Equity In Allocating Identical Objects Through Reserve Categories

Özgür Yılmaz*

First Draft: January 2023, This Draft: April 2024

Abstract

The recent pandemic has highlighted the importance of well-designed mechanisms for rationing identical medical units such as vaccines, ventilators, ICU's or other crucial medical units when resources are in short supply. The major concern is equity and randomization is often inevitable. Thus, an important question follows: How should we design and implement mechanisms for an *equitable* allocation of units? We consider a general framework of *reserve systems*: units are to be distributed among a set of agents through reserve categories, and certain amounts of units are reserved for these categories. We propose and characterize *random* allocation rules to mitigate uneven treatment of agents.

Keywords: Market design, Matching, Reserve Systems, Egalitarianism
JEL Classification Numbers: C71, C78, D02, D47, D63, I10

*Department of Economics, Koç University, Sarıyer, İstanbul, Turkey 34450. E-mail: ozyilmaz@ku.edu.tr; Website: mysite.ku.edu.tr/ozyilmaz

1 Introduction

The recent pandemic has highlighted the importance of well-designed mechanisms for rationing identical medical units when resources are in short supply. The major concern is equity and randomization is often inevitable. A recent such example is the implementation of a weighted lottery mechanism by the Department of Health, Pennsylvania for the allocation of medications to treat COVID-19 (Pennsylvania DH, 2020).[1] Thus, an important question follows: How should we design and implement mechanisms for an *equitable* allocation of vaccines, ICU's or other crucial medical units? For this problem, we propose and characterize random allocation rules to mitigate uneven treatment of agents.

Our model builds on a general framework of *reserve systems*:[2] Agents are grouped into *reserve categories* depending on their occupations and characteristics (e.g. essential workers, disadvantaged communities, etc.) such that each category has its own priority ordering over agents,[3] and for each category, a certain amount of units is reserved. By a well-designed reserve system, policy makers can simultaneously prioritize certain groups like essential or health workers, by reserving higher amounts of units for them, and prevent disadvantaged groups from being deprived of access. Thus, these are versatile systems addressing equity concerns, and providing policy makers with great flexibility to achieve different policy goals. But, there is a downside: since patients are in general beneficiaries of multiple categories, when implemented via a precedence order of processing categories (which is the most common practice), reserve systems have unintended distributional consequences (Pathak, Sönmez, Ünver, and Yenmez, 2021).

Example 1. *Suppose there are three agents, i, j and k. Agents i and j are essential workers, and agents j and k are disadvantaged community members. For essential workers category, j has a higher priority than i, and for disadvantaged community members category, j has a higher priority than k. There are two units in supply and the policy makers' goal is to guarantee one unit for each of these categories (by respecting their priorities) via a reserve system. If essential worker category is processed first, then j and k receive one unit each, otherwise j and i receive one unit each. Thus, any choice of precedence order achieves the policy goal, but (unintentionally) favors one category over the other.*

Our work is motivated by the uneven treatment of categories under reserve systems as demonstrated in Example 1 (this type of unfairness is formalized in Section 6.2). Our goal is twofold: (i) to propose an equity notion, which is formulated to prevent any category being favored purely due to the selection of a precedence order, and (ii) to design a *simultaneous* (rather than sequential) processing of reserve

[1] Their guideline states that 'all patients who meet clinical eligibility criteria should have a chance to receive treatment'. See Section 2 for the details of their lottery procedure.

[2] This model is not limited to pandemic resource allocation, there are other real-life examples (discussed in the related literature below, and in Section 8). We refer to 'agents' in general but whenever more convenient, we use the terminology of 'patients' (pandemic resource allocation) or 'students' (affirmative action in schools' assignment of their seats).

[3] These orderings are over the set of all agents since we are interested in soft reserves to avoid inefficiency. Also, in real-life, since the units are in very short supply, hard quota constraints are usually non-binding.

categories, which complies with this notion.

For the class of rationing problems with reserves, two properties are indispensable (see Section 4). First, priorities of agents under categories should be respected (*respecting priorities*). Second, no unit should be wasted (*efficiency*). We call a (random) allocation *acceptable* if it satisfies these standard axioms. Our first theorem is a characterization of the set of *acceptable* (random) allocation rules (Theorem 2). The insight is that each *acceptable* (random) allocation can be described as a sequential welfare improvement process.

An *acceptable* deterministic allocation does not necessarily treat categories fairly (as Example 1 above demonstrates) and this points to randomization. But, randomization is not a significant alternative under strict priorities: Let C be the set of reserve categories. Suppose priorities are strict. Then, *respecting priorities* implies that the number of agents, who are assigned a unit probabilistically, is at most $|C|$. The rest would be either not assigned at all or assigned a unit deterministically. Thus, given that the number of categories is a small number in many real-life applications, there would be very little scope in randomization and fairness.

