# The Stable Set in Exchange of Discrete Resources[*]

Ömer Faruk Şahin[†] Murat Yılmaz[‡] Özgür Yılmaz[§]

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**Abstract**

A central stability notion for exchange economies with discrete resources is *core*: no coalition credibly objects the allocation by suggesting an alternative allotment that is better for its members and can be implemented by its members alone (that is, no coalition *blocks* the allocation). But, *core* is in general empty. We propose a novel stability notion for this context. We consider an environment where agents foresee that a *Pareto inefficient* allocation is not plausible since it is *blocked* by all agents, and therefore, starting from the endowment, they engage in a process of *Pareto improving* updates. But, only some *Pareto improvements* are *stable*: a *(weak) Pareto improvement* over an assignment is a *stable improvement* if it *weakly blocks* any other *Pareto improvement.* A *stable set* is a set of assignments such that (i) (*external stability*) no assignment outside this set is a *stable improvement* over some assignment in the set, (ii) no strict subset of it is *externally stable.* The *(stable improvement) frontier of the stable set* is the set of assignments in the *stable set,* which do not admit a *stable improvement.* Our main result characterizes the *frontier of the stable set* by the set of outcomes of the well-known *Top Trading Cycles* solution.

**Keywords** : Stable Set, Stable Improvement, Core, Von Neumann-Morgenstern Stable set, Bargaining Set, Top Trading Cycles

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[†]Koç University, Sarıyer, İstanbul, Turkey 34450; *E-mail address:* osahin18@ku.edu.tr.

[‡]Department of Economics, Boğaziçi University, Bebek, İstanbul, 34342; *E-mail address:* muraty@boun.edu.tr.

[§]Corresponding author. Koç University, Sarıyer, İstanbul, Turkey 34450; *E-mail address:* ozyilmaz@ku.edu.tr.

# 1 Introduction

In an exchange economy of discrete resources with private endowments, each agent is endowed with an indivisible good (an object), and monetary transfers are not allowed. A central concept for these economies is *stability*, which captures resilience of an assignment against coalitional deviations. A well-known such property is *core*: an assignment is in the *core* if there is no coalition such that each agent in it prefers reallocating their endowments among themselves (by leaving the economy) over the assignment, that is, if no coalition *blocks* the assignment. Unfortunately, *core* is in general empty.[1] We propose a novel concept for *stability*. A (weak) *Pareto improvement* over an assignment is a *stable improvement* if it *weakly blocks* any other *Pareto improvement*. A *stable set* is a set of assignments such that (i) (*external stability*) no assignment out of this set is a *stable improvement* over some assignment in the set, and (ii) no strict subset of it is *externally stable*. The *(stable improvement) frontier of the stable set* is the set of assignments in the *stable set*, which do not admit a *stable improvement*. We show that the *frontier of the stable set* characterizes the set of outcomes of the well-known *Top Trading Cycles* ($TTC$) solution.

The central notion in our approach is *stable improvement*: Agents have a **farsighted** view on possible outcomes. **First**, they foresee that a *Pareto inefficient* outcome is not possible since such an outcome will be *blocked* by all agents. Thus, a plausible allocation is necessarily an outcome of a process of *Pareto improvements*. **Second**, agents deem a *Pareto improving* blocking of an initial assignment as *stable* (we call it a *stable improvement*) if it *weakly blocks* any other *Pareto improvement* over the initial assignment.[2] Thus, a *stable improvement* is justified in the sense that any potential objection to it (by means of an alternative improvement) is counter-objected by at least one group of agents (that is, there exists at least one coalition which *weakly blocks* the alternative improvement).

This approach to *stability* is essentially a sequential process of *blockings* via *stable improvements*. Thus, while certain *Pareto efficient* assignments survive this process as an outcome of a sequence of *stable improvements*, certain other *Pareto efficient* assignments do not. Specifically, the *frontier of the stable set* characterizes the set of outcomes of the $TTC$ solution (Theorem 2). This provides a theoretical foundation for why the $TTC$ solution excludes certain *Pareto efficient* assignments, from the perspective of *stability*. We next provide an example to illustrate the concept of *stable improvement* and to distinguish the *Pareto efficient* outcomes, which are obtained through a sequence of *stable improvements*, from other *Pareto efficient* outcomes.

[1] *Core* is non-empty only under a very specific structure of the exchange economy (see Section 2 for a discussion of this structure).

[2] *Weak blocking* is when each agent in the blocking coalition is weakly better off.

**An illustrative example:** Let $\{i_1, i_2, i_3, i_4\}$ be the set of agents and $\{o_1, o_2, o_3, o_4\}$ the set of objects. We consider the exchange economy where for each $k$, object $o_k$ is the endowment of agent $i_k$. Let us denote this endowment profile by $e$. The preference profile $R$ is as in the following table in the form of ranking such that the agent is indifferent between the objects in a given set:

| $R_{i_1}$ | $R_{i_2}$ | $R_{i_3}$ | $R_{i_4}$ |
|---|---|---|---|
| $o_2$ | $\{o_1, o_3\}$ | $o_2$ | $o_2$ |
| $\{o_3, o_4\}$ | $o_2$ | $o_1$ | $o_4$ |
| $o_1$ | | $o_3$ | |

Let us denote an assignment by a tuple $\mu = (\mu(i_1), \mu(i_2), \mu(i_3), \mu(i_4))$ where $\mu(i)$ denotes the object assigned to $i$. First, note that the *core* of this economy is empty. Also, there are three *Pareto efficient* assignments: $\mu_1 = (o_2, o_3, o_1, o_4)$, $\mu_2 = (o_3, o_1, o_2, o_4)$, and $\mu_3 = (o_4, o_3, o_1, o_2)$. Both $\mu_1$ and $\mu_2$ are such that the endowments of agents $i_1$, $i_2$ and $i_3$ are assigned to these agents and $i_4$ remains unassigned. Suppose first agents $i_1$ and $i_2$ *Pareto improve* $e$ via $\bar{\mu}_1 = (o_2, o_1, o_3, o_4)$. Since agents $i_1$ and $i_2$ are assigned one of their best objects, they *weakly block* any other *Pareto improvement* over $e$ (that is, each of them *weakly* prefers $\bar{\mu}_1$ over any such assignment). Thus, agents $i_1$ and $i_2$ form a coalition and via $\bar{\mu}_1$, they counter-object to any other *Pareto improvement* over $e$. Thus, $\bar{\mu}_1$ is a *stable improvement*. Moreover, $\mu_1$ is the only *Pareto improvement* over $\bar{\mu}_1$ and thus, a *stable improvement*. Since, assuming the welfare levels conceded under $\bar{\mu}_1$, there is no objection to $\mu_1$, it is a *stable improvement* as well. Thus, $\mu_1$ is achieved by a process of *stable improvements*. Similarly, the assignment $\bar{\mu}_2 = (o_1, o_3, o_2, o_4)$ is a *stable improvement* over $e$, and $\mu_2$ is a *stable improvement* over $\bar{\mu}_2$. Thus, this is another sequence of *stable improvements*. Let us now consider $\mu_3$. There are two *Pareto improvements* over $e$, where $i_4$ is assigned $o_2$: $\mu_3$ and $\bar{\mu}_3 = (o_4, o_1, o_3, o_2)$. Assignment $\bar{\mu}_3$ is not a *stable improvement* over $e$ since it does not *weakly block* $\bar{\mu}_1$, another *Pareto improvement* over $e$. (On the other hand, $\bar{\mu}_3$ *weakly blocks* $\bar{\mu}_2$.) Also, $\mu_3$ is not a *stable improvement* over $e$ since it does not *weakly block* $\mu_1$ and $\bar{\mu}_1$ (nor $\mu_2$ and $\bar{\mu}_2$), which are both *Pareto improvements* over $e$. Interestingly, $\mu_3$ is a *stable improvement* over $\bar{\mu}_3$ since it is the only *Pareto improvement* over $\bar{\mu}_3$. But, since $\bar{\mu}_3$ is not a *stable improvement* over $e$ (actually, not a *stable improvement* over any assignment), $\mu_3$ cannot be obtained as a sequence of *stable improvements*. Moreover, while $\mu_1$ and $\mu_2$ are the outcomes of the $TTC$, $\mu_3$ cannot be obtained by the $TTC$ through any cycle selection rule (see Section 4 for the detailed description of the $TTC$.) In Figure 1, we show each *stable improvement* by a red arrow. The blue arrows represent *Pareto improvements* which are not *stable*. Also, the black arrows and crossed black arrows represent *weakly blocking* and no *weak blocking*, respectively. (For expositional convenience, we include only the relevant arrows of *improvements* and *blockings*.)

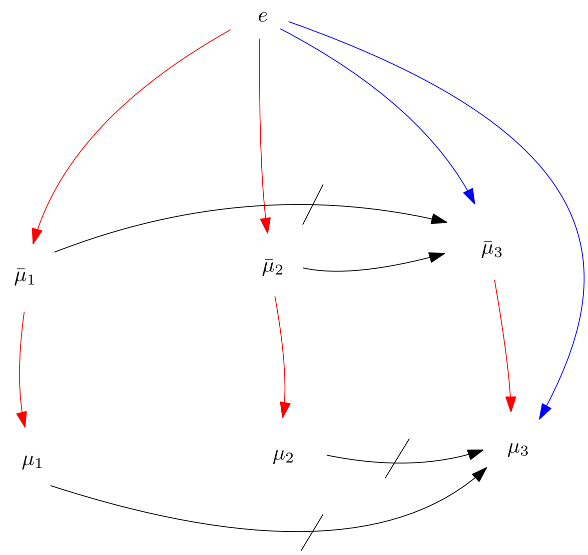


Figure 1: Stable improvements

This example also demonstrates that the structure of welfare-improving blockings embedded in the concept of the *stable set* is essential.[3] If we assign agents $i_1$ and $i_2$ to each other's endowments, then we obtain $\bar{\mu}_1$, which is not *Pareto efficient*. In this case, efficiency improvement is possible by "expanding" this coalition to include $i_3$ through the assignment $\mu_1$ ($i_1$ and $i_2$ remains indifferent and $i_3$ is better off). Note that $\mu_1$ is *blocked* by $\bar{\mu}_2$, but it does not *counter-block* this blocking, since agent $i_3$ would strictly prefer $\bar{\mu}_2$ over $\mu_1$. Thus, by allowing this coalition expansion, $i_1$ becomes worse-off at $\bar{\mu}_2$, and $i_1$ and $i_2$ cannot restore their welfare they achieve through their initial coalition.[4] This contradicts agents' farsighted view on coalition formation to improve welfare. To prevent this, in the definition of *stable improvement*, we impose not to let $i_3$ *block* $\mu_1$ (the assignment of the expanded coalition) by another assignment that would be incompatible with the welfare under the initial coalition of $i_1$ and $i_2$. Thus, we restrict the assignments to those that *Pareto improve* $\bar{\mu}_1$, and prevent $i_3$ from reclaiming $o_2$ after $i_1$ and $i_2$ form a coalition.

