra to the universal enveloping algebra. For a Lie algebra $\mathfrak{g}$ over $\mathbb{Q}$ let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra and $\hat{\mathcal{U}}(\mathfrak{g})$ its completion with respect to its augmentation ideal. We will next express the homology of $\mathfrak{g}$ in terms of the homology of $\mathcal{U}(\mathfrak{g})$.

Let B be an associative algebra over $\mathbb{Q}$ endowed with an augmentation map $\varepsilon : B \to \mathbb{Q}$. Let $C_*(B, \mathbb{Q})$ denote the complex:

$$\cdots \xrightarrow{b} B^{\otimes n} \xrightarrow{b} B^{\otimes(n-1)} \xrightarrow{b} \cdots \xrightarrow{b} \mathbb{Q} \longrightarrow 0,$$

where $b : B^{\otimes n} \to B^{\otimes(n-1)}$ is the map that sends $b_1 \otimes \cdots \otimes b_n$ to

$$\varepsilon(b_1) \cdot b_2 \otimes \cdots \otimes b_n + \sum_{1 \le i \le n-1} (-1)^i b_1 \otimes \cdots \otimes b_i \cdot b_{i+1} \otimes \cdots \otimes b_n + (-1)^n \varepsilon(b_n) \cdot b_1 \otimes \cdots \otimes b_{n-1}.$$

and let $H_*(B, \mathbb{Q})$ denote the homology of this complex.

Then the natural maps

$$H_*(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) \simeq H_*(\mathcal{U}(\mathfrak{t}_\sigma(A, I)), \mathbb{Q}) \simeq H_*(\hat{\mathcal{U}}(\mathfrak{t}_\sigma(A, I)), \mathbb{Q}) \tag{3.4.1}$$

are isomorphisms (Theorem 3.3.2, [16]). Here the first map is induced by the chain map α_{as}, 'as' for anti-symmetrization, that sends $t_1 \wedge \cdots \wedge t_n$ to

$$\sum_{\tau \in S_n} \operatorname{sign}(\tau) \cdot t_{\tau(1)} \otimes \cdots \otimes t_{\tau(n)}.$$

3.5. Malčev theory. If $\sigma \in S_n$, let $T_\sigma(A, I) \subseteq \mathrm{GL}(A)$ denote the group $\{1 + (a_{ij}) \in \mathrm{GL}_n(A) | a_{ij} \in I \text{ if } \sigma(j) \le \sigma(i)\}$. For a discrete group G let $U(G)$ denote group ring of G over $\mathbb{Q}$, and by $\hat{U}(G)$ its completion with respect to the augmentation ideal.

Since $T_\sigma(A, I)$ is a unipotent group with Lie algebra $\mathfrak{t}_\sigma(A, I)$, the natural maps:

$$H_*(\hat{\mathcal{U}}(\mathfrak{t}_\sigma(A, I), \mathbb{Q}) = H_*(\hat{U}(T_\sigma(A, I), \mathbb{Q}) \simeq H_*(U(T_\sigma(A, I)), \mathbb{Q}). \tag{3.5.1}$$

are isomorphisms (§11.3.13, [16]).

Combining (3.3.1), (3.4.1) and (3.5.1) we get a map

$$HC_{*-1}(A, I) \to \sum_\sigma H_*(U(T_\sigma(A, I)), \mathbb{Q}) \to H_*(U(\mathrm{GL}(A)), \mathbb{Q}). \tag{3.5.2}$$

3.6. Group homology. Let G be any (discrete) group and $C_*(G, \mathbb{Q})$ the complex:

$$\cdots \xrightarrow{d} \mathbb{Q}[G^{n+1}] \xrightarrow{d} \mathbb{Q}[G^n] \xrightarrow{d} \cdots \xrightarrow{d} \mathbb{Q}[G] \longrightarrow 0,$$

where $C_n(G, \mathbb{Q}) = \mathbb{Q}[G^{n+1}]$ and the map d is the one that sends $(g_0, g_1, \cdots, g_n)$ to

$$\sum_{0 \le i \le n} (-1)^i (g_0, \cdots, \hat{g}_i, \cdots, g_n).$$

Let G act on this complex by multiplication on the left, i.e. $g \times (g_0, \cdots, g_n) := (g \cdot g_0, \cdots, g \cdot g_n)$ and $H_*(G, \mathbb{Q}) := H_*(C_*(G, \mathbb{Q})_G)$, where the subscript G denotes the space of co-invariants.

The natural map from $C_*(U(G), \mathbb{Q})$ to $C_*(G, \mathbb{Q})$ that sends $g_1 \otimes g_2 \otimes \cdots \otimes g_n$ to

$$(1, g_1, g_1 \cdot g_2, \cdots, g_1 \cdot g_2 \cdots g_n)$$

induces an isomorphism $H_*(U(G),\mathbb{Q}) \to H_*(G,\mathbb{Q})$ (App. C.3, [16]). Applying this to $\mathrm{GL}(A)$ and combining with (3.5.2) we obtain the map

$$HC_{*-1}(A,I) \to H_*(\mathrm{GL}(A),\mathbb{Q}). \tag{3.6.1}$$

3.7. Suslin's stability theorem. The generalization of Suslin's stability theorem [23] by Guin states:

Theorem 3.7.1. *(Guin; §2, [14]) For any $1 \leq n$ and any artinian local algebra A over $\mathbb{Q}$ the map*

$$H_n(\mathrm{GL}_n(A),\mathbb{Q}) \to H_n(\mathrm{GL}(A),\mathbb{Q})$$

induced by the inclusion $\mathrm{GL}_n \hookrightarrow \mathrm{GL}$ is an isomorphism.

Therefore if A is an artinian local algebra over $\mathbb{Q}$ and I is a proper ideal then we have a map

$$\rho_1 : HC_{n-1}(A,I) \to H_n(\mathrm{GL}_n(A),\mathbb{Q}).$$

3.8. Bloch-Suslin map. Let A be an artinian local algebra over $\mathbb{Q}$ with residue field k. In this section we describe the Bloch-Suslin map (§2.6, [10]):

$$\rho_2 : H_3(\mathrm{GL}_3(A),\mathbb{Q}) \to \ker(\delta_A),$$

where $\delta_A : B_2(A)_{\mathbb{Q}} \to \Lambda^2 A^{\times}_{\mathbb{Q}}$ is the differential in the Bloch complex.

Definition 3.8.1. Let V be a finite free module over A and $\tilde{C}_m(V)$ denote the $\mathbb{Q}$-vector space with basis consisting of m-tuples $(x_0,\cdots,x_{m-1})$ of elements of V, that are in general position, i.e. for any $I \subseteq \{0,1,\cdots,m-1\}$, with $|I| \leq \mathrm{rank}(V)$, the set $\{x_i | i \in I\}$ can be extended to a basis of V. Let $C_m(V)$ denote the co-invariants of this space under the natural action of $\mathrm{GL}(V)$. Finally let $\tilde{C}_m(p) := \tilde{C}_m(A^{\oplus p})$ and $C_m(p) := C_m(A^{\oplus p})$.

Remark 3.8.2. Similarly, let $\tilde{C}_m(\mathbb{P}(V))$ denote the $\mathbb{Q}$-space with basis $(v_0,\cdots,v_{m-1})$ of m-tuples of points in $\mathbb{P}(V)$ which are in general position, and $d : \tilde{C}_{m+1}(\mathbb{P}(V)) \to \tilde{C}_m(\mathbb{P}(V))$ be defined by

$$d(v_0,\cdots,v_m) := \sum_{0\leq i\leq m} (-1)^i (v_0,\cdots,\hat{v}_i,\cdots,v_m).$$

Let $C_m(\mathbb{P}(V))$ denote the space of co-invariants of $\tilde{C}_m(\mathbb{P}(V))$ under the natural action of $GL(V)$. Then by identifying $[x]$ with $(0,x,1,\infty) \in C_4(\mathbb{P}(A^{\oplus 2}))$ and by comparing the dilogarithm relation in the definition of $B_2(A)$ to $d(0,x,y,1,\infty) \in C_4(\mathbb{P}(A^{\oplus 2}))$, one sees that

$$B_2(A)_{\mathbb{Q}} = C_4(\mathbb{P}(A^{\oplus 2}))/d(C_5(\mathbb{P}(A^{\oplus 2}))).$$

If $(x_1,\cdots,x_4)$ is a 4-tuple of points in $\mathbb{P}^1_A$ we will denote the corresponding element in $B_2(A)_{\mathbb{Q}}$ by $[x_1,\cdots,x_4]$.

Note that since A is a local ring, a subset of V is in general position if its reduction modulo the maximal ideal is in general position in the k-space $V \otimes_A k$.