We assume that priorities of categories are weak, and when priorities are weak, the scope of fairness is big. We mentioned above the allocation of medications to treat COVID-19 by the Department of Health, Pennsylvania (Pennsylvania DH, 2020). Their allocation mechanism relies on a model where all beneficiaries of a reserve category have the same priority, and it is a lottery mechanism providing patients with probabilistic access to units (actually, no patient is assigned deterministically (see Section 2 for the details of this approach). Thus, randomization and fairness, the focus of the current work, are relevant considerations and have a big scope in our model.

We analyze fairness and equity notions suitable for this context. Motivated by the uneven treatment of categories (as in Example 1), we propose *category-fairness*: if there is no justification for treating a category unfairly, that unfairness should be eliminated (Section 6.2). The second axiom is a standard notion of equity, *egalitarianism*, which requires equating agents' utilities as much as possible (within the constraints of *acceptability*) through the criterion of *Lorenz dominance*. We show that *egalitarianism* is impossible in the current context (see Theorem 3). Moreover, interestingly, even if an *egalitarian* allocation exists for a problem, then it may not even satisfy *equal treatment of equals*. Then, we formalize the notion of *procedural fairness* for the current setting. Recall that *acceptable* random allocations are characterized by the class of welfare improvement processes (Theorem 2). A solution in this class is *procedurally fair* if agents' utilities are chosen in an egalitarian way throughout the sequential welfare improvement process (Section 6.3).[4] *Procedural fairness* is formulated as an intuitive property about the procedure, but this is not enough to justify it as a fairness axiom. Then, we analyze its implications on the final outcome, and it turns out that there are strong normative justifications for

[4]This idea is utilized before: in the class of assignment problems, each *efficient* random allocation is equivalent to a stepwise assignment of probabilities (a class of algorithms), and the procedurally fair one in this class is obtained by equating agents' probabilities at each step (see Section 6.3 for a detailed discussion).

this property: *Procedural fairness* implies both *category-fairness* (Proposition 1) and *equal treatment of equals* (Proposition 2). Thus, we argue that *procedural fairness* is a compelling fairness criterion.

Our second theorem is the characterization of the *procedurally fair* rules, the *Priority-Based Rawlsian (PBR)* (Theorem 4). We prove this result with the help of ideas from graph theory (Appendix A). The *PBR* rules constitute an intuitive class within the set of *acceptable* rules. Basically, they are based on defining a guaranteed utility for each agent and then increasing these utilities sequentially subject to the constraints of *acceptability* and also the Rawlsian principle of prioritizing the most disadvantaged agents. We show how to design this procedure (Section 6.4). We discuss some relevant applications, rationing health care units, vaccines etc. and affirmative action in school choice in Section 8.

Related Literature

Reserve systems with sequential processing has been proposed for affirmative action in school choice (Kominers and Sönmez, 2016).[5] When there are only two types of slots, reserve and open slots, both increasing the reserve quota and raising the precedence order positions of open seats will (weakly) increase the number of reserve-eligible students who are accepted (Dur, Kominers, Pathak, and Sönmez, 2018). For the case of multiple socioeconomic tiers along with the merit tier, the precedence orders for maximizing the number of the most disadvantaged students assigned a seat are characterized as follows: the slots of other tiers precede the merit slots which are succeeded by the slots of the tier for the most disadvantaged students (Dur, Pathak, and Sönmez, 2020).

A model closer to the current setting is when a student is in general a beneficiary at multiple reserve categories, the case of *overlapping reserves*, and the goal is to guarantee maximal compliance with reservations (as many of the reserved positions as possible are to be allocated to the candidates from target groups) (Sönmez and Yenmez, 2020).

Reserve systems have been also relevant in various other contexts: medical rationing (Pathak, Sönmez, Ünver, and Yenmez, 2021), the H-1B visa program (Pathak, Rees-Jones, and Sönmez, 2022), university admissions in India (Sönmez and Yenmez, 2020; Aygün and Turhan, 2020a,b) and Brazil (Aygün and Bo, 2021).