**Overview of our results:** We propose a novel concept of *stability* through an incremental process of *stable improvements*. We define an *externally stable set* as a set of assignments (including the endowment profile) such that no assignment outside this set is a *stable improvement* over some assignment in this set. A *stable set* is the *minimally externally stable set*, which is uniquely defined.[5] Conceptually, our definition of *external stability* is in the same spirit with *external stability* of *Von Neumann-Morgenstern stable set* (see the next section for a discussion). In the example above, as-

[3]In a setting with strict preferences, when agents are endowed with each other's most preferred objects, *stability* would imply that they are assigned each other's endowments since otherwise, they would constitute a coalition such that it *blocks* any other assignment. This is also the reason for why the *TTC* solution characterizes the *core* in the strict preferences domain, as it assigns the endowments of the agents in these "natural" coalitions among themselves. But, this simple structure does not exist in larger domains.

[4]Also, note that allowing any blocking potentially implies a circular process of blockings.

[5]See Remark 3.

signment $\bar{\mu}_3$ is not a *stable improvement* over no assignment, and also $\mu_3$ is a *stable improvement* only over $\bar{\mu}_3$. Thus, these assignments are considered not attainable through a sequence of *stable improvements* and they are outside the *stable set*.

An important remark at this point is that **the process of *stable improvements* is different than the process of *top trading cycles* of the $TTC$ solution.** The *stable set* is not equivalent to the set of assignments obtained through the interim stages of *top trading cycles* of the $TTC$ solution (see Lemma 3 and Remark 5 in Appendix B). Moreover, the *stable set* can get *exponentially larger* compared to the set of assignments that can be achieved in some interim stage of the TTC algorithm (see Remark 6). While these two processes do not coincide, surprisingly, as our main theorem states, they coincide at their frontiers: the *(stable improvement) frontier of the stable set* characterizes the outcomes of the $TTC$ (Theorem 2). This result provides further justifications for the well-known $TTC$ solution, specifically for its strong stability properties. A corollary to our main theorem is that the *core*, whenever non-empty, is equivalent to the *frontier of the stable set* (Corollary 2).

## 2 Literature

There are various concepts of *stability* in the context of allocation problems for discrete resources. We focus on *core*, *bargaining set* and *Von Neumann-Morgenstern stable set*.

**Core.** For an exchange economy with privately endowed discrete resources, *core* is a singleton under strict preferences, and it is the outcome of the $TTC$ solution (Shapley and Scarf, 1974). In the strict preferences domain, the $TTC$ works as follows: Each agent points to her most preferred available object and each object points to its owner. Since all agents and objects point, there is at least one cycle. Also, since preferences are strict no two cycle intersects. The algorithm selects a cycle and assigns to each agent in the cycle her most preferred available object (that is, the object she points at) and removes her with her assigned object. The algorithm terminates when all agents and objects are removed. The $TTC$ (therefore the *core*) solution is characterized by *individual rationality*, *Pareto efficiency* and *strategy-proofness* (Ma (1994), Sönmez (1999)).

The $TTC$ solution has a natural extension for the general preferences domain (Saban and Sethuraman, 2013). Also, there are various *strategy-proof* selections from this class (Jaramillo and Manjunath (2012), Alcalde-Unzu and Molis (2011), Saban and Sethuraman (2013)). But, *core* is not preserved and in general, it is empty in this domain. It is non-empty only under a very special and restrictive structure of the exchange economy, called the *top trading segmentation (TTS)* (Quint and Wako, 2004). This structure is as follows: Let each agent point to her best objects, and each object points to her owner. A set of objects and agents with the following three properties exists: (i) there is no pointing from this set to the outside of it, (ii) no non-empty strict subset of it satisfies property (i), and (iii) it is possible to assign each agent in this set to one of the objects she points to in this set. Such a set is called a *covered minimally self-mapped set* (see Section 3.1 for formal definitions). The agents and objects in this set are removed, and the remaining agents point to their best objects among the remaining objects. This smaller economy also has a *covered minimally self-mapped set* (note that for some agents in smaller economy, their best objects might not be available anymore). Once the second *covered minimally self-mapped set* is removed, there is another one in the remaining economy, and this continues until all agents are removed. Thus, an economy has a *TTS* structure when the economy can be partitioned into a sequence of mutually exclusive smaller economies (that is, into *covered minimally self-mapped sets*) in the specific way described.

**Theorem 1.** *(Quint and Wako, 2004) The core is non-empty if and only if the economy has a TTS structure.*

The *core*, if non-empty, is such that each agent is assigned a best object in the *covered minimally self-mapped set* she is in (Quint and Wako, 2004). The *core* is not necessarily single-valued, but it is *essentially single-valued*: each agent is indifferent between any two assignments in the *core*. These findings imply an immediate characterization result.

**Corollary 1.** *In an exchange economy with a TTS structure, an assignment is in the core if and only if it is an outcome of the TTC.*

**Bargaining set.** An assignment is in the *bargaining set* if blocking by a coalition implies that there is another coalition blocking the assignment resulting from the initial blocking (Aumann and Maschler, 1964). In school choice, this notion provides a weakening of *stability*, the central axiom in matching theory: if a student claims an empty slot at a school and thus, she has an objection to an allocation, then there will be a counter-objection once she is assigned to that school (since the priority of some other student will be violated at that school). A matching is in the *bargaining set* if and only if for each objection to the matching, there exists a counter-objection (referred to as *constrained non-wastefulness* (Ehlers, Hafalir, Yenmez, and Yildirim, 2014)).[6] For an exchange economy with discrete resources, each outcome of the $TTC$ solution is in the *bargaining set*, but not all assignments in the *bargaining set* can be obtained by the $TTC$ (Yılmaz and Yılmaz, 2022). In a market game with a continuum of players, the *bargaining set* characterizes Walrasian allocations (Mas-Colell, 1989). For non-transferable utility games, the *bargaining set* is non-empty

[6] There are other works studying *bargaining set* in the matching context (Ehlers (2010), Kesten (2010), Alcade and Romero-Medina (2017)).

under certain conditions (Vohra, 1991).[7] For an exchange economy with differential information and a continuum of traders, the *bargaining set* characterizes the set of Radner competitive equilibrium allocations (Einy, Moreno, and Shitovitz, 2001). While the *bargaining set* notion in these works takes into account only one step of counter-objection to a blocking coalition, a more refined axiom considers a chain of counter-objections (Dutta, Ray, Sengupta, and Vohra, 1989).

**Von Neumann-Morgenstern stable set.** A *Von Neumann-Morgenstern stable set* (or a set of *stable* allocations) is a set of allocations such that (i) (*internal stability*) no coalition *blocks* any *stable* allocation by suggesting another *stable* allocation; and (ii) (*external stability*) any *unstable* allocation is *blocked* by a coalition by suggesting a *stable* allocation. Clearly, the *core* satisfies *internal stability* but it may violate *external stability*. A set $V$ of matchings is a *Von Neumann-Morgenstern stable set* of a one-to-one matching problem only if $V$ is a maximal set satisfying the following properties: (i) the *core* is a subset of $V$, (ii) $V$ is a distributive lattice, and (iii) the set of unmatched agents is the same for all matchings in $V$ (Ehlers, 2007). While *core* and *Von Neumann-Morgenstern stable set* are myopic notions since only one step deviations are considered, the agents can also be farsighted where coalitional objections can be countered by subsequent counter-objections. A matching $\mu$ *indirectly dominates* $\mu'$ if $\mu$ replaces $\mu'$ in a sequence of matchings, such that at each matching along the sequence, all deviators are strictly better off at $\mu$ than at $\mu'$ (Harsanyi (1974), Chwe (1994)). The *Von Neumann-Morgenstern farsighted stable set* is obtained by replacing the myopic notion of *blocking* in the definition of *Von Neumann-Morgenstern stable set* with the farsighted notion of *indirect dominance* (Mauleon, Vannetelbosch, and Vergote, 2011). A set of matchings is a *Von Neumann-Morgenstern farsightedly stable set* if and only if it is a singleton subset of the *core* (Mauleon, Vannetelbosch, and Vergote, 2011).

**Discussion of our notion of stable set.** *Stable set* in the current work is similar to *Von Neumann-Morgenstern (farsightedly) stable set*. In the *Von Neumann-Morgenstern (farsightedly) stable set*, an assignment is excluded from this set and considered as an unlikely outcome since it is *blocked* (*indirectly dominated*) by an assignment in the *Von Neumann-Morgenstern (farsightedly) stable set* (*external stability*). Our concept of *stable set* relies on the same intuition with a different notion of *exclusion*: if an assignment $\mu$ *weakly Pareto improves* another assignment $\mu'$ in the *stable set* (implying that $\mu$ *weakly blocks* $\mu'$), but $\mu$ does not *block* all *Pareto improvements* over $\mu'$, we consider $\mu$ as not resilient to other *Pareto improving* assignments, thus as not a *stable improvement*. Thus, given a *stable* assignment, our notion tests the resilience of an assignment against group deviations only via other *Pareto improvements*.

[7] There are slight differences in the formulation of the bargaining set defined by Aumann and Maschler (1964) and Mas-Colell (1989). See Vohra (1991) for the differences between these two formulations and also other variants of the notion.