Let $d : \tilde{C}_{m+1}(p) \to \tilde{C}_m(p)$ denote the map

$$d(x_0,x_1,\cdots,x_m) = \sum_{0\leq i\leq m} (-1)^i (x_0,\cdots,\hat{x}_i,\cdots,x_m)$$

and let $d' : \tilde{C}_{m+1}(p) \to \tilde{C}_m(p)$ denote the map

$$d'(x_0, x_1, \cdots, x_m) = \sum_{1 \le i \le m} (-1)^i (x_0, \cdots, \hat{x}_i, \cdots, x_m).$$

Let $\varepsilon : \tilde{C}_1(p) \to \mathbb{Q}$ be the map that sends each term to the sum of its coefficients.

Lemma 3.8.3. *The following complexes are acyclic.*

$$\cdots \xrightarrow{d} \tilde{C}_2(p) \xrightarrow{d} \tilde{C}_1(p) \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0$$

$$\cdots \xrightarrow{d'} \tilde{C}_2(p) \xrightarrow{d'} \tilde{C}_1(p) \longrightarrow 0.$$

Proof. Let $\sum_{j \in J} a_j \cdot (x_0(j), \cdots, x_{m-1}(j))$ be an m-cycle in the first or the second complex. Since the reductions modulo the maximal ideal $\{\overline{x}_0(j), \cdots, \overline{x}_{m-1}(j)\}$ are in general position in $k^{\oplus p}$ and k is an infinite field, we can choose $\alpha \in A$ such that all $\{\overline{x}_0(j), \cdots, \overline{x}_{m-1}(j), \overline{\alpha}\}$ are in general position. Note that if W_i for $1 \le i \le r$ are proper subspaces of a vector space W over an infinite field then $\cup_{1 \le i \le r} W_i \ne W$. We have

$$(-1)^m d(\sum_{j \in J} a_j \cdot (x_0(j), \cdots, x_{m-1}(j), \alpha)) = \sum_{j \in J} a_j \cdot (x_0(j), \cdots, x_{m-1}(j))$$

if $m \ge 2$ and $d \sum_{j \in J} a_j \cdot (x_0(j), \cdots, x_{m-1}(j)) = 0$ or if $m = 1$ and $\sum_{j \in J} a_j = 0$. Similarly,

$$(-1)^m d'(\sum_{j \in J} a_j \cdot (x_0(j), \cdots, x_{m-1}(j), \alpha)) = \sum_{j \in J} a_j \cdot (x_0(j), \cdots, x_{m-1}(j))$$

if $m \ge 2$ and $d' \sum_{j \in J} a_j \cdot (x_0(j), \cdots, x_{m-1}(j)) = 0$ or if $m = 1$.

□

Define maps $\lambda : \tilde{C}_m(p) \to \tilde{C}_m(p)$ by

$$\lambda(x_0, \cdots, x_{m-1}) = \sum_{0 \le i \le m-1} \mathrm{sign}(\sigma(m)^i)(x_{\sigma(m)^i(0)}, \cdots, x_{\sigma(m)^i(m-1)}),$$

where $\sigma(m) := (01 \cdots m-1)$ is the standard m-cylic permutation.

Then $\lambda \circ d = d' \circ \lambda$ and we have a double complex

$$\begin{array}{ccccc} \cdots & \xrightarrow{d} & \tilde{C}_3(3) & \xrightarrow{d} & \tilde{C}_2(3) \\ & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\lambda} \\ \cdots & \xrightarrow{d'} & \tilde{C}_3(3) & \xrightarrow{d'} & \tilde{C}_2(3) \end{array}$$

Definition 3.8.4. Let $\tilde{D}$ be the complex associated to the double complex above. That is $\tilde{D}_0 = \tilde{C}_2(3)$, $\tilde{D}_i = \tilde{C}_{i+2}(3) \oplus \tilde{C}_{i+1}(3)$ and the maps are given by $(x, y) \to (d'(x) + \lambda(y), -d(y))$.

Let $\varepsilon : \tilde{D}_0 \to \mathbb{Q}$ be the map that sends each term to the sum of its coefficients. Then by Lemma 3.8.3 the complex

$$\cdots \longrightarrow \tilde{D}_1 \longrightarrow \tilde{D}_0 \xrightarrow{\varepsilon} \mathbb{Q} \longrightarrow 0$$

is acyclic.

If we endow $\tilde{D}$ with its natural $\mathrm{GL}_3(A)$ action and $\mathbb{Q}$ with the trivial action then the complex above is an acyclic complex of $\mathrm{GL}_3(A)$-modules. Therefore we get a canonical map

$$H_3(\mathrm{GL}_3(A), \mathbb{Q}) \to H_3(D), \tag{3.8.1}$$

where $D := \tilde{D}_{\mathrm{GL}_3(A)}$ is the complex of co-invariants of $\tilde{D}$.

Next we define a map from $H_3(D)$ to $B_2(A)_\mathbb{Q}$. This will be a slight modification of Goncharov's map (§2.6, [10]).

We are interested in the following part

$$\begin{array}{ccccc} C_6(3) & \xrightarrow{d} & C_5(3) & \xrightarrow{d} & C_4(3) \\ \lambda\downarrow & & \lambda\downarrow & & \lambda\downarrow \\ C_6(3) & \xrightarrow{d'} & C_5(3) & \xrightarrow{d'} & C_4(3) \end{array}$$

of the double complex above.

We define a map ϕ from this double complex to the double complex

$$\begin{array}{ccccc} 0 & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & B_2(A)_\mathbb{Q} & \longrightarrow & \Lambda^2 A^\times_\mathbb{Q} \end{array}.$$

In ϕ the only nontrivial map

$$\begin{array}{ccc} C_5(3) & \xrightarrow{d'} & C_4(3) \\ \downarrow & & \downarrow \\ B_2(A)_\mathbb{Q} & \xrightarrow{\delta} & \Lambda^2 A^\times_\mathbb{Q} \end{array} \tag{3.8.2}$$

is a composition of the following two maps:

(i) The first map is:

$$\begin{array}{ccc} C_5(3) & \xrightarrow{d'} & C_4(3) \\ -p\downarrow & & p\downarrow \\ C_4(2) & \xrightarrow{d} & C_3(2), \end{array}$$

where $p : C_{m+1}(3) \to C_m(2)$ is the map that sends $(v_0, v_1, \cdots, v_{m-1})$ to $(\tilde{v}_1, \cdots, \tilde{v}_{m-1})$. Here $\tilde{v}_i$ denotes the image of v_i in $A^{\oplus 3}/\langle v_0\rangle$ and the term $(\tilde{v}_1, \cdots, \tilde{v}_{m-1})$ is identified with an element of $C_m(2)$ by choosing any isomorphism between $A^{\oplus 3}/\langle v_0\rangle$ and $A^{\oplus 2}$.

(ii) The second map is:

$$\begin{array}{ccc} C_4(2) & \xrightarrow{d} & C_3(2) \\ \alpha\downarrow & & \beta\downarrow \\ B_2(A)_\mathbb{Q} & \xrightarrow{\delta} & \Lambda^2 A^\times_\mathbb{Q}, \end{array} \tag{3.8.3}$$

where α is the map that sends (v_0, v_1, v_2, v_3) to $[\underline{v}_0, \underline{v}_1, \underline{v}_2, \underline{v}_3]$, here $\underline{v}_i$ denotes the image of v_i in $\mathbb{P}(A^{\oplus 2})$, and $[\underline{v}_0, \underline{v}_1, \underline{v}_2, \underline{v}_3]$ denotes the image of $(\underline{v}_0, \underline{v}_1, \underline{v}_2, \underline{v}_3)$ under the map

$$C_4(\mathbb{P}(A^{\oplus 2})) \to B_2(A)_{\mathbb{Q}},$$

as in Remark 3.8.2. And β is the map that sends (v_0, v_1, v_2) to

$$\Big(\frac{v_0 \wedge v_1}{v_1 \wedge v_2}\Big) \wedge \Big(\frac{v_0 \wedge v_2}{v_1 \wedge v_2}\Big).$$

The next three lemmas imply that the maps defined so far can be extended to a map ϕ of the double complexes.

Lemma 3.8.5. *The map*

$$C_6(3) \xrightarrow{d'} C_5(3) \xrightarrow{-p} C_4(2) \xrightarrow{\alpha} B_2(A)_{\mathbb{Q}},$$

is zero.

Proof. This follows from the fact that $-p \circ d'(v_0, v_1, v_2, v_3, v_4, v_5) = d(v_1, v_2, v_3, v_4, v_5)$, and that this maps to zero in $B_2(A)_{\mathbb{Q}}$, by Remark 3.8.2. □

Lemma 3.8.6. *The map*

$$C_5(3) \xrightarrow{\lambda} C_5(3) \xrightarrow{-p} C_4(2) \xrightarrow{\alpha} B_2(A)_{\mathbb{Q}}$$

is zero.