Another strand of literature, to which the current work belongs as well, is the approach of processing reserves simultaneously. A recently proposed axiom in this setting is *category neutrality*: An allocation is *category neutral* if an agent who qualifies for multiple categories receives the same amount of capacity from all of them (Delacrétaz, 2021). In the context of hard reserves (only the beneficiaries of a

[5] Affirmative action in school choice has been widely studied. Controlled choice models provide choice to parents while maintaining the racial and ethnic balance at schools through type-specific reserves and quotas (Abdulkadiroğlu and Sönmez, 2003; Ehlers, Hafalir, Yenmez, and Yıldırım, 2014), or through adjusted priorities under minority reserves (Hafalir, Yenmez, and Yıldırım, 2013). A recent work studies how to minimize priority violations for a setting when there is only one ordering of students and there are type-specific reserves and quotas. A particular choice rule, where all applicants are first considered for units reserved for their own types, uniquely minimizes priority violations in this class (Abdulkadiroğlu and Grigoryan, 2021).

given reserve category are eligible for the units under that category), every random allocation satisfying *efficiency*, *respecting priorities* and *category neutrality* assigns to each agent the same amount of probability of receiving a unit in aggregate, and a polynomial-time algorithm exists to compute these allocations (Delacrétaz, 2021). The difference between our approach and this work can be summarized as follows: while *category neutrality* requires that for an agent, the probability of being assigned a unit is the same across all categories for which she is eligible, *procedural fairness* requires equating utilities across agents (procedurally and subject to the constraints of *acceptability*). Clearly, these two axioms and ideas are not only independent but also fundamentally different.

An alternative approach is to apply a *Probabilistic Serial* (PS) mechanism (Bogomolnaia and Moulin, 2001), the *Rationing Eating (RE)* rule, to the current setting: Categories are treated as pseudo-agents and the agents as pseudo-items, as if categories are 'consuming' agents. The pseudo-agents categories now have 'preferences' over the pseudo-items that are derived from the priorities of the corresponding categories. Then, the *PS* rule is implemented on this pseudo-market (Aziz, 2021): beginning from time zero, each pseudo-agent 'eats' at each time the best available pseudo-item with respect to their 'preferences' at the same rate until each pseudo-agent achieves the consumption level given as the amount of units reserved for the corresponding category. The *RE* rule retains the fairness property (*sd-envy-freeness*) in this pseudo-market: *category sd-envy-freeness.* Since we do treat categories as pseudo-agents, this work and the current one are also fundamentally different. The following example demonstrates this point.

Example 2. *Suppose there are four agents, i, j, k and l. Agents i and j are essential workers, and agents i, k and l are disadvantaged community members. The policymakers implement a reserve system where one unit is reserved for essential workers and two units are reserved for disadvantaged community members. For essential worker category, i has a higher priority than j, and for disadvantaged community member category, k has a higher priority than i, and i has a higher priority than l. Suppose the PS mechanism is implemented for this problem where the two categories are pseudo-agents, and the agents are pseudo-items. The PS rule applies to this pseudo-market as follows: Pseudo-agent essential worker 'consumes' one unit of pseudo-item 'i', and pseudo-agent disadvantaged community member 'consumes' one unit of pseudo-item 'k'. Then, pseudo-agent disadvantaged community member has one more unit to 'consume' and 'consumes' pseudo-item l. Thus, this approach gives the following allocation: from the essential worker category, agent i is assigned one unit, and from the disadvantaged community member category, agents k and l are assigned one unit each. Note that this is the same allocation when the reserve system is implemented with the precedence order of essential worker category being processed first. Our approach is substantially different. When agents i and k are assigned one unit each, the reserve requirement is met: one unit for the essential worker category, and two units for the disadvantaged community member category. Also, one unit remains to be allocated. Our approach is to allocate this unit by considering equity among agents rather than treating disadvantaged community member category as a pseudo-agent entitled to 'consume' it. We revisit this example in Section 6.3 to illustrate our approach.*

Our fairness notions do not allow us to apply the *PS* rule directly for the categories (pseudo-agents) over the agents (pseudo-items). Actually, our approach is the opposite: agents 'consume' categories (not the other way around). This leads to an analytical challenge: We should keep track of who can be assigned to units from which categories at a given instance of the random allocation rule. We explain this technical challenge, and propose a methodology for overcoming this difficulty (Section 5).

The idea of *egalitarianism* and the principle of maximizing the minimum welfare are studied in several other contexts of discrete allocation models.[6] Recently, another such work analyzes the incentive schemes designed for plasma donation (Kominers, Pathak, Sönmez, and Ünver, 2020). Plasma donors are given priorities for prospective plasma therapies of their loved ones (*pay-it-backward*), and patients receive priority access for plasma therapy in exchange for a pledge to donate her own plasma in the near future (*pay-it-forward*).[7] The authors also design a mechanism, *plasma pooling procedure*, which guarantees an *egalitarian* distribution of plasma therapy by making non-prioritized patients' welfare as equal as possible across different blood types within *efficiency* constraint.[8]

2 Systems for rationing discrete medical units

Priority systems. Under priority systems, patients are ranked with respect to a single priority ordering, which is obtained by means of a scoring function incorporating a single principle[9] or a set of multiple principles.[10] Priority systems are widely criticized mostly because they could leave certain groups of patients with no or very little access to medical units. For example, the consequence of prioritizing frontline healthcare workers could be that other groups systematically would be deprived access.