The notion of *stable set* is also inspired by the insight of *bargaining set*: a *stable improvement* $\mu$ over $\mu'$, by definition, is a counter-objection to other assignments, but, unlike *bargaining set*, these counter-objections are not to **any** other assignment, but only to certain assignments, that is, other *Pareto improvements* over $\mu'$. In the example given in the introduction, assignment $\bar{\mu}_1$ has the following property: it *blocks* any other *Pareto improving* assignment over the endowment. One such assignment is $\bar{\mu}_2$ and actually, $\bar{\mu}_2$ blocks $\bar{\mu}_1$. Thus, as in the *bargaining set*, $\bar{\mu}_1$ *counter-blocks* $\bar{\mu}_2$. In other words, any *stable improvement* $\mu$ over an initial assignment $\mu'$ *counter-blocks* any other *stable improvement* over $\mu'$, which blocks $\mu$. Thus, it belongs to the *bargaining set* when the set of *blockings* are restricted to the set of other *Pareto improvements*.

# 3 Model

Let $\mathcal{A}$ be a set of agents and $O$ a set of objects such that each agent is endowed with one object. An **endowment profile** is a bijection $e : \mathcal{A} \to O$. We call each set of agents $A \subseteq \mathcal{A}$ a **coalition** and denote the set of endowments of coalition $A$ by $e(A)$. Each agent $i$ has a complete and transitive preference relation $R_i$ on $O$; that is, we allow for indifferences. For each $i$, let $P_i$ and $I_i$ denote the strict and indifferences parts of $R_i$, respectively. Let $R = (R_i)_{i\in\mathcal{A}}$ be a preference profile.

An **assignment problem** is allocating objects in $O$ to agents in $\mathcal{A}$ in such a way that each agent receives exactly one object. We fix $\mathcal{A}$ and $O$ throughout the paper, and denote an *assignment problem* (or simply a *problem*) by a pair $(e, R)$.

An **assignment** $\mu$ is a bijection $\mu : \mathcal{A} \to O$. An *assignment* $\mu$ is **individually rational** if for each $i \in \mathcal{A}$, $\mu(i)\ R_i\ e(i)$. For each problem $(e, R)$, we denote the set of all *individually rational assignments* by $\mathcal{M}(e, R)$. Clearly, for each problem $(e, R)$, $e \in \mathcal{M}(e, R)$. Given an assignment $\mu \in \mathcal{M}(e, R)$, the **updated endowment profile under** $\mu$, denoted by $e_\mu$, is the assignment $\mu$, that is, for each $i \in \mathcal{A}$, $e_\mu(i) = \mu(i)$.

An *assignment* $\mu$ **Pareto dominates** $\mu'$, denoted by $\mu \succ \mu'$, if for each $i \in \mathcal{A}$, $\mu(i)\ R_i\ \mu'(i)$ and for some $j \in \mathcal{A}$, $\mu(j)\ P_j\ \mu'(j)$. Also, $\mu$ is **Pareto indifferent** to $\mu'$, denoted by $\mu \sim \mu'$, if for each $i \in \mathcal{A}$, $\mu(i)\ I_i\ \mu'(i)$. Whenever $\mu \succ \mu'$ or $\mu \sim \mu'$, we denote it by $\mu \succeq \mu'$. Also, $\mu$ **strictly Pareto dominates** $\mu'$, if for each $i \in \mathcal{A}$, $\mu(i)\ P_i\ \mu'(i)$. An *assignment* $\mu$ is **Pareto efficient** if there does not exist another assignment which *Pareto dominates* $\mu$. For each problem $(e, R)$, we denote the set of *Pareto efficient assignments* by $\mathcal{E}(e, R)$.

### 3.1 Graph theoretical representation

The definitions and the notation in this section follow closely existing works by Quint and Wako (2004) and Yılmaz and Yılmaz (2022). Let $G = (V, E)$ be a directed graph, where $V$ is the set of *vertices* and $E$ is the set of *directed edges*, that is a family of ordered pairs from $V$. For each $U \subset V$, let $\delta^{in}(U)$ be the set of edges $(u, v) \in E$ such that $u \in V \setminus U$ and $v \in U$ (i.e. the set of edges **entering** $U$) and $\delta^{out}(U)$ be the set of edges $(u, v) \in E$ such that $u \in U$ and $v \in V \setminus U$ (i.e. the set of edges **leaving** $U$). If $U$ is a singleton, say $U = \{v\}$, then we use $\delta^{in}(v)$ (and $\delta^{out}(v)$) instead of $\delta^{in}(U)$ (and $\delta^{out}(U)$). A **subgraph** of $G$ is any directed graph $G' = (V', E')$ with $\emptyset \neq V' \subseteq V$ and $E' \subseteq E$ and each edge in $E'$ consisting of vertices in $V'$. For a set of vertices $T \subseteq V$, the **subgraph of** $G$ **induced by** $T$ is the subgraph $(T, E')$ such that $E' = \{(u, v) \in E : u, v \in T\}$. A sequence of vertices $\{v_1, \dots v_m\}$ is a **path from** $v_1$ **to** $v_m$ if (i) $m \geq 1$, (ii) $v_1, \dots, v_m$ are distinct (except for possibly $v_1 = v_m$), and (iii) for each $k = 1, \dots, m-1$, $(v_k, v_{k+1}) \in E$. A **cycle** is a path $\{v_1, \dots v_m\}$ is a cycle if $m \geq 2$ and $v_1 = v_m$.

A set of vertices $T \subseteq V$ is **strongly connected** if the subgraph induced by $T$ is such that for any $u, v \in T$, there is a path from $u$ to $v$. A **self-mapped set** is a set of vertices $S \subseteq V$ such that $S = \bigcup_{v \in S} \delta^{out}(v)$[8]. A **minimally self-mapped set** is a *self-mapped set* such that no strict and non-empty subset of it *self-mapped*. The next observation follows from Proposition 2.2 by Quint and Wako (2004).

**Remark 1.** *Let $G = (V, E)$ be a directed graph. A set of vertices $S \subseteq V$ is non-empty and strongly connected such that $\delta^{out}(S) = \emptyset$ if and only if $S$ is a minimally self-mapped set.*

Whenever convenient, we refer to this equivalence result and say that a set of vertices $S$ is a *minimally self-mapped set* if (i) for any two vertices in $S$, there is a path from one to the other, and (ii) there is no path from any vertex $u \in S$ to any vertex $v \notin S$. The following follows directly from Remark 1 and the $\mathcal{MSMS}$ algorithm introduced by Quint and Wako (2004).

**Remark 2.** *Let $G = (V, E)$ be a directed graph. If for each $v \in V$, $\delta^{out}(v) \neq \emptyset$, then a minimally self-mapped set exists.*

Let $w : E \to \Re$ be a function. We denote $\sum_{e \in F \subseteq E} w(e)$ by $w(F)$. A function $f : E \to \Re$ is called a **circulation** if for each $v \in V$, $f(\delta^{in}(v)) = f(\delta^{out}(v))$. Let $d, c : E \to \Re$ with $d \leq c$. A *circulation* $f$ **respects d and c** if for each edge $e$, $c(e) \geq f(e) \geq d(e)$. A *minimally self-mapped set* $S$ is **covered** if there exists an integer-valued *circulation* $f$ such that for each $v \in S$, $f(e) = 1$ for some edge $e$ entering $v$.

[8] Note that $\bigcup_{v \in S} \delta^{out}(v)$ and $\delta^{out}(S)$ are different sets in general.

## 4 The class of the Top Trading Cycles (TTC) rules

The $TTC$ class is a set of assignment rules as an extension of the well-known $TTC$ mechanism defined on the strict domain. Agents have indifference classes and point to more than object. Thus, the problem is to select a particular cycle among many intersecting cycles. This is the crux in defining a particular mechanism.

Let $F$ be a selection rule: for each *minimally self-mapped set* that is not *covered*, $F$ selects one of the cycles in the *minimally self-mapped sets*. The $TTC$ updates the endowment profile by assigning each agent in the cycle to the object that she points to in the same cycle. Let $e_1 = e$ and for $n \geq 1$, the steps below are repeated until all agents and objects are removed.

**The $TTC$ Algorithm:**

**Step n.** Let each agent point to her best objects among the remaining objects[9] and each remaining object points to its owner according to the endowment profile $e_n$. Select a *minimally self-mapped set* $T_n$ in this digraph.

**(n.1)** If $T_n$ is *covered*, then each agent in $T_n$ is removed by assigning her one of the best objects in $T_n$.

**(n.2)** Otherwise, select one of the cycles in the *minimally self-mapped set* using the selection rule $F$, and update the endowment profile in the cycle to obtain $e_{n+1}$.

For each problem $(e, R)$, we denote the set of assignments that can be achieved with the $TTC$ algorithm by $TTC(e, R)$

## 5 Stable Set

A central concept in exchange economies is immunity to *coalition* formations where agents reallocate their endowments among themselves such that they are better off than the proposed assignment.

Let $(e, R)$ be a problem. A *coalition* $A \subseteq \mathcal{A}$ **weakly blocks** an assignment $\mu$ with $\mu'$ if $i \notin A$ implies $\mu'(i) = e(i)$, and for each $i \in A$, $\mu'(i)\ R_i\ \mu(i)$. Also, *coalition* $A$ **blocks** $\mu$ with $\mu'$, if it *weakly blocks* $\mu$ with $\mu'$ and for some $j \in A$, $\mu'(j)\ P_j\ \mu(j)$. The **core** is the set of assignments that are not *blocked* by any *coalition*. For each problem $(e, R)$, we denote the *core* by $C(e, R)$.

[9] At Step 1, the set of remaining objects is the set of all objects, $O$.

*Core* is in general empty and we need a different approach for *stability*. Our notion is as follows: Let $\mu'$ be an assignment and consider a *(weak) Pareto improvement* $\mu$ over $\mu'$. This *(weak) Pareto improvement* is *stable* if and only if it *weakly blocks* any other *Pareto improvement* over $\mu'$. In other words, any other *Pareto improving* alternative (or objection) to $\mu$ is counter-objected with $\mu$ by some coalition.