Proof. (c.f. Lemma 2.18, [10]) Let $(v_0, \cdots, c_4) \in C_5(3)$. Then there is a conic Q passing through the images of the five points v_0, v_1, v_2, v_3, v_4 in the projective plane. Choosing any isomorphism, we identify Q with $\mathbb{P}^1_A$. Let the images of v_i be $x_i \in \mathbb{P}^1_A$ under this isomorphism. The composition of the maps in the statement of the lemma then maps $(v_0, \cdots, v_4)$ in $C_5(3)$ to

$$-\sum_{0 \le i \le 4} [x_i, x_{i+1}, \cdots, x_{i+3}],$$

in $B_2(A)_{\mathbb{Q}}$, where the indices are modulo 5.

Claim 3.8.7. *In $B_2(A)_{\mathbb{Q}}$ we have $[x_1, x_2, x_3, x_4] = \operatorname{sign}(\sigma) \cdot [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}]$ for any $\sigma \in S_4$.*

Proof of the claim. Note that since we are working with $\mathbb{Q}$-modules we have $[0, x, 1, \infty] = -[0, x/(x-1), 1, \infty]$, (Lemma 1.2, Lemma 1.5, [23]), hence $[0, x, 1, \infty] = -[x, 0, 1, \infty]$; $[0, x, 1, \infty] = -[0, 1/x, 1, \infty]$ (Lemma 1.2, [23]), hence $[0, x, 1, \infty] = -[0, 1, x, \infty]$; and again since $[0, x, 1, \infty] = -[0, x/(x-1), 1, \infty]$, we have $[0, x, 1, \infty] = -[0, x, \infty, 1]$.

Therefore the formula in the statement holds for the transpositions (12), (23), and (34). Since these generate S_4, the statement follows. □

Finally, by the claim $\sum_{0 \le i \le 4} [x_i, x_{i+1}, \cdots, x_{i+3}] = \sum_{0 \le i \le 4} (-1)^i [x_0, \cdots, \hat{x}_i, \cdots, x_4]$ and the right hand side is zero in $B_2(A)_{\mathbb{Q}}$ by Remark 3.8.2. □

Lemma 3.8.8. *The map*

$$C_4(3) \xrightarrow{\lambda} C_4(3) \xrightarrow{p} C_3(2) \xrightarrow{\beta} \Lambda^2 A^{\times}_{\mathbb{Q}}$$

is zero.

Proof. First note that β sends (v_0, v_1, v_2) to

$$\Big(\frac{v_0 \wedge v_1}{v_1 \wedge v_2}\Big) \wedge \Big(\frac{v_0 \wedge v_2}{v_1 \wedge v_2}\Big) = \Big(\frac{\ell(v_0 \wedge v_1)}{\ell(v_1 \wedge v_2)}\Big) \wedge \Big(\frac{\ell(v_0 \wedge v_2)}{\ell(v_1 \wedge v_2)}\Big),$$

where $\ell : \det_A(A^{\oplus 2}) \to A$ is any surjective A-linear map. Therefore since we are looking at configurations in general position the composition $\beta \circ p$ maps $(y_0, y_1, y_2, y_3) \in C_4(3)$ to

$$\Big(\frac{y_0 \wedge y_1 \wedge y_2}{y_0 \wedge y_2 \wedge y_3}\Big) \wedge \Big(\frac{y_0 \wedge y_1 \wedge y_3}{y_0 \wedge y_2 \wedge y_3}\Big).$$

This implies the statement by direct computation.

□

Therefore ϕ is a map of double complexes which induces a map $H_3(D) \to ker(\delta)$ of the homology of the associated complexes. Combining this with the map

$$H_3(\mathrm{GL}_3(A), \mathbb{Q}) \to H_3(D)$$

in (3.8.1), we obtain a map

$$\rho_2 : H_3(\mathrm{GL}_3(A), \mathbb{Q}) \to \ker(\delta).$$

Therefore applying §3.1-3.7 to $(A, I) = (k[\varepsilon]_n, (\varepsilon))$ proves:

Proposition 3.8.9. *The composition* $T := \rho_2 \circ \rho_1$ *defines a natural map*

$$T : HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)} \hookrightarrow HC_2(k[\varepsilon]_n, (\varepsilon)) \to B_2(k[\varepsilon]_n)_{\mathbb{Q}},$$

whose image lies in $\ker(\delta_n)$.

4. Nonvanishing of $Li_{2,n}$ on $HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$

4.1. This section shows that $Li_{2,n}$ is the correct map, as we show that it does not vanish on $HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$. First we describe $HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)}$ and define some elements α_w in it on which we will evaluate the additive dilogarithm.

4.1.1. Note that $HC_*(k[\varepsilon]_n, (\varepsilon))$ is a $k^\times$-abelian group, where $\lambda \in k^\times$ acts as the map induced by the k-algebra automorphism of $k[\varepsilon]_n$ that sends ε to $\lambda \cdot \varepsilon$. This action is compatible with the decomposition (Remark 3.1.2) of

$$HC_2(k[\varepsilon]_n, (\varepsilon)) = HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)} \oplus HC_2(k[\varepsilon]_n, (\varepsilon))^{(2)}$$

(pp. 593-594, [7]) and

$$HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)} = \oplus_{n+1 \leq w \leq 2n-1} HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)}_{\langle w \rangle},$$

where each summand is isomorphic to k (loc. cit.); and

$$HC_2(k[\varepsilon]_n, (\varepsilon))^{(2)} = \oplus_{1 \leq w \leq n-1} HC_2(k[\varepsilon]_n, (\varepsilon))^{(2)}_{\langle w \rangle},$$

where each summand is isomorphic to $\Omega^2_{k/\mathbb{Q}}$ (loc. cit.).

4.1.2. Let $\chi : \mathbb{N} \to \mathbb{N}$ be defined by $\chi(n) = 0$ if n is even, and $\chi(n) = 1$ if n is odd. For $n+1 \leq w \leq 2n-1$, let

$$\alpha_w := \sum_{0 \leq j < (2n-1-w)/2} (\varepsilon^{n-1-j}, \varepsilon^{w-n+j}, \varepsilon) + (1/2) \cdot \chi(w) \cdot (\varepsilon^{(w-1)/2}, \varepsilon^{(w-1)/2}, \varepsilon)$$

in $C_2^{\lambda}(k[\varepsilon]_n)$. Since α_w is a cycle, as can be checked by direct computation, with $k^{\times}$-weight w, it defines an element $\alpha_w \in HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)}_{\langle w \rangle}$ by §4.1.1.

4.2. Computation of $Li_{2,n}$ on HC_2.

Our aim in this section is to compute $Li_{2,n}(T(\alpha_w))(= Li_{2,n,w}(T(\alpha_w)))$. This we will do in several steps.

4.2.1. *From $\mathfrak{gl}_3(k[\varepsilon]_n)$ to $\mathfrak{gl}_2(k[\varepsilon]_n)$.*
We begin with the 2-chain $(\varepsilon^a, \varepsilon^b, \varepsilon) \in C_2^{\lambda}(k[\varepsilon]_n)$ in the Connes' complex, where $a + b \geq n$. By the map in §3.2, on the chain complex level, $(\varepsilon^a, \varepsilon^b, \varepsilon)$ goes to $\beta_{a,b} := \varepsilon^a e_{12} \wedge \varepsilon^b e_{23} \wedge \varepsilon e_{31} \in C_3(\mathfrak{gl}_3(k[\varepsilon]_n))_{\mathfrak{gl}_3(\mathbb{Q})}$. Therefore we need to compute the image of

$$\beta_w := \sum_{0 \leq j < (2n-1-w)/2} \beta_{n-1-j, w-n+j} + (1/2)\chi(w)\beta_{(w-1)/2,(w-1)/2}$$

in k. Let $\gamma_{a,b} := \varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{11}$, and

$$\gamma_w := \sum_{0 \leq j < (2n-1-w)/2} \gamma_{n-1-j, w-n+j} + (1/2)\chi(w)\gamma_{(w-1)/2,(w-1)/2}.$$

We defined T as the composition

$$\begin{aligned} HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)} &\to \mathrm{Prim} H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})} \simeq \mathrm{Prim} H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q}) \\ &\to H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q}) \to H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q}) \to ker(\delta). \end{aligned}$$

Let $T' : \mathrm{Prim} H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}(\mathbb{Q})} \to ker(\delta)$ and $T'' : H_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q}) \to ker(\delta)$ be the obvious compositions.

The following lemma enables us to work in the homology of $\mathfrak{gl}_2(k[\varepsilon]_n)$ rather than that of $\mathfrak{gl}_3(k[\varepsilon]_n)$:

Lemma 4.2.1. *We have $(Li_{2,n,w} \circ T')(\beta_w) = (Li_{2,n,w} \circ T')(\gamma_w)$.*

Proof. First note that

$$\begin{aligned} d(e_{13} \wedge \varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{31}) &= -\beta_{a,b} + \gamma_{a,b} \\ &\quad -\varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33} - e_{13} \wedge \varepsilon^{a+1} e_{32} \wedge \varepsilon^b e_{21}; \end{aligned}$$

$\varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33}$ is a cycle; and $e_{13} \wedge \varepsilon^{a+1} e_{32} \wedge \varepsilon^b e_{21}$ is a boundary in $C_*(\mathfrak{gl}(k[\varepsilon]_n))_{\mathfrak{gl}(\mathbb{Q})}$, since this element corresponds to the element $(1, \varepsilon^{a+1}, \varepsilon^b)$ in the Connes' complex and $d(1, \varepsilon^{a+1}, \varepsilon^b, 1) = (1, \varepsilon^{a+1}, \varepsilon^b)$.