Reserve systems. Reserve systems are proposed to eliminate the shortcomings of the priority systems due to a single priority ordering. The main goal is to provide a fair access to medical units across different interest groups. Basically, all units are divided into *reserve categories* (disadvantaged communities, essential workers etc.), where a certain number of units is reserved for each category and each category has its own priority ordering of patients.

Reserve categories are usually processed in a precedence order in deterministic applications of these systems and patients are, in general, eligible to receive a medical unit through multiple categories. This has an important implication: sequential processing of categories has distributional consequences for

[6] See Bogolomania and Moulin (2004), Roth, Sönmez, and Ünver (2005) and Yılmaz (2011).

[7] A different health care setting where similar incentive schemes are analyzed is a kidney exchange model where compatible pairs are incentivized to participate in kidney exchange by insuring their patients against future renal failure via increased priority in the deceased-donor queue (Sönmez, Ünver, and Yenmez, 2020).

[8] This method is also based on graph theoretical ideas and in particular, on parametric flows (see also Katta and Sethuraman (2006)).

[9] An example is the 2015 New York State Ventilator Guideline: Eligible patients are ranked with respect to a regularly reevaluated mortality risk (Zucker, Adler, Berens, Bleich, Brynner, Butler, et al., 2015)

[10] Such a system aggregates several ethical criteria to obtain a score for each patient and a single ordering of patients (White, Katz, Luce, and Lo, 2009).

patients (Pathak, Sönmez, Ünver, and Yenmez, 2021). This leads to considerations of lottery systems for a fair division of medical units.

Lottery systems. The Department of Health, Pennsylvania has been recently implementing a weighted lottery mechanism for the allocation of medications to treat COVID-19 (Pennsylvania DH, 2020). As outlined in the "Pandemic Guidelines for the Interim Pennsylvania Crisis Standards of Care", this framework is designed such that "all patients who meet clinical eligibility criteria should have a chance to receive treatment". In the preliminary step, the number of available courses of the COVID-19 therapy is determined and the number of eligible patients (for which the drug is allotted) is *estimated*. By dividing the first number by the second, the chances for each eligible "general community" patient to receive the drug is determined. In the second step, patients' characteristics relevant to the weighted lottery are determined to adjust the *general community chances* found in the preliminary step (Table 1). Then, a

Group	**Chances to receive treatment**
Disadvantaged community member (c_1)	1.25 x (general community chances)
Essential worker (c_2)	1.25 x (general community chances)
Death likely within 1 year (c_3)	0.5 x (general community chances)
Disadvantaged community member + Essential worker	1.5 x (general community chances)
Disadvantaged community member + death likely within 1 year	0.75 x (general community chances)
Essential worker + death likely within 1 year	0.75 x (general community chances)

Table 1: Probabilities in the weighted lottery

lottery number between 1 and 100 is randomly selected for each eligible patient. If the lottery chances for the patient is x out of 100 and the patient's randomly drawn lottery number is less than or equal to x, they should be offered the scarce drug. If the lottery number is greater than x, then they should not be offered the scarce drug.[11]

Our goal is to provide a general model for lottery systems. We introduce a methodology for processing reserves simultaneously and probabilistically.[12]

[11] There are two issues with this mechanism. First, the implementation of the lottery (i.e. single patient-single lottery) does not imply a probability distribution. Second, since these probabilities are fixed and do not depend on the number of patients in each group, *target ratios between the weights of each pair of patient groups* (Table 1) are not feasible in general.

[12] While our solution is more general than aiming particular chances to receive treatment for interest groups, it can be specifically applied to the case introduced by the Department of Health, Pennsylvania by correcting the infeasibility in their mechanism (see Section 8).

3 Model

There is a set of **agents** $\mathcal{I}$ and a set of **reserve categories** (or shortly, **categories**) $\mathcal{C}$. For each $c \in \mathcal{C}$, q_c identical units are reserved, and there is a **weak priority order** π_c over $\mathcal{I}$. The strict and indifference parts of π_c are denoted by π_c^P and π_c^I, respectively. For each c, the set of agents in the k−th indifference class of π_c is $\mathcal{I}_{\pi_c}(k)$ such that for $k' > k''$, $i \in \mathcal{I}_{\pi_c}(k')$ and $j \in \mathcal{I}_{\pi_c}(k'')$ imply $j \ \pi_c^P \ i$. The set of agents in the first k indifference classes is denoted by $UCS_{\pi_c}(k)$, thus, $UCS_{\pi_c}(k) = \bigcup_{k'=1}^{k} \mathcal{I}_{\pi_c}(k')$.