**Definition 1.** *An assignment $\mu$ is a* ***stable improvement*** *over $\mu'$ if $\mu \succeq \mu'$ and for each $\mu^* \succ \mu'$, $\mu$ weakly blocks $\mu^*$.*

We denote $\mu$ being a *stable improvement* over $\mu'$ by $\mu \succeq^s \mu'$. We also write $\mu \sim^s \mu'$ if $\mu \succeq^s \mu'$ and $\mu \sim \mu'$, and $\mu \succ^s \mu'$ if $\mu \succeq^s \mu'$ and $\mu \succ \mu'$. We define the set of *externally stable* assignments $M$ such that no assignment in $\mathcal{M}(e,R) \setminus M$ is a *stable improvement* over some assignment in $M$. Thus, for each assignment $\mu' \ in\ M$, each *(weak) Pareto improving* assignment over $\mu'$ is *weakly blocked* by another *Pareto improving* assignment over $\mu'$.

**Definition 2.** *Let $(e,R)$ be a problem. A set of assignments $M \subseteq \mathcal{M}(e,R)$ is* ***externally stable*** *if*

*(i)* $e \in M$, *and*

*(ii)* $\mu' \in M$ *and* $\mu \succeq^s \mu'$ *implies* $\mu \in M$.

*A* ***stable set*** *is an externally stable set such that no strict subset of it externally stable.*

### 5.1 Discussion of the stable set

The **first** point we should clarify is that our definition of *blocking* is more restrictive than the standard notion: whenever a coalition $A$ *(weakly) blocks* an assignment $\mu'$ with $\mu$, we impose that each agent $i \notin A$ is assigned to their endowment under $\mu$. We argue that this is the most natural restriction in the current context: *Core* as a stability notion requires that there are no coalitional deviations. Thus, since *blocking* is considered as a one-time deviation without further *blockings*, 'which objects are assigned to agents outside the *blocking* coalition' is irrelevant. But, in our context where there are sequential *blockings*, this is relevant and the least restrictive and natural assumption is that agents outside a *blocking* coalition are assigned to their endowments. The reason is that whenever a coalition *blocks* an assignment, existing exchange cycles of endowments are potentially broken, and these broken cycles necessarily imply that agents in these cycles, who are outside the *blocking* coalition, are necessarily left with their endowments. An important point here is that whenever an exchange cycle is not broken, our notion of *(weak) blocking* is not restrictive (see Example 1 in Appendix A). Moreover, this stronger *(weak) blocking* notion implies a stronger notion of *stable improvement*, thus a stronger notion of *stable set*. We discuss and argue for the implication of this strengthening in Section 6.

The **second** point is about the justification of *weak blocking* in the definition of *stable improvement* (rather than *blocking*). The reason is straightforward: if, instead, we use *blocking*, the *stable set* is essentially empty (contains only the endowment profile) in general (see Example 2 in Appendix A). Although there are other (weaker) versions of *stable set* (see Example 3 in Appendix A), we argue for the current (stronger) version (Definition 2) in Section 6.

The **third** point is about the relationship between the **stable set and core**. The *core* can be defined by the set of assignments that *block* any other assignments.[10] If the problem did not involve any cases where the natural coalitions must be expanded, the *stable set* would not take an incremental form, and the *stable efficient* set would be the efficient stable improvements of the endowments. In such a case, the definition would correspond to the set of assignments that block any other assignment, which would correspond to the definition of the *core*. Therefore, the *stable set* can be thought of as the incremental restrictions of the *core* to each expansion of coalitions.

**Proposition 1.** *The core is non-empty if and only if there is an essentially-unique path of stable improvements, that is, for each* $\mu_1, \mu_2 \in \mathcal{S}(e,R)$, *there exists stable improving sequences* $(\mu_k^1)_{0\leq k\leq n}$ *and* $(\mu_k^2)_{0\leq k\leq m}$ *such that* $\mu_0^1 = \mu_1$, $\mu_0^2 = \mu_2$ *and* $\mu_n^1 = \mu_m^2$.

*Proof.* See Appendix A. □

The **fourth** point is about the relationship between the **stable set and bargaining set**. For each problem $(e,R)$, let

$$B(e,R) = \{\mu\colon \mu \textit{ weakly blocks any assignment } \mu' \in M(e,R)\}$$

to be the weaker version of the *bargaining set* that allows *Pareto inefficient* assignments to be inside the *bargaining set* if they can *weakly block* any of their *Pareto improvements*. For any set of endowment profiles $E$, let $B(E,R)$ denote the image of $E$ in $B$ in the usual sense. Let $B^2(e,R)$ denote $B(B(e,R),R)$ and let $B^n(e,R) = B(B^{n-1}(e,R),R)$ denote the $n$-times composition of $B$ with itself, so that suppressing the preferences $R$, we have $B^n(e) = B(B(B(..B(e))$. Since for any $(e,R)$, we have $e \in B(e,R)$, by a simple induction we see that $B^n(e,R) \subset B^{n+1}(e,R)$. Thus, $\lim B^n(e,R)$ is defined an is equal to $\bigcup B^n(e,R)$. Now we can further see the connection between our definition of the *stable set* and the *bargaining set*:

**Proposition 2.** *Let* $B^\infty(e,R) = \lim B^n(e,R)$. *Then,* $\mathcal{S}(e,R) = B^\infty(e,R)$.

*Proof.* See Appendix A. □

[10] The formal statement would be somewhat more nuanced and the definition of blocking would be slightly different, nevertheless the intuition would be preserved.

It is worth noting that the statement is not trivial, in the sense that it does not follow from the definition (and also is not true in general) that $\mu \succeq^s \mu'$ if and only $\mu \in B(\mu', R)$. Although both definitions require that $\mu$ weakly blocks any assignment $\mu^* \succ \mu'$, in the former case, the (weak) blocking is done with the starting endowments $e$, however in the latter case the (weak) blocking must be done with the new endowments $\mu'$. Nevertheless, the statement does turn out to be true if we further assume $\mu' \in \mathcal{S}(e, R)$, which will be sufficient for our proof.

This observation follows from the definition of the *stable set*, and it gives us the connection between the *stable set* and the *bargaining set*: the *stable set* characterizes the *bargaining set* in each expansion of coalitions. Thus, it can be viewed as the repeated application of the *bargaining set*.

## 6 Main result

The following observation states that there is a unique *stable set*.

**Remark 3.** *For each problem $(e, R)$, there exists a unique stable set.*

*Proof.* Let $(e, R)$ be a problem and $S$ a *stable set* for $(e, R)$. Let $S_0(e, R) = \{e\}$ and for $k \geq 1$, let $S_k(e, R) = \{\mu\colon \mu \succeq^s \mu' \textit{ for some } \mu' \in S_{k-1}(e, R)\}$ be the set of all *stable improvements* over each assignment in $S_{k-1}(e, R)$. Let $S^*(e, R) = \bigcup_{k=0}^{\infty} S_k(e, R)$. Since $e \in S_0$, we have $e \in S^*(e, R)$. If $\mu' \in S^*(e, R)$ and $\mu \succeq^s \mu'$, then $\mu' \in S_k(e, R)$ for some $k$, and by definition, $\mu \in S_{k+1}(e, R)$. Then, we have $\mu \in S^*(e, R)$. Thus, $S^*(e, R)$ is *externally stable*. Since $S$ is *stable*, it is *externally stable* and we have $S_0 = \{e\} \subseteq S$. By *external stability* of $S$, $S_k \subseteq S$ implies $S_{k+1} \subseteq S$. Then, for each $k$, $S_k \subseteq S$. Thus, $S^*(e, R) \subseteq S$. Since $S$ is *stable* and $S^* \subseteq S$ is *externally stable*, $S = S^*(e, R)$. We conclude that there exists a unique *stable set* and it is given by $S^*(e, R)$. □

For each problem $(e, R)$, we denote the *stable set* by $\mathcal{S}(e, R)$. The **stable improvement frontier of the stable set**, or simply the **frontier of the stable set** is the set of assignments in the *stable set*, which do not admit a *stable improvement*. We denote the *frontier of the stable set* by $\mathcal{S}^f(e, R)$. Our main theorem states that the *frontier of the stable set* characterizes the $TTC$.

**Theorem 2.** *For each problem $(e, R)$, $\mathcal{S}^f(e, R) = TTC(e, R)$.*

*Proof.* See Appendix B. □

As this result shows, the notion of *stable set* is strong enough to characterize the $TTC$ and weak enough to be non-empty. This is important, since as far as the $TTC$ is concerned, the right stability notion has been an open question: First, *core* is empty (too strong). Second, both *weak core* and the *weak bargaining set* (see Yılmaz and Yılmaz (2022)) are weak enough to be non-empty but at the same time, not strong enough to characterize the $TTC$ (Yılmaz and Yılmaz, 2022). Moreover, it turns out that strengthening of *(weak) blocking* (through a restriction on the agents outside the deviating coalition) is not only natural and intuitive in the current context (see Section 5.1), but also a key aspect in characterizing the $TTC$ solution, along a stronger version of *stable set*[11]

Finally, the following corollary is a direct consequence of Theorem 2.

**Corollary 2.** *For each problem* $(e, R)$*, if* $C(e, R) \neq \emptyset$*, then* $\mathcal{S}^f(e, R) = C(e, R)$.

Note that the $TTC$ is equivalent (under the strict preferences) to the *core* (the most intuitive stability notion). Thus, our notion is strong enough to reduce to the *core* in the strict domain and weak enough to be non-empty even in the most general domain (as opposed to the *core*) and to characterize the $TTC$.