Therefore since β_w is a cycle so is γ_w and to prove the lemma it suffices to show that $(Li_{2,n,w} \circ T')(\varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33}) = 0$, for $a + b \geq n$.

Note that since

$$
\begin{aligned}
d(e_{12} \wedge \varepsilon^a e_{11} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33}) &= -\varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{33} \\
&\quad + \varepsilon^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33} - \varepsilon^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}
\end{aligned}
$$

it is sufficient to show that

$$(Li_{2,n,w} \circ T')(\varepsilon^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33}) = (Li_{2,n,w} \circ T'')(\varepsilon^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33})$$

and

$$(Li_{2,n,w} \circ T')(\varepsilon^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}) = (Li_{2,n,w} \circ T'')(\varepsilon^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33})$$

are 0. The equalities above follow immediately from the fact that $\varepsilon^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33}$ and $\varepsilon^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}$ are cycles not only in $C_*(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})}$ but also in $C_*(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})$.

Recall that for $x \in B_2(k[\varepsilon]_n)$, $Li_{2,n,w}(x) = (li_{2,(2n-1,n),w} \circ \delta_{2n-1})(\tilde{x})$ (Proposition 2.2.1 and 2.2.2), where $\tilde{x} \in B_2(k[\varepsilon]_{2n-1})$ is any lift of x.

Let $\tilde{\alpha} \in \{\varepsilon^a e_{11} \wedge \varepsilon^b e_{11} \wedge \varepsilon e_{33}, \varepsilon^a e_{11} \wedge \varepsilon^b e_{22} \wedge \varepsilon e_{33}\} \subseteq C_3(\mathfrak{gl}(k[\varepsilon]_{2n-1}), \mathbb{Q})$, and α the reduction of $\tilde{\alpha}$ to $C_3(\mathfrak{gl}(k[\varepsilon]_n), \mathbb{Q})$. Then

$$Li_{2,n,w}(T''(\alpha)) = (li_{2,(2n-1,n),w} \circ \delta_{2n-1})(\underline{T}''(\tilde{\alpha})).$$

Here by $\underline{T}''$, we denote the chain map, which maps $C_3(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to $B_2(k[\varepsilon]_{2n-1})_{\mathbb{Q}}$ and $C_2(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to $\Lambda^2 V_{2n-1}$, that induces T''. The map $\underline{T}''$ depends on certain choices (see the next paragraph).

Let us recall how $\underline{T}''(\tilde{\alpha})$ is defined in §3. First through the anti-symmetrization map α_{as} (§3.4) and the use of the exponential map (§3.5; §11.3.13 [16]), we get a chain map $C_*(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})^\circ \to C_*(\hat{U}(\mathrm{GL}_3(k[\varepsilon]_{2n-1})), \mathbb{Q})$. In fact, it is immediately seen that the image of $\tilde{\alpha}$ under these maps lies inside the image of $C_*(U(\mathrm{GL}_3(k[\varepsilon]_{2n-1})), \mathbb{Q})$ in $C_*(\hat{U}(\mathrm{GL}_3(k[\varepsilon]_{2n-1})), \mathbb{Q})$. From $C_*(U(\mathrm{GL}_3(k[\varepsilon]_{2n-1})), \mathbb{Q})$ to $C_*(\mathrm{GL}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$ we pass via the map described in §3.6. Bypassing the need for stabilization since we are already in GL_3, and using the fact that $\tilde{D}$ is an acyclic complex of $\mathrm{GL}_3(k[\varepsilon]_{2n-1})$ modules we get a (non-canonical) map from $C_*(\mathrm{GL}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to $\tilde{D}$. Finally taking $\mathrm{GL}_3(k[\varepsilon]_{2n-1})$ co-invariants we end up in the complex D and using the map of double complexes (induced by (3.8.2)), we pass from D to the complex

$$\gamma_{k[\varepsilon]_{2n-1}}(2)_{\mathbb{Q}} : B_2(k[\varepsilon]_{2n-1})_{\mathbb{Q}} \xrightarrow{\delta_{2n-1}} \Lambda^2 V_{2n-1}.$$

Since $\underline{T}''$ is a map of complexes, $\delta_{2n-1}(\underline{T}''(\tilde{\alpha})) = \underline{T}''(d(\tilde{\alpha})) = 0$, as $d(\tilde{\alpha}) = 0$ in $C_*(\mathfrak{gl}_3(k[\varepsilon]_{2n-1}), \mathbb{Q})$. This implies that $Li_{2,n,w}(T''(\alpha)) = li_{2,(2n-1,n),w}(\delta_{2n-1}(T''(\tilde{\alpha}))) = 0$ and finishes the proof of the lemma. □

The next lemma will help us to reduce the computation to $\mathfrak{gl}_2$:

Lemma 4.2.2. *The chain γ_w, as defined above, is a cycle in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ and hence defines an element in $H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$.*

Proof. We already know that γ_w defines a cycle in $C_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})}$. Since $C_i(\mathfrak{gl}_m(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_m(\mathbb{Q})} = (\Lambda^i \mathfrak{gl}_m(k[\varepsilon]_n))_{\mathfrak{gl}_m(\mathbb{Q})}$ (§3.2.2) and

$$(\Lambda^i \mathfrak{gl}_m(k[\varepsilon]_n))_{\mathfrak{gl}_m(\mathbb{Q})} = (\Lambda^i \mathfrak{gl}_i(k[\varepsilon]_n))_{\mathfrak{gl}_i(\mathbb{Q})},$$

for $m \geq i$, (Corollary 9.2.8 and (10.2.10.1), [16]), we have

$$d(\gamma_w) = 0 \in C_2(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_3(\mathbb{Q})} = C_2(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}.$$

□

4.2.2. *From* $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ *to* $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$.

(i) In order to continue with the computation of $Li_{2,n,w}(T'(\gamma_w))$, we need to compute the image of γ_w in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n, \mathbb{Q})$. This would be a very long computation, in fact we will see in this section that we can get away with much less. The following proposition will be crucial for this.

Proposition 4.2.3. *For any* $\mathbb{Q}$*-algebra* A*, let* $\mathfrak{gl}_n(\mathbb{Q})$ *act on* $\mathfrak{gl}_n(A)$ *by its adjoint action. Let* $C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})}$ *be the subcomplex of* $C_*(\mathfrak{gl}_n(A), \mathbb{Q})$ *on which* $\mathfrak{gl}_n(\mathbb{Q})$ *acts trivially. Then the canonical map*

$$C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})} \to C_*(\mathfrak{gl}_n(A), \mathbb{Q}) \to C_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})}$$

is an isomorphism and there is a canonical direct sum of complexes

$$C_*(\mathfrak{gl}_n(A), \mathbb{Q}) = C'_*(\mathfrak{gl}_n(A), \mathbb{Q})_{\mathfrak{gl}_n(\mathbb{Q})} \oplus L_*, \tag{4.2.1}$$

with $\mathfrak{gl}_n(\mathbb{Q})$*-action, such that* L_* *is acyclic.*

Proof. This is Proposition 10.1.8 in [16], taking for $\mathfrak{g} = \mathfrak{gl}_n(A)$ and for $\mathfrak{h} = \mathfrak{gl}_n(\mathbb{Q})$, and noting the reductivity of $\mathfrak{gl}_n(\mathbb{Q})$ (10.2.9, [16]). □

To continue with the computation we need to compute the image γ'_w of γ_w in $H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})$. Then we should lift γ'_w to a chain $\tilde{\gamma}'_w$ in $C'_3(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ and continue just as in the last part of the proof of Lemma 4.2.1. Namely

$$\begin{aligned} Li_{2,n,w}(T'(\gamma_w)) &= Li_{2,n,w}(\underline{T}''(\tilde{\gamma}'_w)) = li_{2,(2n-1,n),w}(\delta_{2n-1}(\underline{T}''(\tilde{\gamma}'_w))) \\ &= li_{2,(2n-1,n),w}(\underline{T}''(d(\tilde{\gamma}'_w))). \end{aligned}$$

Let $\tilde{\gamma}^*_w$ be any chain in $C_3(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$ such that its image in $C_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$ (under the canonical maps) is a cycle and lifts γ_w. Then the first component $\tilde{\gamma}^{*(1)}_w$ of $\tilde{\gamma}^*_w$ under the decomposition in (4.2.1) is a lifting of the element γ'_w, by Proposition 4.2.3. Therefore we can choose $\tilde{\gamma}'_w := \tilde{\gamma}^{*(1)}_w$, and to continue the computation we need to compute

$$d(\tilde{\gamma}^{*(1)}_w) = d(\tilde{\gamma}^*_w)^{(1)}.$$

For the rest of the computation, we will let $\tilde{\gamma}^*_w := \tilde{\gamma}_w$, where

$$\tilde{\gamma}_w := \sum_{0 \leq j < (2n-1-w)/2} \tilde{\gamma}_{n-1-j,w-n+j} + (1/2)\chi(w)\tilde{\gamma}_{(w-1)/2,(w-1)/2},$$

and $\tilde{\gamma}_{a,b} := \varepsilon^a e_{12} \wedge \varepsilon^b e_{21} \wedge \varepsilon e_{11}$.