A **(rationing) problem** is a tuple $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$. Let $\mathcal{R}$ denote the set of all problems. We consider a setting where units are assigned to agents probabilistically such that for each $R \in \mathcal{R}$, the probability with which an agent is assigned a unit is at most one and for each $c \in \mathcal{C}$, at most q_c units are assigned to agents.

Definition 1. *Given a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, a* ***random allocation*** *is a stochastic $|\mathcal{I}| \times |\mathcal{C}|$ matrix Z where for each i and c, z_{ic} is the probability with which agent i is assigned one unit from category c such that*

i. for each $i \in \mathcal{I}$, $\sum_{c\in\mathcal{C}} z_{ic} \leq 1$,
ii. for each $c \in \mathcal{C}$, $\sum_{i\in\mathcal{I}} z_{ic} \leq q_c$.

Let $\mathcal{Z}(R)$ denote the set of all random allocations for a problem R, and $\mathcal{Z} = \bigcup_{R\in\mathcal{R}} \mathcal{Z}(R)$ the set of all random allocations. A **rule** is a mapping $\varphi : \mathcal{R} \to \mathcal{Z}$ such that for each problem R, $\varphi(R) \in \mathcal{Z}(R)$.

Since all units are identical, only the probability of receiving a unit is relevant for agents, not the specific categories through which they are (randomly) assigned a unit. Let $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$ be a problem and $Z \in \mathcal{Z}(R)$ a random allocation. The **utility of agent** i is given by $u_Z(i) = \sum_{c\in\mathcal{C}} z_{ic}$. The vector $u_Z = (u_Z(i))_{i\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ is the **utility profile**. We also say that a utility profile u is **generated** by a random allocation Z if $u = u_Z$. Random allocations Z and Z' are **welfare equivalent** if $u_Z = u_{Z'}$. Similarly, rules φ and φ' are *welfare equivalent* if for each problem R, random allocations $\varphi(R)$ and $\varphi'(R)$ are *welfare equivalent.*

Remark 1. *The theoretical contribution of our model is* ***not*** *the assumption of weak priority orders. Even under strict priorities, all the analytical and conceptual challenges (handled in this work) remain. Nevertheless, the assumption of weak priorities is crucial for the scope of fairness in most of the real-life applications, as we discussed in the Introduction.*

4 Axioms

There are two indispensable requirements: (1) Resources should not be wasted (*efficiency*), and (2) an agent can be assigned a unit under a category only if each agent with a strictly higher priority for that category is assigned a unit with probability one (*respecting priorities*).

The first axiom states that no unit should be wasted. If there are agents demanding a unit and that unit is available, then it should not remain as unassigned.

Definition 2. *For a problem $R = (\mathcal{I}, C, (\pi_c)_{c\in C}, (q_c)_{c\in C})$, a random allocation $Z \in \mathcal{Z}(R)$ is* ***non-wasteful****, if for any $c \in C$,*

$$\sum_{i\in\mathcal{I}} z_{ic} < q_c \implies \text{for each } i \in \mathcal{I}, \sum_{c'\in C} z_{ic'} = 1.$$

A rule φ is ***non-wasteful*** *if for any problem R, random allocation $\varphi(R)$ is non-wasteful.*

The only case for a unit remaining (partially) unassigned under *non-wastefulness* is when each agent is assigned a unit with probability one. For expositional simplicity, we exclude these cases: A problem is **non-trivial**, if it is not possible to assign each agent a unit. We assume that each problem in $\mathcal{R}$ is *non-trivial.*[13] Clearly, *non-wastefulness* and *non-triviality* together imply that Condition *(ii)* of Definiton 1 holds with equality.

The second axiom is about priorities: an agent cannot be (probabilistically) assigned a unit from a category if there is another agent with a strictly higher priority and a utility less than one.