[11]As we discuss in Section 5.1, this stronger version is through using *weak blocking* rather than *blocking* in the definition of *stable improvement.*

## Appendix A  Alternative formulations of the stable set

**Example 1.** *When is our definition of blocking not restrictive?*

*Suppose there are two intersecting trading cycles $C_1$ and $C_2$ such that it is not possible to assign all agents in these cycles their best objects simultaneously, and a third cycle $C_3$, which does not intersect with $C_1$ or $C_2$. Let $\mu' = C_1 \circ C_3$ and $\mu = C_2 \circ C_3$, the assignments obtained by executing the cycles $C_1$ and $C_3$, and the cycles $C_2$ and $C_3$, respectively. Assignments $\mu'$ and $\mu$ reflect a conflict interest between coalitions $C_1$ and $C_2$, and we would expect them to block each other. Since the additional restriction of our definition implies that agents outside the blocking coalition must be assigned to their endowments, (i) blocking by the coalition $C_2$ would imply $\mu^* = C_2$ (only the agents in $C_2$ are assigned to their best objects and the rest to their endowments), and (ii) $C_2$ cannot block $\mu'$ with $\mu$. But, since all agents in $C_3$ are indifferent between $\mu'$ and $\mu$, the coalition $C_1 \cup C_3$ blocks $\mu'$ with $\mu$. Thus, in such cases, our definition is not restrictive.*

**Example 2.** *The stable set would be empty if we use blocking (instead of weak blocking) in the definition of stable improvement.*

*By definition of a Pareto improvement, an assignment $\mu$ can not block another assignment $\mu'$, if $\mu'$ Pareto improves $\mu$. Thus, if an assignment $\mu \succeq \mu'$ is not Pareto efficient, it can not block all assignments $\mu^* \succeq \mu'$, as there will be one such $\mu^*$ that Pareto improves $\mu$. Next, we refer to the example in the Introduction: agents $i_1$ and $i_2$ are endowed with each other's most preferred objects and can trade them to achieve $\bar{\mu}_1 \succ e$. But, since $\mu_1 \succ \bar{\mu}_1$, we would not deem $\bar{\mu}_1$ as a stable improvement over $e$, if, instead of weak blocking, we use blocking in the definition of a stable improvement. Moreover, even though $\mu_1$ is Pareto efficient, it can not block $\mu_2$, as agent $i_3$ strictly prefers $\mu_2$ over $\mu_1$. Thus, $\mu_1$ would not be a stable improvement over $e$. By a symmetric argument, neither $\mu_2$ nor $\bar{\mu}_2$ would be stable improvements over $e$. Thus, the stable set would consist only of $e$ and the efficiency frontier of the stable set would be the empty set.*

**Example 3.** *A weaker version of stable set.*

*Let us consider the following version of stable improvement: $\mu$ is a stable improvement over $\mu'$ if and only if for each $\mu^* \succ \mu'$ such that $\mu^* \not\succeq \mu$, $\mu$ blocks $\mu^*$. It is easy to see that our definition is strictly stronger. If $\mu^* \not\succeq \mu$ and $\mu$ weakly blocks $\mu^*$, then there exists an agent $i$, who strictly prefers $\mu$ over $\mu^*$. But, since both assignments are individually rational, we have $\mu(i) \neq e(i)$. This means that agent $i$ is part of the weakly blocking coalition of $\mu$. Thus, $\mu$ blocks $\mu^*$. Thus, the blocking condition in the definition can be rewritten as follows: "For each $\mu^* \succ \mu'$ with $\mu^* \not\succeq \mu$, $\mu$ blocks $\mu^*$ and for each $\mu^* \succeq \mu \succeq \mu'$, $\mu$ weakly blocks $\mu^*$". Thus, using weak blocking implies a stronger (and non-empty) definition of stable improvement. Thus, we choose the compact version, and say "$\mu$ weakly blocks* *any assignment $\mu^* > \mu'$". Also, to let $\mu^* > \mu$ instead of $\mu^* \geq \mu'$ is without loss of generality, since for each $\mu^* \sim \mu'$, we have $\mu \geq \mu' \sim \mu^*$, thus, $\mu$ weakly blocks $\mu^*$.*

**Proof of Proposition 1.** Suppose the core is non-empty. Then, by Theorem 2, $TTC(e,R) = \mathcal{S}^f(e,R) = C(e,R)$. Let $\mu_1,\ \mu_2 \in \mathcal{S}(e,R)$. By the structure of the *stable set*, there exist *stable improving* sequences $(\mu_k^1)_{0\leq k\leq n}$ and $(\mu_k^2)_{0\leq k\leq m}$ such that $\mu_n^1 \in \mathcal{S}^f = C$ and $\mu_m^2 \in \mathcal{S}^f = C$. Since the *core* is essentially unique, we have $\mu_n^1 \sim \mu_m^2$. Since the *stable frontier* is *Pareto efficient*, there exists no $\mu^* > \mu_m^2$. Thus, we have $\mu_n^1 \geq^s \mu_m^2$. Thus, letting $\mu_{m+1}^2 = \mu_n^1$, we are done.

Similarly suppose the condition is satisfied. Let $\mu_1,\ \mu_2 \in \mathcal{S}^f(e,R)$. Then, since both assignments are *Pareto efficient*, the stable improving sequences $(\mu_k^1)_{0\leq k\leq n}$ and $(\mu_k^2)_{0\leq k\leq m}$ must be such that $\mu_k^1 \sim \mu_{k-1}^1$ and $\mu_k^2 \sim \mu_{k-1}^2$. Thus we have $\mu_1 \sim \mu_2$. Thus the *stable frontier* is essentially unique. Suppose at some point in the $TTC$ algorithm we have an *uncovered minimally self-mapped set*. Then there exist two agents $i$ and $j$ such that they can not be assigned to the object they point to in the graph, but each of them can be assigned to that object separately, depending on the cycle selection rule. Let $\mu_1$ be an assignment in which agent $i$ is assigned to his most preferred object and let $\mu_2$ be an assignment in which agent $j$ is assigned to his most preferred object. Then by our previous observation, we must have $\mu_1 \nsim \mu_2$. But our main theorem implies that $\mu_2, \mu_1 \in \mathcal{S}^f$, which contradict *stable frontier* being essentially unique. Thus, at each step of the $TTC$ algorithm, we must get a *covered self-mapped set*. This means the problem has a *TTS* structure, thus the *core* is non-empty.

**Proof of Proposition 2.** We first show that for any $\mu' \in \mathcal{S}(e,R)$, we have $\mu \geq^s \mu'$ if and only if $\mu \in B(\mu',R)$. First assume $\mu \geq^s \mu'$. Let $\mu^* > \mu'$. Then for any agent $i \in \mathcal{A}$ such that $\mu(i) \neq e(i)$, we have $\mu(i)\ R_i\ \mu^*(i)$. We need to show for any $i \in \mathcal{A}$ such that $\mu(i) \neq \mu'(i)$, we have $\mu(i)\ R_i\ \mu^*(i)$. Let $i^*$ be such an agent. If $\mu(i^*) \neq e(i^*)$, then we are done by assumption $\mu \geq^s \mu'$. Suppose $\mu(i^*) = e(i^*) \neq \mu'(i^*)$. Then, by the argument in the proof of Lemma 6, we must have $\mu(i^*)\ I_i\ \mu^*(i^*)\ I_i\ \mu'(i^*)$, so we conclude $\mu(i^*)\ R_i\ \mu^*(i^*)$. Next assume that $\mu \in B(\mu',R)$, let $\mu^* \in \mathcal{M}(\mu',R)$ so that $\mu^* \geq \mu'$. Then for any agent $i \in \mathcal{A}$ such that $\mu(i) \neq \mu'(i)$, we have $\mu(i)\ R_i\ \mu^*(i)$. We need to show for any $i \in \mathcal{A}$ such that $\mu(i) \neq e(i)$, we have $\mu(i)\ R_i\ \mu^*(i)$. Let $i^*$ be such an agent. If $\mu(i) \neq \mu'(i)$, then we are done by assumption. Suppose $\mu(i) = \mu'(i) \neq e(i)$. Again by the argument in the proof of Lemma 6, we have $\mu(i) = \mu'(i)\ R_i\ \mu^*(i)$. Which proves the statement.

Now that we have established for any $\mu' \in \mathcal{S}(e,R)$, we have $\mu \geq^s \mu'$ if and only if $\mu \in B(\mu',R)$, we will show $B^\infty(e,R) = \mathcal{S}(e,R)$ by referring to the construction of the stable set in the proof of Remark 3. Notice that the set since $\bigcup S_k(e,R) = \mathcal{S}(e,R)$, we have $S_k \subset \mathcal{S}$ for each $k$. Thus, if $\mu' \in S_k$, we have $\mu \geq \mu'$ if and only if $\mu \in B(\mu',R)$. Thus, $S_{k+1} = \{\mu\colon \mu \geq^s \mu' \textit{ for some } \mu' \in S_k\} = \{\mu\colon \mu \in B(\mu',R) \textit{ for some } \mu' \in S_k\} = B(S_k,R)$. Let $B^0 := S_0 = \{e\}$. Then, $B(e,R) = S_1(e,R)$. By a simple induction using the identity above, we obtain $B^n(e,R) = S_n(e,R)$. Thus, $B^\infty(e,R) = \bigcup B^n(e,R) = \bigcup S_n(e,R) = \mathcal{S}(e,R)$.

## Appendix B Proof of Theorem 2

Let $(e,R)$ be a problem. We fix $(e,R)$ throughout the proof, and for ease of notation, we drop the argument $(e,R)$ whenever relevant.

**Definition 3.** *An assignment $\mu$ is in the **stable reach** of another assignment $\mu'$ if there exists a sequence of assignments $(\mu_0 = \mu', \mu_1, ...\mu_n = \mu)$ such that $\mu_k \geq^s \mu_{k-1}$ for any $1 \leq k \leq n$. We write $\mu' \rightarrow^S \mu$ if $\mu$ is in the stable reach of $\mu'$.*

**Remark 4.** *By construction of the set $S^*(e,R)$ in the proof of Remark 3, an assignment $\mu$ is in the stable set if and only if it is in the stable reach of $e$, that is, $\mathcal{S}(e,R) = \{\mu\colon e \rightarrow^S \mu\}$.*

By definition of the $TTC$ algorithm, at each step, a cycle $C$ is chosen and executed to obtain a new updated endowment profile such that each agent in $C$ is assigned the object that they point to in $C$. For convenience, for agent $i$ in $C$, we denote the object that they point to by $C(i)$. For a cycle $C$, the cycle $C^{-1}$ is such that $x$ points to $y$ in $C$ if and only if $y$ points to $x$ in $C^{-1}$. We say that a cycle $C$ is **improving** if for some agent $i$ in $C$, $C(i)\ P_i\ e(i)$, and **non-improving** if for each agent $i$ in $C$, $C(i)\ I_i\ e(i)$. Also, an assignment which is obtained at Step $n$ of a member of the $TTC$ class by executing a sequence of cycles, say $C_1, C_2, \ldots, C_n$, is denoted by $C_1 \circ C_2 \circ .. \circ C_n$. We refer to any such assignment as a **sub-TTC assignment** and for each problem $(e,R)$, denote the set of *sub-TTC assignments* by $STTC(e,R)$, or by $STTC$ whenever convenient.