Combining the above we have

$$Li_{2,n,w}(T'(\gamma_w)) = li_{2,(2n-1,n),w}(\underline{T}''(d(\tilde{\gamma}_w)^{(1)})) \tag{4.2.2}$$

(ii) Next we will compute $d(\tilde{\gamma}_{a,b})^{(1)}$. For any $\mathbb{Q}$-algebra A there is a canonical isomorphism for $i \geq n$:

$$C_n(\mathfrak{gl}_i(A), \mathbb{Q})_{\mathfrak{gl}_i(\mathbb{Q})} = (\Lambda^n \mathfrak{gl}_i(A))_{\mathfrak{gl}_i(\mathbb{Q})} \to (\mathbb{Q}[S_n] \otimes A^{\otimes n})_{S_n}, \tag{4.2.3}$$

where S_n acts on $\mathbb{Q}[S_n]$ by conjugation and on $A^{\otimes n}$ by permuting the factors and multiplying with sign (10.2.10.1, [16]).

Letting

$$\Gamma_{x,y} := xe_{12} \wedge ye_{21} + xe_{21} \wedge ye_{12} + \frac{1}{2}x(e_{22} - e_{11}) \wedge y(e_{22} - e_{11}),$$

for $x, y \in A$, we see by direct computation that $\Gamma_{x,y} \in C_2'(\mathfrak{gl}_2(A), \mathbb{Q})_{\mathfrak{gl}_2(\mathbb{Q})}$.

Under the map (4.2.3), $\Gamma_{x,y}$ is sent to $(3 \cdot \tau) \otimes (x \otimes y)$; $x(e_{11} - e_{22}) \wedge ye_{11}$ to $(1 \cdot \tau) \otimes (x \otimes y)$; $xe_{21} \wedge ye_{12}$ to $(1 \cdot \tau) \otimes (x \otimes y)$; and $xe_{12} \wedge ye_{21}$ to $(1 \cdot \tau) \otimes (x \otimes y)$, where $S_2 = \{id, \tau\}$. Therefore, using Proposition 4.2.3, we have

$$(x(e_{11} - e_{22}) \wedge ye_{11})^{(1)} = (xe_{21} \wedge ye_{12})^{(1)} = (xe_{12} \wedge ye_{21})^{(1)} = \frac{1}{3}\Gamma_{x,y}.$$

Since

$$d(\tilde{\gamma}_{a,b}) = \varepsilon^{a+b}(e_{11} - e_{22}) \wedge \varepsilon e_{11} - \varepsilon^b e_{21} \wedge \varepsilon^{a+1} e_{12} - \varepsilon^a e_{12} \wedge \varepsilon^{b+1} e_{21},$$

we have

$$d(\tilde{\gamma}_{a,b})^{(1)} = \frac{1}{3}(\Gamma_{\varepsilon^{a+b},\varepsilon} - \Gamma_{\varepsilon^b,\varepsilon^{a+1}} - \Gamma_{\varepsilon^a,\varepsilon^{b+1}}). \tag{4.2.4}$$

4.2.3. *Fixing a choice for $\underline{T}''$.*

We need to fix a choice for the restriction of $\underline{T}''$ to $C_*(\mathfrak{gl}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$. In order to do this, recalling the description in the last part of the proof of Lemma 4.2.1, we need to fix the map from $C_*(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q}) \to \tilde{D} \to D \to \gamma_{k[\varepsilon]_{2n-1}}(2)_{\mathbb{Q}}$, in degree 2.

Fixing v_1, v_2, v_3 any three vectors in $k[\varepsilon]_{2n-1}^{\oplus 2}$ in general position, we define a map that sends $(g_1, g_2, g_3) \in C_2(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q})$ to

$$(w, g_1v_1, g_2v_2, g_3v_3) - (w, g_1v_1, g_2v_2, g_2v_3) - (w, g_1v_1, g_1v_2, g_3v_3) + (w, g_1v_1, g_1v_2, g_2v_3)$$

in $\tilde{C}_4(k[\varepsilon]_{2n-1}^{\oplus 3}) = \tilde{C}_4(3) \subseteq \tilde{C}_4(3) \oplus \tilde{C}_3(3)$, where we view $k[\varepsilon]_{2n-1}^{\oplus 2} = \{(a_1, a_2, a_3) \in k[\varepsilon]_{2n-1}^{\oplus 3} | a_3 = 0\}$, and we let $w = (0, 0, 1)$. It is seen without difficulty that this map can be extended to a map of complexes $C_*(\mathrm{GL}_2(k[\varepsilon]_{2n-1}), \mathbb{Q}) \to \tilde{D}$.

Composing with the remaining map given in (3.8.2) this gives a map that sends (g_1, g_2, g_3) to

$$\beta((g_1v_1, g_2v_2, g_3v_3) - (g_1v_1, g_2v_2, g_2v_3) - (g_1v_1, g_1v_2, g_3v_3) + (g_1v_1, g_1v_2, g_2v_3))$$

in $\Lambda^2 V_{2n-1}$, where β is the map in (3.8.3). From now on we fix $v_1 := (1, 1)$, $v_2 := (0, 1)$ and $v_3 := (1, 0)$ and denote the resulting map by $\underline{T}''$.

4.2.4. *Computing $li_{2,(2n-1,n),w}(\underline{T}''(\Gamma_{\varepsilon^p,\varepsilon^q}))$.*

Because of (4.2.2) and (4.2.4) we realize that we need to compute

$$li_{2,(2n-1,n),w}(\underline{T}''(\Gamma_{\varepsilon^p,\varepsilon^q})),$$

for $p + q = w$. We will do this in a few steps.

Lemma 4.2.4. *For $i = 1, 2$, and $p + q = w$, with $p, q \geq 1$:*

$$li_{2,(2n-1,n),w}(\underline{T}''(\varepsilon^p e_{ii} \wedge \varepsilon^q e_{ii})) = 0.$$

Proof. The element $\varepsilon^p e_{ii} \wedge \varepsilon^q e_{ii}$ maps to

$$\varepsilon^p e_{ii} \otimes \varepsilon^q e_{ii} - \varepsilon^q e_{ii} \otimes \varepsilon^p e_{ii} \in C_2(\mathcal{U}(\mathfrak{gl}_2(k[\varepsilon]_{2n-1})), \mathbb{Q}).$$

Since $\varepsilon^x e_{ii} = \log(1-(1-\exp(\varepsilon^x e_{ii}))) = -\sum_{1\le k} \frac{(1-\exp(\varepsilon^x e_{ii}))^k}{k}$, for $x \geq 1$, we see that $\varepsilon^p e_{ii} \otimes \varepsilon^q e_{ii}$ is a $\mathbb{Q}$-linear combination of terms of the form

$$\exp(\varepsilon^s e_{ii})^u \otimes \exp(\varepsilon^t e_{ii})^v.$$

Let $g_1 := \exp(\varepsilon^s e_{ii})^u$ and $g_2 := \exp(\varepsilon^t e_{ii})^v$ then $g_1 \otimes g_2$ maps to $(1, g_1, g_1 g_2)$ which maps to

$$(v_1, g_1v_2, g_1g_2v_3) - (v_1, g_1v_2, g_1v_3) - (v_1, v_2, g_1g_2v_3) + (v_1, v_2, g_1v_3). \tag{4.2.5}$$

Since, depending on i, $g_1(v_2) = v_2$ or $g_1(v_3) = g_1g_2(v_3) = v_3$, we see that the last expression is 0. ☐

Lemma 4.2.5. *The value of $li_{2,(2n-1,n),w}$ on the image of the element $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ in $\Lambda^2 V_{2n-1}$, under the chain map that we fixed in §4.2.3, is 0, if $p+q \neq w$, and $p, q \geq 1$.*