Definition 3. *For a problem $R = (\mathcal{I}, C, (\pi_c)_{c\in C}, (q_c)_{c\in C})$, a random allocation $Z \in \mathcal{Z}(R)$* ***respects priorities****, if for any $i \in \mathcal{I}$, and $c \in C$,*

$$i \; \pi_c^P \; j \text{ and } u_Z(i) < 1 \implies z_{jc} = 0$$

A rule φ ***respects priorities*** *if for any problem R, random allocation $\varphi(R)$ respects priorities.*

5 Acceptable random allocations

Our analysis throughout the paper is based on the simple idea of sequentially updating the probabilities with which agents are assigned a unit. We refer to these probability vectors as **reservation profiles** in general, generically denoted by $v = (v_i)_{i\in\mathcal{I}}$. Although the *utility profile* u_Z under a random allocation Z and the *reservation profile* $v = (v_i)_{i\in\mathcal{I}}$ are mathematically the same type of objects, there is an important difference between them: While a *utility profile* represents agents' utilities induced by a random allocation, the interpretation of a *reservation profile* $v = (v_i)_{i\in\mathcal{I}}$ is that agent i is guaranteed a utility level at least as much as v_i, without any implication of a specific random allocation and agents' utilities. A reservation profile v is **feasible** if there exists a random allocation Z such that $v = u_Z$.

We consider only the rules, which satisfy the axioms in Section 4. For any problem R, a random allocation $Z \in \mathcal{Z}(R)$ is **acceptable** if it satisfies *non-wastefulness* and *respects priorities.* We denote the set of

[13] There is no loss of generality in assuming *non-triviality*: the definitions and results hold also for *trivial* problems.

acceptable random allocations by $\mathcal{Z}^a(R)$. A rule φ is **acceptable** if for each problem R, $\varphi(R) \in \mathcal{Z}^a(R)$.

We first characterize *acceptable* rules. These rules are based on a procedure of sequential improvement of agents' utilities by (probabilistically) assigning units simultaneously. This is a simple idea but its design is surprisingly complicated for several reasons. Next, we analyze these challenges.

First, since agents can receive units from different categories, it is not clear which agents should have access to a given category at a given instance of improving utilities.

Example 3. *(Determination of agents' access to categories)*
Let $\mathcal{I} = \{i, j, k\}$ and $\mathcal{C} = \{c_1, c_2\}$ such that one unit is reserved for each category. The priority orders for categories are given below with each set in the table being an indifference class (in all the examples, we use the same type of representation for a problem and we present only the first few indifference classes that matter for the argument):

π_{c_1}	π_{c_2}
$\{i\}$	$\{i, j\}$
$\{k\}$	$\{k\}$

A plausible argument is the following: Respecting priorities implies that a unit should be assigned to i with probability one (since otherwise, k is assigned a unit under c_1 with a positive probability, violating priorities). Also, since j has a higher priority than k at c_2, the remaining unit should be assigned to j, which is an acceptable allocation. On the other hand, for each $\lambda \in [0, \frac{1}{2}]$, the following random allocation is also acceptable:

$$Z = \begin{array}{c|cc} & c_1 & c_2 \\ i & \frac{1}{2}+\lambda & \frac{1}{2}-\lambda \\ j & 0 & \frac{1}{2}+\lambda \\ k & \frac{1}{2}-\lambda & 0 \end{array}$$

In Example 3, since the unit under c_2 can also be (probabilistically) assigned to i (along with the unit under c_1), there is room for the unit under c_1 to be (probabilistically) assigned to k. This demonstrates the first difficulty: how should agents' access to categories be defined in the most comprehensive way?

Definition 4. *Let $v = (v_i)_{i \in \mathcal{I}}$ be a reservation profile. Agent i is a **claimant** for category c under v if $i \in \mathcal{I}_{\pi_c}(k)$ such that either $k = 1$ or for each $j \in UCS_{\pi_c}(k-1)$, $v_j = 1$. The set of claimants for category c under v is denoted by $\Gamma_c(v)$.*

Whenever the first $k-1$ indifference classes consist of only agents with reservation value one, all these agents and the agents in the kth indifference class are *claimants* for the corresponding category.[14] This implies that these agents (*claimants*) can receive units from these categories.

[14] In Example 3, i and k are *claimants* for c_1, while i and j are *claimants* for c_2 (note that k is not a *claimant* for c_2).

Second, there is an exception to this intuition: as the following example demonstrates, it is not always the case that units under a category can be assigned to all of its *claimants*.

Example 4. *(Claimants cannot always receive a positive share.)*
Let $\mathcal{I}=\{i,j,k\}$ *and* $C=\{c_1,c_2\}$ *such that one unit is reserved for each category. Consider the following category priorities:*

π_{c_1}	π_{c_2}
$\{i\}$	$\{j\}$
$\{k\}$	$\{k\}$

For the reservation profile $v=(v_i,v_j,v_k)=(1,1,0)$*, all agents are claimants for all categories. But, any random allocation such that a unit is (probabilistically) assigned to* k *does not respect priorities. Thus, although* k *is a claimant of both categories at the given reservation profile, at an acceptable allocation, she cannot be assigned any unit from these categories.*

In Example 4, *respecting priorities* implies that agents i and j are assigned all units, one unit each. Thus, the units under c_1 and c_2 should be 'exclusively reserved' for agents i and j.