**Lemma 1.** *Let $\mu = C_1 \circ C_2 \circ .. \circ C_n \in STTC$ and $\mu'$ be an assignment. Suppose that, for each $i \in \bigcup_{k=1}^{m(<n)} C_k$, we have $\mu(i)\ I_i\ \mu'(i)$. Then, for each agent $i \in C_{m+1}$, $\mu(i)\ R_i\ \mu'(i)$.*

*Proof.* First, note that at Step $m+1$, when agents in $C_{m+1}$ are assigned to each other's endowments, they are pointing to their most preferred object. Thus, if for an agent $i \in C_{m+1}$, $\mu'(i)\ P_i\ \mu(i)$, then that object belongs to some agent in $C_{m'}$ where $C_{m'}$ is part of a *covered self-mapped set* with $m' < m+1$. But, for each agent $j \in C_{m'}$, we have $\mu(j)\ I_j\ \mu'(j)$. Thus, if agent $i$ receives an object from $C_{m'}$, then an agent $i' \in C_{m'}$ receives the endowment of $i$. Thus, at Step $m'$, agent $i'$ points to an agent in $C_{m+1}$. By definition of a *self-mapped set*, this means that agent $i$ is part of the *self-mapped set* of agents in $C_{m'}$. Thus, since that *self-mapped set* is *covered*, $i \in C_{m+1}$ is removed together with the agents in $C_{m'}$, which is a contradiction. □

**Corollary 3.** *Let $\mu = C_1 \circ C_2 \circ .. \circ C_n \in STTC$ and $\mu'$ be an assignment such that $\mu' \succ \mu$. Then, each agent in $\bigcup_{k=1}^{n} C_k$ is indifferent between $\mu$ and $\mu'$.*

**Lemma 2.** *Let* $\mu = C_1 \circ C_2 \circ .. \circ C_n \in STTC$ *and let* $\mu' = C_1 \circ C_2 \circ .. \circ C_{n-1}$. *Then we have* $\mu \succeq^s \mu'$.

*Proof.* By definition of the $TTC$ algorithm, $\mu \succeq \mu'$. Let $\mu^*$ be an assignment such that $\mu^* \succ \mu'$. By Corollary 3, agents in $\bigcup_{k=1}^{n-1} C_k$ are indifferent between $\mu^*$, $\mu'$ and $\mu$, and by Lemma 1, for each agent $i \in C_n$, $\mu(i)\ R_i\ \mu^*(i)$. Thus, for each agent $i \in \bigcup_{1 \le i \le n} C_i$, we have $\mu(i)\ R_i\ \mu^*(i)$. Since $i \notin \bigcup_{1 \le i \le n} C_i$ implies $\mu(i) = e(i)$, $\mu$ *weakly blocks* $\mu^*$. Thus, we have $\mu \succeq^s \mu'$. □

**Lemma 3.** *For each problem* $(e, R)$, $STTC(e, R) \subseteq \mathcal{S}(e, R)$.

*Proof.* Let $(e, R)$ be a problem. Let $\mu = C_1 \circ C_2 \circ .. \circ C_n \in STTC$. Let $\mu_k = C_1 \circ C_2 \circ .. \circ C_k$ and let $\mu_0 = e$. Then by Lemma 2, we have $\mu_k \succeq^s \mu_{k-1}$ for any $1 \le k \le n$. Thus, we have $e \rightarrow^S \mu$. By Remark 4, we have $\mu \in \mathcal{S}(e, R)$. □

**Remark 5.** *The inclusion in Lemma 3 is in general strict. There are problems for which the stable set contains an assignment which cannot be obtained at some interim stage of any cycle selection rule F of the TTC algorithm.*

*Proof.* Let $\mathcal{A} = \{i_1, i_2, i_3, i_4\}$ and $\mathcal{O} = \{o_1, o_2, o_3, o_4\}$. We consider the following problem $(e, R)$: Let the endowment profile $e$ be such that for each $k$, $e(i_k) = o_k$. Let $R$ be the following preferences profile:

| $R_{i_1}$ | $R_{i_2}$ | $R_{i_3}$ | $R_{i_4}$ |
|---|---|---|---|
| $o_2$ | $o_1$ | $o_2$ | $o_3$ |
| $o_1$ | $o_2$ | $o_4$ | $o_4$ |
| | | $o_3$ | |

For ease of notation, we denote an assignment $\mu$ as $(\mu(i_1), \mu(i_2), \mu(i_3), \mu(i_4))$. It is easy to see that $\mathcal{M}(e, R) = \{e, (o_2, o_1, o_3, o_4), (o_1, o_2, o_4, o_3), (o_2, o_1, o_4, o_3)\}$. The $TTC$ algorithm chooses the cycle where $i_1$ and $i_2$ pointing each other's endowments, and since all preferences are strict, agents $\{i_1, i_2\}$ and their endowments are removed. Then, the $TTC$ chooses the cycle where $i_3$ and $i_4$ pointing each other's endowments, and agents $\{i_3, i_4\}$ and their endowments are removed. Thus, we have $STTC(e, R) = \{e, (o_2, o_1, o_3, o_4), (o_2, o_1, o_4, o_3)\}$ and $TTC(e, R) = C(e, R) = \{(o_2, o_1, o_4, o_3)\}$. But, it is easy to see each *individually rational* assignment except $e$ is a *stable improvement* over $e$. Thus, we have $\mathcal{S}(e, R) = \mathcal{M}(e, R)$. Thus, $(o_1, o_2, o_4, o_3) \in \mathcal{S}(e, R)$ but $(o_1, o_2, o_4, o_3) \notin STTC(e, R)$.[12] □

**Remark 6.** *In general, the number of stable improvements can get arbitrarily large compared to number of TTC improvements, and the stable set can get exponentially large compared to STTC.*

[12]Note that, for this example, $\mathcal{SE}(e, R) = TTC(e, R) = C(e, R) = \{(o_2, o_1, o_4, o_3)\}$, which is consistent with our main result (Theorem 2).

*Proof.* We will expand the example given above. Let $\mathcal{A} = \{i_k : k = 1, 2, ..., 2n\}$. Let $e(i_k) = o_k$, let the preferences be

| $R_{i_1}$ | $R_{i_2}$ | $R_{i_{2k+1}}$ | $R_{i_{2k+2}}$ |
|---|---|---|---|
| $o_2$ | $o_1$ | $o_{2k}$ | $o_{2k+1}$ |
| $o_1$ | $o_2$ | $o_{2k+2}$ | $o_{2k+2}$ |
| | | $o_{2k+1}$ | |

So that we expand the example above to $n$ pair of agents. Then the only TTC improvement over $e$ is still $\mu = (12)$, where as for any subset of $P = \{(12), (34), ...((2n-1)2n)\}$, we have a stable improvement over $e$. So there are $2^n$ elements in the stable set. However, TTC algorithm has $n$ many steps and reaches $n$ many assignments. Therefore, $\mathcal{S}(e, R)$ is potentially exponentially larger then $STTC$. □

Let $\mu'$ be an assignment. Let $G_{\mu'}$ denote the graph where each agent points to their most preferred objects and each object points to its owner in $e' = \mu'$, that is, each object $o$ points to agent $i$ such that $\mu'(i) = o$. Thus, for an *(updated) endowment profile* $e' = \mu'$, the $TTC$ algorithm starts with graph $G_{\mu'}$. Let $k$ be the first step, at which an agent is assigned a strictly preferred object than they own under $\mu'$. Thus, by definition of the $TTC$ algorithm, at each Step $k' \leq k-1$, a *covered set* is removed where each agent in that *covered set* already points to their endowment, and no welfare gain is obtained until Step $k$. Let $G^1_{\mu'}$ denote the graph after these removals, that is, the graph, where each object points to the agent whom it is assigned at the end of Step $k-1$, and each remaining agent points to their most preferred available objects.

**Lemma 4.** *Let $\mu \succeq^s \mu'$. Let $T$ be a minimal self-mapped set in $G_{\mu'}$ and $i \in T$ be an agent such that $\mu(i) \neq e(i)$. Then, in $G_{\mu'}$, agent $i$ points to object $\mu(i)$.*

*Proof.* Since $i \in T$ for some *minimal self-mapped set* $T$, and by Remark 1, $T$ is *strongly connected*, there exists a cycle $C \subset T$ such that $i \in C$. Let

$$\mu^*(j) = \begin{cases} C(j) & \text{if } j \in C \\ \mu'(j) & \text{if } j \notin C, \end{cases}$$

where $C(j)$ denotes the object $j$ points to in $C \subset T$. Since each agent points to their most preferred objects in $G_{\mu'}$, under $\mu^*$, each agent in $C$ is assigned to one of their most preferred objects. Thus, $\mu^* \succeq \mu'$. Suppose $\mu^* \succ \mu'$. Then, since $\mu \succeq^s \mu'$, and $\mu^* \succ \mu'$, $\mu$ *weakly blocks* $\mu^*$. Then, since $\mu(i) \neq e(i)$, by definition of *weak blocking*, $\mu(i)\ R_i\ \mu^*(i)$. Suppose $\mu^* \sim \mu'$. Then, since $\mu \succeq^s \mu'$, we have $\mu(i)\ R_i\ \mu'(i)$. Thus, $\mu(i)\ R_i\ \mu^*(i)$.

Since $\mu^*(i)$ is a most preferred object of $i$, this implies that, $\mu(i)$ is also a most preferred object of $i$. Thus, in $G_{\mu'}$, agent $i$ points to object $\mu(i)$. □

**Lemma 5.** *Let $\mu \succeq^s \mu'$. Let $T$ be a minimal self-mapped set in $G^1_{\mu'}$. Let $i \in T$ be an agent. If $\mu(i) \neq e(i)$, then $i$ points to object $\mu(i)$ in $G^1_{\mu'}$.*

*Proof.* Let $C \subset T$ be an arbitrary cycle such that $i \in C$. Let $\mu^*$ be the assignment defined in Lemma 4. Since $\mu \succeq^s \mu'$, and $\mu^* \geq \mu'$ (see the proof of Lemma 4), we have $\mu(i)\ R_i\ \mu^*(i)$.