Proof. Note that by Proposition 2.1.2 to compute the value of $li_{2,(2n-1,n),w}$ on the image of $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ in $\Lambda^2 V_{2n-1}$, we first need to project that image to $S_k(2n-1,n)_{\langle w\rangle}$. But for $\lambda \in \mathbb{Q}$, replacing ε with $\lambda\varepsilon$ multiplies $\varepsilon^p e_{ij} \otimes \varepsilon^q e_{kl}$ by λ^{p+q}, whereas the projection of its image to $S_k(2n-1,n)_{\langle w\rangle}$ gets multiplied by λ^w. Therefore this projection is 0. Hence the statement in the lemma. ☐

Lemma 4.2.6. *For $p+q = w$, and $p, q \geq 1$:*

$$li_{2,(2n-1,n),w}(\underline{T}''(\varepsilon^p e_{22} \wedge \varepsilon^q e_{11})) = li_{2,(2n-1,n),w}((1+\varepsilon^q) \wedge (1+\varepsilon^p)).$$

Proof. The expression $\varepsilon^p e_{22} \wedge \varepsilon^q e_{11}$ maps to

$$\varepsilon^p e_{22} \otimes \varepsilon^q e_{11} - \varepsilon^q e_{11} \otimes \varepsilon^p e_{22}. \tag{4.2.6}$$

The $k^\times$-weight w component of

$$\varepsilon^p e_{ii} \otimes \varepsilon^q e_{jj} \tag{4.2.7}$$

is the same as the $k^\times$-weight w component of

$$\exp(\varepsilon^p e_{ii}) \otimes \exp(\varepsilon^q e_{jj}) - \exp(\varepsilon^p e_{ii}) \otimes 1 - 1 \otimes \exp(\varepsilon^q e_{jj}). \tag{4.2.8}$$

Therefore by Lemma 4.2.5, (4.2.7) and (4.2.8) have the same image. Note that terms of the form $1 \otimes g$ and $g \otimes 1$ map to 0, because of the computation in (4.2.5). Hence the left hand side of the expression in the statement of the lemma is equal to the image of

$$\exp(\varepsilon^p e_{22}) \otimes \exp(\varepsilon^q e_{11}) - \exp(\varepsilon^q e_{11}) \otimes \exp(\varepsilon^p e_{22}).$$

Since $\exp(\varepsilon^q e_{11})v_2 = v_2$, using the expression (4.2.5) we see that $\exp(\varepsilon^q e_{11}) \otimes \exp(\varepsilon^p e_{22})$ maps to 0. Again using (4.2.5) and the definition of β and $li_{2,(2n-1,n),w}$ we see that $\exp(\varepsilon^p e_{22}) \otimes \exp(\varepsilon^q e_{11})$ maps to $li_{2,(2n-1,n),w}((1+\varepsilon^q) \wedge (1+\varepsilon^p))$. ☐

Lemma 4.2.7. *For $p+q = w$, and $p, q \geq 1$:*

$$li_{2,(2n-1,n),w}(\underline{T}''(\varepsilon^p e_{12} \wedge \varepsilon^q e_{21})) = li_{2,(2n-1,n),w}((1-\varepsilon^p) \wedge (1-\varepsilon^q)).$$

Proof. Exactly as in the proof of Lemma 4.2.6, we see that the left hand side of the expression in the statement of the lemma is equal to the image of

$$\exp(\varepsilon^p e_{12}) \otimes \exp(\varepsilon^q e_{21}) - \exp(\varepsilon^q e_{21}) \otimes \exp(\varepsilon^p e_{12}).$$

As $\exp(\varepsilon^q e_{21})(v_2) = v_2$, we see, using (4.2.5), that $\exp(\varepsilon^q e_{21}) \otimes \exp(\varepsilon^p e_{12})$ maps to 0. Finally using (4.2.5), and the definition of β and $li_{2,(2n-1,n),w}$ we see that

$$\exp(\varepsilon^p e_{12}) \otimes \exp(\varepsilon^q e_{21})$$

maps to $li_{2,(2n-1,n),w}((1-\varepsilon^p) \wedge (1-\varepsilon^q))$. □

Lemma 4.2.8. *For $p+q=w$, and $p,q \geq 1$:*

$$li_{2,(2n-1,n),w}(\underline{T}''(\Gamma_{\varepsilon^p,\varepsilon^q})) = 3li_{2,(2n-1,n),w}((1-\varepsilon^p) \wedge (1-\varepsilon^q)).$$

Proof. This follows from Lemma 4.2.4, Lemma 4.2.6, and Lemma 4.2.7, together with the fact, which is immediate from the definition of $li_{2,(2n-1,n),w}$, that

$$li_{2,(2n-1,n),w}((1-\varepsilon^p) \wedge (1-\varepsilon^q)) = li_{2,(2n-1,n),w}((1+\varepsilon^p) \wedge (1+\varepsilon^q)).$$

□

Let $[|\cdot|]$ denote the greatest integer function.

Theorem 4.2.9. *With the notation as in §4.1.2, $Li_{2,n}(T(\alpha_w)) =$*

$$-([|\frac{2n-1-w}{2}|] + w - n + 1 + \frac{\chi(w)}{2}),$$

if $w \neq 2n-1$; and $-\frac{2n-1}{2}$, if $w = 2n-1$.

Proof. Since $Li_{2,n}(T(\alpha_w)) = Li_{2,n,w}(T'(\beta_w))$, using Lemma 4.2.1, (4.2.2), (4.2.4) we see that the left hand side of the expression in the statement of the theorem equals to $(1/3)li_{2,(2n-1,n),w} \circ \underline{T}''$ evaluated on

$$\sum_{0 \leq j < (2n-1-w)/2} (\Gamma_{\varepsilon^{w-1},\varepsilon} - \Gamma_{\varepsilon^{w-n+j},\varepsilon^{n-j}} - \Gamma_{\varepsilon^{n-1-j},\varepsilon^{w-n+j+1}})$$
$$+ \frac{1}{2}\chi(w)(\Gamma_{\varepsilon^{w-1},\varepsilon} - 2\Gamma_{\varepsilon^{(w-1)/2},\varepsilon^{(w+1)/2}}).$$

Using Lemma 4.2.8 and the definition of $li_{2,(2n-1,n),w}$ we see that:

(i) if $w \neq 2n-1$: then the contribution from $j=0$ is $-(w-n+1)$; the contribution from each of the terms where $0 < j$ is -1; and from the last term is $-\frac{1}{2}\chi(w)$

(ii) if $w = 2n-1$: there is only one contribution, coming from the last term, and this is $\frac{1}{2}\chi(2n-1)(-1-2(n-1)) = -\frac{1}{2}(2n-1)$. □

4.3. Proof of Theorem 1.3.2. By Goodwillie's theorem (Theorem 3.1.1), Remark 3.1.2, §4.1.1 and §4.1.2, all we need to show is that $Li_{2,n,w} : (k \simeq) HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)}_{\langle w \rangle} \to k$ is an isomorphism. We know that this map is nonzero by Theorem 4.2.9, and replacing ε by $\lambda\varepsilon$ has the effect of multiplication by λ^w, using the vector space structures on both sides (Proposition 8.1, [15]). This immediately implies the theorem when k is algebraically closed. In the general case, we just need to use Theorem 1.3.2 for $\overline{k}$, and the equivariance of $Li_{2,n,w}$ with respect to $Gal(\overline{k}/k)$ and take galois invariants on both sides.

5. The complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$

5.1. To compute the kernel of δ_n in Theorem 1.3.1, we will need the following proposition. Following Suslin's notation, let $T_m(A) \subseteq \mathrm{GL}_m(A)$ denote the subgroup of diagonal matrices.

Proposition 5.1.1. *The map $\rho_2 : H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \to ker(\delta_n)$, defined in §3.8, has the property that*

$$\rho_2(H_3(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q}) = ker(\delta_n)$$

and

$$H_3(T_3(k[\varepsilon]_n), \mathbb{Q}) \subseteq ker(\rho_2).$$

Proof. The first statement is proved in the case of fields in §2 of [23]. The same proof works for $k[\varepsilon]_n$, if in the first line of p. 222 in [23], we use Theorem 2.2.2 of [14] to show that $H_*(T_2(k[\varepsilon]_n), \mathbb{Q}) = H_*(UT_2(k[\varepsilon]_n), \mathbb{Q})$, where $UT_2(A)$ denotes upper triangular matrices in $\mathrm{GL}_2(A)$ (this is denoted by $B_2(A)$ in [23]). We note that there is a slight difference between the construction of our map ρ_2 and the corresponding map in [23]. Namely Suslin uses configurations in the projective space rather than the affine space, but the resulting maps $H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \to ker(\delta_n)$ are the same.