Let $v=(v_i)_{i\in\mathcal{I}}$ be a reservation profile. For each i with $v_i>0$, let $C(i,v)$ denote the set of categories, for which agent i is a *claimant* under the reservation profile v. Let $C(I,v)=\bigcup\limits_{i\in I} C(i,v)$.

Definition 5. *Given a reservation profile* $v=(v_i)_{i\in\mathcal{I}}$*, if for a set of agents* I*, we have* $\sum\limits_{i\in I} v_i=\sum\limits_{c\in C(I,v)} q_c$*, then the units under the categories in* $C(I,v)$ *(or shortly, the categories in* $C(I,v)$*) are* ***exclusively reserved*** *for* I.

For a given reservation profile, *exclusively reserved* categories correspond to binding feasibility constraints. The subtle point here is that under *exclusively reserved* categories for I, while each agent in I can be assigned probabilities from **all** the categories for which she is a *claimant*, not **all** the *claimants* of these categories can receive units from them. Thus, a case of *exclusively reserved* categories is an exception to the general idea that any *claimant* of a category should be able to receive units from that category. Moreover, it turns out that, in the process of sequential improvement of agents' reservation values, (i) as long as there are no *exclusively reserved* categories, the reservation values of all *claimants* can be increased, and (ii) when a reservation profile is achieved, for which there are *exclusively reserved* categories, that reservation profile is *feasible*. This insight is given by the following characterization theorem (see Appendix B for how we utilize this important insight to prove the characterization of the *acceptable* rules (Theorem 2)).

Theorem 1. *(The Supply-Demand Theorem (Gale, 1957))*[15]
Let $v=(v_i)_{i\in\mathcal{I}}$ *be a reservation profile. There is a random allocation* Z *such that (i) for each* $i\in\mathcal{I}$, $u_z(i)\geq$

[15] This is a generalization of Hall's Set Representation Theorem (Hall, 1935), which holds only for integers.

v_i*, and (ii)* $z_{ic} > 0$ *implies* $i \in \Gamma_c(v)$*, if and only if, for each subset* I *of agents*

$$\sum_{i \in I} v_i \leq \sum_{c \in C(I,v)} q_c. \tag{1}$$

Third, the approach of sequentially updating the reservation values requires keeping track of changes in the set of *claimants*: while an agent may not be a *claimant* for a category at a given *reservation profile*, as the agents' reservation values possibly go up, she might be a *claimant* for it at a different one.

Example 5. *(Sequential improvement of agents' access to categories)*
Let $\mathcal{I} = \{i, j, k, m\}$ *and* $C = \{c_1, c_2, c_3\}$ *such that one unit is reserved for each category. Consider the following priorities:*

π_{c_1}	π_{c_2}	π_{c_3}
$\{i, j\}$	$\{k\}$	$\{k\}$
	$\{i, m\}$	$\{j, m\}$

Under the reservation profile $v = (v_i, v_j, v_k, v_m) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0)$*, agents* i, m *and* j, m *are not claimants for* c_2 *and* c_3*, respectively, but under* $v' = (\frac{1}{2}, \frac{1}{2}, 1, 0)$*, they are. Thus, as agents' probabilities at categories increase sequentially, the set of claimants might change.*

The fact that the set of *claimants* changes with respect to the reservation profile further complicates the implementation of the simple idea of 'improving agents' utilities sequentially'. Note that in the last example, under the reservation profile v, category c_1 is *exclusively reserved* for i and j. But, when the reservation value of another agent (agent k) is increased to one (so that the reservation profile becomes v'), there is no *exclusively reserved* category anymore: when i, m and j, m become *claimants* for c_2 and c_3, respectively, the condition for *exclusively reserved* categories in Definition 5 does not hold.

We characterize the set of *acceptable* random allocations by a sequential allocation procedure: the **Priority-Based Sequential Welfare Improvement (PBSWI)** solution, or in short **sequential improvement** solution. The design relies on careful treatment of the difficulties discussed above. The idea is to sequentially update agents' access to categories through the criterion of being a *claimant* by keeping track of instances such that there are *exclusively reserved* categories.

The PBSWI Class:

Step 0. Let the reservation profile be $v^0 = (v_i^0)_{i \in \mathcal{I}}$ such that for each $i \in \mathcal{I}$, $v_i^0 = 0$.

For each $n \geq 1$ and the reservation profile v^{n-1}, the following steps are executed.