Suppose $\mu(i)\ P_i\ \mu^*(i)$. Then, since, in $G^1_{\mu'}$, agent $i$ points to one of their most preferred objects, object $\mu(i)$ is removed before Step $k$. Then, it is part of a *covered minimal self-mapped set* at some Step $k' \leq k-1$. But, by the same argument in Lemma 1, an agent in that *covered minimal self-mapped set* must be assigned $e'(i) = \mu'(i)$. This implies that $i$ is also part of that *minimal self-mapped set*, and thus, removed at the same step. Since this contradicts with $i \in G^1_{\mu'}$, we have $\mu(i)\ I_i\ \mu^*(i)$. Since $\mu^*(i)$ is one of the most preferred object for $i$ in $G^1_{\mu'}$, $\mu(i)$ is also one of their most preferred objects and $i$ points to $\mu(i)$ in $G^1_{\mu'}$. □

**Lemma 6.** *Let $\mu \succeq^s \mu'$ and $\mu' \in \mathcal{S}(e,R)$. Let $T$ be a minimal self-mapped set in $G^1_{\mu'}$. Let $i \in T$ be an agent. If $\mu(i) \neq \mu'(i)$, then $i$ points to object $\mu(i)$ in $G^1_{\mu'}$.*

*Proof.* If $\mu(i) \neq e(i)$, the claim directly follows from Lemma 5. Suppose $\mu(i) = e(i) \neq \mu'(i)$. Since both assignments are *individually rational*, we must have $\mu(i) = e(i)\ I_i\ \mu'(i)$. Since $\mu'(i) \in \mathcal{S}(e,R)$, $\mu' \succeq^s \mu''$ for some $\mu'' \in \mathcal{S}(e,R)$. Then, for each $\mu^* > \mu'$, we have $\mu^* > \mu''$ and by definition of a *stable improvement*, $\mu'$ weakly blocks $\mu^*$. Since $\mu'(i) \neq e(i)$, this means $\mu^*(i)\ I_i\ \mu'(i)$. Thus, for each $\mu^* > \mu'$, we have $\mu(i)\ I_i\ \mu^*(i)\ I_i\ \mu'(i)$, . Then, by constructing $\mu^*$ in the proof of Lemma 4, we complete the proof. □

**Lemma 7.** *Let $\mu \succeq^s \mu'$ and $\mu' \in \mathcal{S}(e,R)$. Let $T$ be a minimal self-mapped set in $G^1_{\mu'}$. Let $i \in T$ be an agent such that $\mu(i) \neq \mu'(i)$. Then, there exists a cycle $C \subset T$ with $i \in C$ such that $\mu(j) = C(j)$ for any agent $j \in C$ and objects in $C$ points to their owners in $\mu'$. Thus, $C$ is a* top trading cycle *in $T$.*

*Proof.* By Lemma 6, agent $i$ points to the object $\mu(i)$ in graph $G^1_{\mu'}$. Then, since $T$ is *self mapped* and $i \in T$, owner of the object $\mu(i)$ in graph $G^1_{\mu'}$ is also in $T$. Let this agent be $j$, so that $\mu'(j) = \mu(i)$. Since object $\mu'(j)$ is assigned to $i$ under $\mu$, we have $\mu'(j) \neq \mu(j)$. Then owner of $\mu(j)$ in graph $G^1_{\mu'}$ is also in $T$. Continuing this way, we can construct a cycle $C \subset T$ of agents and objects, such that $C(o) = \mu'^{-1}(o)$ and $C(i) = \mu(i)$. □

**Lemma 8.** *Let $(e,R)$ be a problem. Let $T$ be a minimal self-mapped set in the graph $G^1_e$. Let $C \subset T$ be a non-improving cycle in $T$. Then, $T$ is also a minimal self-mapped set in $G^1_e$ after $C$ is executed.*

*Proof.* Since the cycle $C$ is *non-improving*, each agent in the cycle points to its own object as well as the object it is assigned under $C$. Then, after $C$ is executed, it is easy to see that $C^{-1}$ is a cycle in the new graph. Then, let $(a_1, a_2, ...a_n)$ be a path in $T$. If this path does not intersect agents in $C$, then it is also a path in $T \circ C$. Suppose it intersects $C$. Let $a_k$ and $a_l$ be the first and last agents in $C$, respectively. Then, agents in $(a_1, ...a_{k-1}) \cup (a_{l+1}, a_{l+2}, ...a_n)$ are not in $C$. Since both $a_k$ and $a_l$ are in $C$, and $C^{-1}$ is a cycle in $T$, using $C^{-1}$ we can construct a path from $a_k$ to $a_l$ consisting only of agents in $C$. Let this path be $(c_1 = a_k, c_2, ...c_m = a_l)$. Then, since all of these agents are in $C$, and none of the agents in $(a_1, ...a_{k-1}) \cup (a_{l+1}, a_{l+2}, ...a_n)$ are in $C$, the path $(a_1, ...a_{k-1}, c_1 = a_k, c_2, ...c_m = a_l, a_{l+1}, a_{l+2}, ...a_n$ is a path from $a_1$ to $a_n$, which does not intersect itself. Since the path was arbitrary, we conclude that $T$ being *strongly connected* implies $T \circ C$ being *strongly connected.* Since executing the cycle $C \subset T$ does not change $\delta^{out}(T) = \emptyset$, we also have $\delta^{out}(T \circ C) = \emptyset$. Then, by Remark 1, $T \circ C$ is also a *minimal self-mapped* set. □

**Lemma 9.** *Let $\mu \geq^s \mu'$. Let $T$ be a minimal self-mapped set in $G^1_{\mu'}$ such that, for each agent $i \in T$, $\mu(i) = \mu'(i)$. Let $C \subset T$ be a cycle. Then, $\mu \circ C \geq^s \mu' \circ C$.*

*Proof.* Let $\mu^* > \mu' \circ C$. Then, $\mu^* > \mu'$. Thus, $\mu$ *weakly blocks* $\mu^*$. Let $i$ be an agent such that $(\mu \circ C)(i) \neq e(i)$. If $i \notin C$, then $(\mu \circ C)(i) = \mu(i)\ R_i\ \mu^*(i)$ by *weak blocking* of $\mu$. If $i \in C$, then since $C$ is a *top trading cycle*, by the argument in Lemma 1, we must have $(\mu \circ C)(i) = C(i)\ R_i\ \mu^*(i)$. Then $\mu \circ C$ *weakly blocks* $\mu^*$. Since $\mu \geq \mu'$ implies $\mu \circ C \geq \mu' \circ C$, we conclude that $\mu \circ C \geq^s \mu' \circ S$. □

**Lemma 10.** *Let $\mu \geq^s \mu'$. Let $T$ be a minimal self-mapped set in $G^1_{\mu'}$. Let $C \subset T$ be a cycle such that for each agent $i \in C$, $\mu(i) = C(i)$. Then, we have $\mu \geq^s \mu' \circ C$*

*Proof.* Let $\mu^* > \mu' \circ C$. Then, $\mu^* > \mu'$ as well. Then, since $\mu \geq^s \mu'$, $\mu$ *weakly blocks* $\mu^*$. For each agent $i$, if $i \in C$, we have $\mu(i) = (\mu' \circ C)(i) = C(i)$, thus $\mu(i)\ I_i\ \mu'(i)$. If $i \notin C$, then since $\mu \geq \mu'$, we have $\mu(i)\ R_i\ \mu'(i)$. Thus, we have $\mu \geq \mu' \circ C$. Then, we conclude $\mu \geq^s \mu' \circ C$. □

**Lemma 11.** *If $\mu \in \mathcal{SE}$, then $\mu \in TTC$.*

*Proof.* First, we define a partial order on $STTC$. Let $\geq^{TTC}$ be such that (i) $\mu \geq^{TTC} \mu'$ if and only $\mu$ is obtained as a *sub-TTC assignment* for the problem $(e_{\mu'}, R)$, and (ii) $\mu >^{TTC} \mu'$ if and only if $\mu \geq^{TTC} \mu'$ and $\mu \geq \mu'$.

Let $\mu \in \mathcal{SE}$. Let $\underline{S}^\mu := \{\mu' \in STTC\colon \mu' \to^S \mu\}$. Since, by Remark 4, $e \to^S \mu$, we have $e \in \underline{S}^\mu$, thus $\underline{S}^\mu \neq \emptyset$. Since $\underline{S}^\mu \subseteq STTC$ and $\underline{S}^\mu \neq \emptyset$, there exist $>^{TTC}$-maximal assignments in $\underline{S}^\mu$. Formally, there exists $\mu' \in \underline{S}^\mu$ such that for each $\mu'' \in \underline{S}^\mu$, we have $\mu'' \not>^{TTC} \mu'$. Let $\mu'$ be such an assignment. Since $\mu' \to^S \mu$, by definition of *stable reach*, we have sequence $(\mu_0 = \mu', \mu_1, \mu_2, ...\mu_n = \mu)$ with $\mu_k \geq^s \mu_{k-1}$ for any $1 \leq k \leq n$.

Suppose $\mu \notin TTC$. If $\mu'$ is *Pareto efficient*, then $\mu' \in TTC$. But, since $\mu' \to^S \mu$, this implies that, for each agent $i$, $\mu(i)\ I_i\ \mu'(i)$. By the argument in the proof of Lemma 1, for each *covered self-mapped set* $T$ obtained with the TTC algorithm to reach $\mu'$, we have $\mu(T) = e(T)$ as well. But, since the TTC algorithm allows for each assignment to the agents when those agents are in some *covered self-mapped set*, we can assign each agent in $i \in T$ to $\mu(i)$, and thus we can obtain $\mu$ with the TTC algorithm as well. But since $\mu$ is *Pareto efficient*, this means $\mu \in TTC$, contradicting our assumption. Thus, $\mu'$ is not *Pareto efficient*. Then, the graph $G^1_{\mu'}$ is non-empty, in particular, it contains *minimal self-mapped* sets, which contain *improving* cycles. Clearly, for each *minimal self-mapped* set $T$ in $G^1_{\mu'}$, and each cycle $C \subseteq T$, if $C$ is an *improving* cycle, then $\mu' \circ C \succ^{TTC} \mu'$.