The proof of Proposition 3.1 in [23] works for $k[\varepsilon]_n$ as well, proving the second statement. □

Proposition 5.1.2. *The map $T : HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)} \to ker(\delta_n)^{\circ}$ (c.f. Notation 2.0.1), defined in Proposition 3.8.9, is surjective.*

Proof. Because of Proposition 5.1.1, Theorem 3.1.1 and Remark 3.1.2, it suffices to show that the image of $K_3(k[\varepsilon]_n)^{(2)}_{\mathbb{Q}}$ in $H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}/H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}$, under the composition of the maps

$$\begin{aligned} K_3(k[\varepsilon]_n)^{(2)}_{\mathbb{Q}} &\to K_3(k[\varepsilon]_n)_{\mathbb{Q}} \to H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q}) \simeq H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \\ &\to H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}/H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^{\circ} \end{aligned}$$

contains the image of $H_3(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q})$ in $H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}/H_3(T_3(k[\varepsilon]_n), \mathbb{Q})^{\circ}$.

For a graded vector space V, let ΛV, denote the graded symmetric algebra over V. By the Milnor-Moore theorem $H_*(\mathrm{GL}(A), \mathbb{Q}) \simeq \Lambda((K_*(A)_{\mathbb{Q}})_{>0})$, (Corollary 11.2.12, [16]); and the stability theorem $H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) = H_3(\mathrm{GL}(k[\varepsilon]_n), \mathbb{Q})$ (§2, [14]). Combining these we obtain:

$$H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) = \Lambda^3 K_1(k[\varepsilon]_n)_{\mathbb{Q}} \oplus (K_1(k[\varepsilon]_n)_{\mathbb{Q}} \otimes K_2(k[\varepsilon]_n)_{\mathbb{Q}}) \oplus K_3(k[\varepsilon]_n)_{\mathbb{Q}}.$$

The first two components of the decomposition lie inside

$$H_1(\mathrm{GL}_1(k[\varepsilon]_n), \mathbb{Q}) \otimes H_2(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q}) \subseteq H_3(T_3(k[\varepsilon]_n), \mathbb{Q}),$$

(by the proof of Lemma 4.2, [23]; [14]). Therefore it suffices to prove that the image of $K_3(k[\varepsilon]_n)^{(2)}_{\mathbb{Q}}$ under the canonical projection

$$H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \to \mathrm{Prim} H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}) \to (\mathrm{Prim} H_3(\mathrm{GL}_3(k[\varepsilon]_n), \mathbb{Q}))^{\circ}$$

contains the image of $H_3(\mathrm{GL}_2(k[\varepsilon]_n), \mathbb{Q})$.

By the construction of ρ_1 in §3.2-3.7 and Remark 3.1.2 the last statement translates to showing that the image $im(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q}))$ of $H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})$ in

$$(\mathrm{Prim}H_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q}))^\circ = HC_2(k[\varepsilon]_n)^\circ = HC_2(k[\varepsilon]_n)^{\circ(1)} \oplus HC_2(k[\varepsilon]_n)^{\circ(2)}$$

is contained in $HC_2(k[\varepsilon]_n)^{\circ(1)}$.

First note that α_w for $n+1 \leq w \leq 2n-1$ form a basis for $HC_2(k[\varepsilon]_n)^{\circ(1)}$ by Theorem 4.2.9 and §4.1.1. By Lemma 4.2.1, Lemma 4.2.2 and Proposition 4.2.3 and the discussion following it, the image of α_w in $H_3(\mathfrak{gl}_3(k[\varepsilon]_n))^\circ$ is equal to that of an element $\gamma'_w \in H_3(\mathfrak{gl}_2(k[\varepsilon]_n))^\circ$. This implies immediately that $HC_2(k[\varepsilon]_n)^{\circ(1)} \subseteq im(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q}))$.

On the other hand Theorem 10.3.4 and Theorem 4.6.8 in [16] and Remark 6.10 in [17] imply that there is a natural map

$$(\mathrm{Prim}H_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q}))^\circ / im(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})) \to HC_2(k[\varepsilon]_n)^{\circ(2)}$$

which induces an automorphism of $HC_2(k[\varepsilon]_n)^{\circ(2)}$ when precomposed with

$$HC_2(k[\varepsilon]_n)^{\circ(2)} \to (\mathrm{Prim}H_3(\mathfrak{gl}_3(k[\varepsilon]_n), \mathbb{Q}))^\circ / im(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})).$$

These imply that $im(H_3(\mathfrak{gl}_2(k[\varepsilon]_n), \mathbb{Q})) = HC_2(k[\varepsilon]_n)^{\circ(1)}$ and hence the proposition.

□

The corollary below computes the infinitesimal part of the first cohomology of the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$. Note that $H^1(\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}})^\circ = ker(\delta_n)^\circ$.

Corollary 5.1.3. *The maps* $T : HC_2(k[\varepsilon]_n, (\varepsilon))^{(1)} \to ker(\delta_n)^\circ$ *and* $Li_{2,n} : ker(\delta_n)^\circ \to k^{\oplus n-1}$ *are isomorphisms.*

Proof. This follows immediately from the fact that T is surjective (Proposition 5.1.2) and that $Li_{2,n} \circ T$ is an isomorphism (Theorem 1.3.2).

□

Proposition 5.1.4. *There are natural isomorphisms*

$$H^2(\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}})^\circ \simeq HC_1(k[\varepsilon]_n)^\circ = HC_1(k[\varepsilon]_n))^{\circ(1)} \simeq \oplus_{1 \leq i \leq n-1} \Omega^1_k.$$

Proof. Note that by the definition of Milnor K-theory (11.1.16, [16])

$$H^2(\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}) = K_2^M(k[\varepsilon]_n). \tag{5.1.1}$$

Since

$$K_2^M(k[\varepsilon]_n) = K_2(k[\varepsilon]_n) \tag{5.1.2}$$

(§4.2, [14]),

$$K_2(k[\varepsilon]_n)^\circ = HC_1(k[\varepsilon]_n)^\circ = \Omega^1_{k[\varepsilon]_n} / (\Omega^1_k + d(k[\varepsilon]_n)) \simeq \oplus_{1 \leq i \leq n-1} \Omega^1_k$$

(Proposition 2.1.14, [16]).

Finally that $HC_1(k[\varepsilon]_n) = HC_1(k[\varepsilon]_n)^{(1)}$ follows from Theorem 4.6.7 [16].

□

5.2. Proof of Theorem 1.3.1. Over k this is the main theorem in [23]. However, note that there the indecomposable quotient $K_3(k)_{ind,\mathbb{Q}}$ of $K_3(k)_{\mathbb{Q}}$ appears instead of $K_3(k)^{(2)}_{\mathbb{Q}}$. To see that these two groups are canonically isomorphic see p. 207 in [18]. Therefore we only need to compute the cohomology of the infinitesimal part of the complex $\gamma_{k[\varepsilon]_n}(2)_{\mathbb{Q}}$. And this is done in Corollary 5.1.3 and (5.1.1) and (5.1.2).

6. Comparison with the additive dilogarithm of Bloch-Esnault

In this section we compare the complex $\gamma_{k[\varepsilon]_n}(2)^{\circ}_{\mathbb{Q}}$ to the complex $T_n\mathbb{Q}(2)(k)$ of Bloch-Esnault.

6.1. The reduced complex. In order to make the comparison we first define a subcomplex of $\gamma_{k[\varepsilon]_n}(2)^{\circ}_{\mathbb{Q}}$: let $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ be the subcomplex of $\gamma_{k[\varepsilon]_n}(2)^{\circ}_{\mathbb{Q}}$ whose degree 2 term is $k^{\times} \otimes V_n^{\circ} \subseteq (\Lambda^2 V_n)^{\circ}$ and whose degree 1 term is the inverse image

$$\delta_n^{-1}(k^{\times} \otimes V_n^{\circ}) \subseteq B_2(k[\varepsilon]_n)^{\circ}_{\mathbb{Q}}.$$

Denote this last group by $B_2(k[\varepsilon]_n)'_{\mathbb{Q}}$. Then we have

$$\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}} : B_2(k[\varepsilon]_n)'_{\mathbb{Q}} \to k^{\times} \otimes V_n^{\circ}.$$

We will need the following lemma to compute the cohomology of this reduced complex:

Lemma 6.1.1. *The natural map $(k^{\times})^{\otimes(i-1)} \otimes k[\varepsilon]_n^{\times} \to K_i^M(k[\varepsilon]_n)$ is a surjection.*

Proof. By the definition of Milnor K-theory, it is clear that it suffices to prove the lemma for $i = 2$. In this case the lemma follows from the isomorphism

$$K_2(k[\varepsilon]_n) \simeq K_2(k) \oplus \frac{\Omega^1_{k[\varepsilon]_n}}{\Omega^1_k + d(k[\varepsilon]_n)},$$

(Theorem 3, [13]) and the observation that $k^{\times} \otimes k[\varepsilon]_n^{\times}$ surjects onto the expression on the right, under this isomorphism. Note that $K_2^M(k[\varepsilon]_n) = K_2(k[\varepsilon]_n)$ [14]. □

Proposition 6.1.2. *The inclusion $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}} \to \gamma_{k[\varepsilon]_n}(2)^{\circ}_{\mathbb{Q}}$ is a quasi-isomorphism.*

Proof. The only thing that needs a justification is the surjectivity of the induced map on the degree 2 cohomology groups or equivalently the surjectivity of the composition

$$k^{\times}_{\mathbb{Q}} \otimes_{\mathbb{Q}} V_n^{\circ} \to (\Lambda^2 V_n)^{\circ} \to \Omega^1_{k[\varepsilon]_n}/(\Omega^1_k + d(k[\varepsilon]_n)),$$

where the last map is the one in the proof of Proposition 5.1.4. But this is exactly Lemma 6.1.1. □

6.2. The construction of Bloch-Esnault with higher modulus. For the rest of the section we assume that k is algebraically closed. In [5], Bloch and Esnault construct the additive weight 2 complex with modulus 2, the proof of Bloch-Esnault goes through to give a construction for all moduli $n \geq 2$. We describe the properties of this complex below. The proofs and the details of the construction can be found in §2 of [5].