Let $T$ be a *minimal self-mapped* set in $G^1_{\mu'}$. Note that by definition of the graph $G^1_{\mu'}$, there exists an agent in $T$, say $i'$, pointing to an object preferred to their endowment. Suppose for each agent $i \in T$, $\mu(i)\ I_i\ \mu'(i)$ (that is, no agent in $T$ is improved throughout the *stable improvements* to obtain $\mu$). Then, since, by definition of a *self-mapped set*, the set $T$ contains a cycle with $i'$, this implies that there exist improvable agents at $\mu$, contradicting $\mu$ being *Pareto efficient*. Then, there exists $\mu_{k_1}$ such that for each agent $i \in T$, and for $k < k_1$, we have $\mu_k(i) = \mu'(i)$, but for some agent $j \in T$, we have $\mu_{k_1}(j) \neq \mu_{k_1-1}(j) = \mu'(j)$. Then, $T$ is a *minimal self-mapped* set in the graph $G^1_{\mu_{k_1-1}}$. By Lemma 7, there exists a *top trading cycle* $S_1 \subset T$ with $j \in S_1$ such that, under $\mu_{k_1}$, each agent in $S_1$ is assigned the object that they point to in cycle $S_1$. Note that since $S_1 \subseteq T$ and $k_1$ is the first step an agent in $T$ is assigned, for each $k < k_1$, no agent in $S_1$ is re-assigned at $\mu_k$.

By Lemma 9, we have $\mu_k \circ S_1 \succeq^s \mu_{k-1} \circ S_1$, and by Lemma 10, we have $\mu_{k_1} \succeq^s \mu_{k_1-1} \circ S_1$. Thus we can construct the sequence $(\mu' \circ S_1, \mu_1 \circ S_1, ...\mu_{k_1-1} \circ S_1, \mu_{k_1}, \mu_{k_1+1}, ...\mu_n = \mu)$ of *stable improvements* to $\mu$. Then $\mu' \circ S_1 \to^S \mu$.

If $S_1$ is an *improving* cycle, we have $\mu' \circ S_1 \succ^{TTC} \mu'$ and $\mu' \circ S_1 \to^S \mu$, which contradicts that $\mu'$ is $\succ^{TTC}$-maximal in $\underline{S}^\mu$. Then, $S_1$ is a *non-improving* cycle. But, by Lemma 8, $T$ is also a *minimal self-mapped* set in $G^1_{\mu' \circ S_1}$. Then, applying the same reasoning, some agent in $T$ is re-assigned in the sequence $(\mu' \circ S_1, \mu_1 \circ S_1, ...\mu_{k_1-1} \circ S_1, \mu_{k_1}, \mu_{k_1+1}, ...\mu_n = \mu)$ as well. But, since $k_1$ is the first step at which some agent is assigned in the previous sequence, and agents in $S_1$ hold the same object throughout $k_1 > k \geq 1$ in the new sequence, for each agent $i \in T$ and $k < k_1$, we have $(\mu_k \circ S_1)(i) = (\mu' \circ S_1)(i)$. We again take the first assignment, say $\mu''$, in this sequence at which an agent, say $j$, is assigned a different object than the previous assignment in this sequence. By the above argument, for each $i \in T$, and each $k \leq k_1 - 1$, $(\mu_k \circ S_1)(i) = (\mu' \circ S_1)(i)$. Thus, $\mu'' \in \{\mu_{k_1}, \mu_{k_1+1}, ...\mu_n\}$. Then, $\mu'' = \mu_{k_2}$ for some $k_2 \in \{k_1, k_1+1, \ldots, n\}$. Thus, $k_2 \geq k_1$. Again by the same reasoning, let $S_2$ be the cycle in which agent $j$ is assigned. Then, we add this cycle to each assignment in the sequence of *stable improvements* to obtain $(\mu' \circ S_1 \circ S_2, \mu_1 \circ S_1 \circ S_2, ...\mu_{k_1-1} \circ S_1 \circ S_2, \mu_{k_1} \circ S_2, \mu_{k_1+1} \circ S_2, ...\mu_{k_2-1} \circ S_2, \mu_{k_2}, ..., \mu_n = \mu)$. Again, if $S_2$ is an *improving* cycle, we have

$\mu' \circ S_1 \circ S_2 \succ^{TTC} \mu'$ and $\mu' \circ S_1 \circ S_2 \rightarrow^S \mu$, which contradicts that $\mu'$ is $\succ^{TTC}$-maximal in $\underline{S}^{\mu}$.

Thus, $S_2$ is a *non-improving* cycle. But, by Lemma 8, $T$ is also a *minimal self-mapped* set in $G^1_{\mu' \circ S_1 \circ S_2}$, and it contains an *improving* cycle. We then continue with choosing $S_k$'s, and adding them to $\mu'_{k-1} = \mu' \circ S_1 \circ S_2 \circ ...S_{k-1}$ to construct $\mu'_k$. Notice if this procedure is finite, then we must choose an *improving* cycle $S_k$ at some point, which implies that $\mu'_k \succ^{TTC} \mu'$ and $\mu'_k \rightarrow^S \mu$, contradicting that $\mu'$ is $\succ^{TTC}$-maximal in $\underline{S}^{\mu}$. Then, it suffices to show the process is finite. Since by the above argument, for $n \geq 1$, we have $k_n \leq k_{n+1}$, it is enough to show that $k_n = k_{n+1}$ for only finitely many $n \geq 1$. Suppose that this is not the case. Then, there exists $N$ such that for $n \geq N$, we have $k_n = k_{n+1} = k^*$. But, the number of agents for which $(\mu_{k_n} \circ S_1 \circ S_2 \circ ...S_n)(i) \neq \mu'_n(i)$ strictly decreases. This contradicts with the number of agents being finite. Then, at some Step $k$, we must choose an *improving* cycle $S_k$ with $\mu'_k \succ^{TTC} \mu'$. But, this contradicts that $\mu'$ is $\succeq^{TTC}$-maximal in $\underline{S}^{\mu}$. Thus, $\mu \in TTC$. □

Let $(e,R)$ be a problem. By Lemma 11, $\mathcal{SE}(e,R) \subseteq TTC(e,R)$. Also, by Lemma 3, $STTC(e,R) \subseteq \mathcal{S}(e,R)$. Then, $STTC(e,R) \cap \mathcal{E}(e,R) = TTC(e,R) \subseteq \mathcal{S}(e,R) \cap \mathcal{E}(e,R) = \mathcal{SE}(e,R)$. Thus, $\mathcal{SE}(e,R) = TTC(e,R)$.

## References

ALCADE, J., AND A. ROMERO-MEDINA (2017): "Fair Student Placement," *Theory and Decision*, 83, 293–307.

ALCALDE-UNZU, J., AND E. MOLIS (2011): "Exchange of indivisible goods and indifferences: The Top Trading Absorbing Sets mechanisms," *Games and Economic Behavior*, 73(1), 1–16.

AUMANN, R., AND M. MASCHLER (1964): "The bargaining set for cooperative games," in *Advances in Game Theory*, ed. by L. S. M. Dresher, and A. Tucker, no. 52 in Annals of Mathematics Studies, pp. 443–476. Princeton University Press, Princeton, NJ.

CHWE, M. S. Y. (1994): "Farsighted Coalitional Stability," *Journal of Economic Theory*, 63, 299 – 325.

DUTTA, B., D. RAY, K. SENGUPTA, AND R. VOHRA (1989): "A Consistent Bargaining Set," *Journal of Eonomic Theory*, 49, 1913–1946.

EHLERS, L. (2007): "Von Neumann-Morgenstern Stable sets in Matching Problems," *Journal of Economic Theory*, 134, 537–547.

——— (2010): "School choice with control," unpublished mimeo.

EHLERS, L., I. E. HAFALIR, M. B. YENMEZ, AND M. A. YILDIRIM (2014): "School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds," *Journal of Economic Theory*, 153, 648–683.

EINY, E., D. MORENO, AND B. SHITOVITZ (2001): "The Bargaining Set of a Large Economy with Differential Information," *Economic Theory*, 18, 473–484.

HARSANYI, J. C. (1974): "An Equilibrium-point Interpretation of Stable Sets and a Proposed Alternative Definition," *Management Science*, 20, 1472 – 1495.

JARAMILLO, P., AND V. MANJUNATH (2012): "The difference indifference makes in strategy-proof allocation of objects," *Journal of Eonomic Theory*, 147(5), 1913–1946.

KESTEN, O. (2010): "School Choice with Consent," *Quarterly Journal of Economics*, 125, 1297–1348.

MA, J. (1994): "Strategy-proofness and the strict core in a market with indivisibilities," *International Journal of Game Theory*, 23, 75–83.

MAS-COLELL, A. (1989): "An Equivalence Theorem for a Bargaining Set," *Journal of Mathematical Economics*, 18, 128–139.

MAULEON, A., V. J. VANNETELBOSCH, AND W. VERGOTE (2011): "Von Neumann-Morgenstern Farsightedly Stable Sets in Two-sided Matching," *Theoretical Economics*, 6, 499–521.

QUINT, T., AND J. WAKO (2004): "On Houseswapping, the Strict Core, Segmentation, and Linear Programming," *Mathematics of Operations Research*, 29(4), 861–877.

SABAN, D., AND J. SETHURAMAN (2013): "House Allocation with Indifferences: A Generalization and a Unified View," ACM EC, 2013.

SHAPLEY, L., AND H. SCARF (1974): "On Cores and Indivisibility," *Journal of Mathematical Economics*, 1, 23–37.

SÖNMEZ, T. (1999): "Strategy-Proofness and Essentially Single-Valued Cores," *Econometrica*, 67, 677–690.

VOHRA, R. (1991): "An Existence Theorem for a Bargaining Set," *Journal of Mathematical Economics*, 20, 19–34.

YILMAZ, M., AND O. YILMAZ (2022): "Stability of an Allocation of Objects," *Review of Economic Design*, 26, 561–580.