Following the notation of [5], let R be the local ring of 0 in $\mathbb{A}^1_k$. The localization (away from 0) sequence for the pair $(k[t],(t^n))$, splits into the exact sequences:

$$K_2(k[t],(t^n)) \to K_2(R,(t^n)) \xrightarrow{\partial} \oplus_{x\in k^\times} K_1(k) \to K_1(k[t],(t^n)) \to 0$$

and

$$0 \to K_1(R,(t^n)) \xrightarrow{\partial} \oplus_{x\in k^\times} K_0(k) \to K_0(k[t],(t^n)) \to 0,$$

since $K_0(R,(t^n)) = 0$; and the map $K_1(R,(t^n)) \to \oplus_{x\in k^\times} K_0(k)$ is injective, as $K_1(R,(t^n)) = 1+(t^n)$ and the map is given by the divisor of the function (Appendix, [18]). This description also gives a canonical identification

$$K_0(k[t],(t^n)) = (k[t]_n^\times)^\circ.$$

Using the product structure on K-theory let

$$T_nB_2(k) := (K_2(R,(t^n))/im(K_1(k)\cdot K_1(R,(t^n)))_{\mathbb{Q}},$$

and let $T_nH^1_M(k,2)$ be the image of $K_2(k[t],(t^n))_{\mathbb{Q}}$ in $T_nB_2(k)$. Then the above exact sequences give the following exact sequence:

$$(6.2.1)\quad 0 \to T_nH^1_M(k,2) \to T_nB_2(k) \to k^\times \otimes V_n^\circ \to K_1(k[t],(t^n))_{\mathbb{Q}} \to 0.$$

We let

$$T_n\mathbb{Q}(2)(k) : T_nB_2(k) \to k^\times \otimes V_n^\circ$$

denote the middle part of this sequence. This is the exact generalization of the complex considered by Bloch and Esnault in [5] (the complex described in §1.2) to higher moduli.

We will try to express the cohomology groups of $T_n\mathbb{Q}(2)(k)$ in terms of the groups $K_*(k[t]_n,(t))_{\mathbb{Q}}$.

First note that the long exact sequence for the pair $(k[t],(t^n))$, together with the homotopy invariance of K-theory gives canonical isomorphisms

$$K_{*+1}(k[t]_n,(t)) \simeq K_*(k[t],(t^n)),$$

and therefore that there is a surjection

$$(6.2.2)\ \Big(\frac{K_3(k[t]_n,(t))}{K_1(k)\cdot K_2(k[t]_n,(t))}\Big)_{\mathbb{Q}} \simeq \Big(\frac{K_2(k[t],(t))}{K_1(k)\cdot K_1(k[t],(t))}\Big)_{\mathbb{Q}} \to T_nH^1_M(k,2).$$

Lemma 6.2.1. *There is a canonical surjection $K_3(k[t]_n,(t))^{(2)}_{\mathbb{Q}} \to T_nH^1_M(k,2)$.*

Proof. By p. 191, [18], $K_3(k[t]_n)_{\mathbb{Q}} = K_3(k[t]_n)^{(2)}_{\mathbb{Q}} \oplus K_3^M(k[t]_n)_{\mathbb{Q}}$, and by Lemma 6.1.1, the image of $K_1(k)\otimes K_2(k[t]_n)$ in $K_3(k[t]_n)$ is $K_3^M(k[t]_n)$. Hence that (6.2.2) is a surjection proves the lemma.

□

Let

$$\rho : T_nB_2(k) = \Big(\frac{K_2(R,(t^n))}{K_1(k)\cdot K_1(R,(t^n))}\Big)_{\mathbb{Q}} \to \Big(\frac{K_2(k[t]_{2n-1},(t^n))}{K_1(k)\cdot K_1(k[t]_{2n-1},(t^n))}\Big)_{\mathbb{Q}} =: N$$

denote the map induced by reduction modulo (t^{2n-1}). We will prove that ρ behaves like an additive dilogarithm in this setting.

Proposition 6.2.2. *The composition* $K_3(k[t]_n,(t))^{(2)}_{\mathbb{Q}} \to T_nH^1_M(k,2) \to N$, *induced by the inclusion* $K_3(k[t]_n,(t))^{(2)}_{\mathbb{Q}} \to K_3(k[t]_n,(t))_{\mathbb{Q}}$, *(6.2.2), and* ρ *is an isomorphism.*

Proof. This map is induced by the long exact sequence of the pair $(k[t]_{2n-1},(t^n))$:

$$\cdots \to K_3(k[t]_n,(t)) \to K_2(k[t]_{2n-1},(t^n)) \to K_2(k[t]_{2n-1},(t)) \to \cdots .$$

By Goodwillie's theorem, Remark 3.1.2 and §4.1.1, the map $K_3(k[t]_{2n-1},(t))^{(2)}_{\mathbb{Q}} \to K_3(k[t]_n,(t))^{(2)}_{\mathbb{Q}}$ is equivalent to a map $k^{\oplus(2n-2)} \to k^{\oplus(n-1)}$, where the $k^\times$-weights in the source range in [2n,4n-3] whereas in the target they range in $[n,2n-1]$. Therefore this last map is zero and hence

$$K_3(k[t]_n,(t))^{(2)}_{\mathbb{Q}} \to K_2(k[t]_{2n-1},(t^n))_{\mathbb{Q}}$$

is injective.

By Theorem 1.11 in [22], $K_2(k[t]_{2n-1},(t^n))_{\mathbb{Q}} \simeq k^{\oplus(n-1)} \oplus (\Omega^1_k)^{\oplus(n-1)}$, and $K_1(k) \otimes K_1(k[t]_{2n-1},(t^n)) \to K_2(k[t]_{2n-1},(t^n))_{\mathbb{Q}}$ has image exactly $(\Omega^1_k)^{\oplus(n-1)}$. This proves the proposition

□

Corollary 6.2.3. *There are canonical isomorphisms:*

$$H^1(T_n\mathbb{Q}(2)(k)) \simeq K_3(k[t]_n,(t))^{(2)}_{\mathbb{Q}} \simeq HC_2(k[t]_n,(t))^{(1)},$$

$$H^2(T_n\mathbb{Q}(2)(k)) \simeq K_2(k[t]_n,(t))_{\mathbb{Q}} \simeq HC_1(k[t]_n,(t)).$$

Proof. The first isomorphism is an immediate consequence of Lemma 6.2.1 and Proposition 6.2.2, and the second one a consequence of the isomorphism

$$K_2(k[t]_n,(t)) \simeq K_1(k[t],(t^n)),$$

which follows from the long exact sequence for $(k[t],(t^n))$ and the homotopy invariance of K-theory.

□

6.3. Proof of Corollary 1.4.1. First we note that the degree 2 terms of $T_n\mathbb{Q}(2)(k)$ and $\gamma_{k[\varepsilon]_n}(2)'_{\mathbb{Q}}$ are both equal to $k^\times \otimes V_n^\circ$ and that the cohomology groups of the two complexes are canonically isomorphic (Theorem 1.3.1, Proposition 6.2.2, and Corollary 6.2.3). Moreover in both case the projection from $k^\times \otimes V_n^\circ$ to the degree 2 cohomology is induced by the composition

$$k^\times \otimes V_n^\circ \to K_2^M(k[\varepsilon]_n) \to \Omega^1_{k[\varepsilon]_n}/(\Omega^1_k + d(k[\varepsilon]_n))$$

(cf. proof of Lemma 6.1.1). Therefore the images of $T_nB_2(k)$ and of $B_2(k[\varepsilon]_n)'_{\mathbb{Q}}$ in $k^\times \otimes V_n^\circ$ are the same. The exact sequence (6.2.1) and Proposition 6.2.2 give a splitting of $T_nB_2(k)$; and Theorem 1.3.1 and 1.3.2 give a splitting of $B_2(k[\varepsilon]_n)'_{\mathbb{Q}}$. This proves the corollary.

We would like to emphasize that the isomorphism given in the statement of the corollary uses the additive dilogarithm in both constructions and thus should not be considered as natural.

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