Step n.1 For each set of agents I such that categories in $C(I, v^{n-1})$ are *exclusively reserved* for I,

i. for each $i \in I$, let $v_i^n = v_i^{n-1}$, and
ii. mark each category in the set $C(I, v^{n-1})$ as *unavailable*.

Let A_n denote the set of *available* categories.

Step n.2 If $A_n = \emptyset$, then let $Z^\star$ with $u_{Z^\star} = v^{n-1}$ be the outcome. Otherwise, proceed to Step n.3.

Step n.3 *(Welfare improvement)* Select a *feasible* reservation profile $v^n \neq v^{n-1}$ such that for each i, $v_i^n = v_i^{n-1} + \lambda_i^n$ where $\lambda_i^n \in [0,1]$, and for each $i \notin \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, $\lambda_i^n = 0$.

The *PBSWI* selects a welfare improvement at each step, and it is a class of rules since each sequence of these selections implies a different random allocation. To define a rule in the *PBSWI* class, it is sufficient to specify the selection rule of welfare improvement at Step n.3. (We define such a rule in Section 6.4.) For each problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, let *PBSWI*$(R)$ denote the set of all random allocations obtained by the class *PBSWI*.

Theorem 2. *For a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, a random allocation Z is acceptable if and only if $Z \in$ PBSWI(R).*

Proof. See Appendix B □

This result provides an insight on how to describe an *acceptable* random allocation by means of a sequence of welfare improvement profiles. We use this insight later when we characterize the set of 'equitable' (see Section 6) allocations by means of allocation rules (see Section 6.4).

6 Enhancing equity

As we argue via Example 1 in the Introduction, equity in reserve systems is crucial. There are two possibly sensible formulations of equity in the current setting, which we formally analyze in the following sections: (1) Agents' utilities should be equalized as much as possible *(egalitarianism)*. (2) Categories should be treated fairly unless there is a justification for unfairness *(category-fairness)*.

Our first observation is that an *egalitarian* rule does not exist in the current context (Theorem 3 in Section 6.1). Moreover, even if an *egalitarian* random allocation exists for some problem, it may not even treat equals as equal, the most fundamental principle of fairness (Section 6.1). Thus, *egalitarianism* is not plausible in the current setting. We propose *category-fairness,* an axiom which requires fair treatment of categories unless there is a justification for unfairness (Section 6.2). Then, we formulate a different notion, *procedural fairness* (Section 6.3). This axiom is in the spirit of *egalitarianism.* Yet, it is independent from it, and it is stronger than both *category-fairness* and *equal treatment of equals* (Section 6.3). Finally, we characterize the rules satisfying *procedural fairness* in the class of *acceptable* rules (Section 6.4).

6.1 Egalitarianism

The standard formulation of egalitarian access to resources is 'equating utilities as much as possible' through the *Lorenz dominance* criterion. For any vector $u \in \mathbb{R}^{|\mathcal{I}|}$, let $u^\star$ be the vector obtained upon rearranging the coordinates of u increasingly. Given a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$ and $Z,\ Z' \in \mathcal{Z}(R)$, Z **Lorenz dominates** Z' if

$$\text{for each } l = 1, \ldots, |\mathcal{I}| : \sum_{m=1}^{l} ((u_Z^\star)_m - (u_{Z'}^\star)_m) \geq 0.$$

The question of defining equitable access is entangled with the indispensability of the axioms in Section 4. Fortunately, it can easily be adapted to the current context.

Definition 6. *A random allocation $Z \in \mathcal{Z}^a(R)$ is **egalitarian** if it is Lorenz dominant in the set $\mathcal{Z}^a(R)$. A random allocation $Z \in \mathcal{Z}^a(R)$ is **weakly egalitarian** if it is not Lorenz dominated by another allocation in the set $\mathcal{Z}^a(R)$. A rule φ is **(weakly) egalitarian** if for any problem R, random allocation $\varphi(R)$ is (weakly) egalitarian.*

There are two important issues regarding an *egalitarian* random allocation: First, it turns out that a *Lorenz dominant* allocation may not exist in the set of *acceptable* random allocations.[16]

Theorem 3. *No rule is egalitarian.*

Proof. See Appendix C. □

Second, even if an *egalitarian* random allocation exists for a problem, it does not necessarily '*treat equals as equal*', as the following example demonstrates. (Clearly, this observation holds also for *weakly egalitarian* allocations.)

Example 6. *(A (weak) egalitarian random allocation does not necessarily 'treat equals as equal'.)*
Let $\mathcal{I} = \{i, j, i_1, i_2, j_1, j_2, k, l\}$ and $\mathcal{C} = \{c_1, c_2\}$ such that three units are reserved for each category. Consider the following category priorities:

π_{c_1}	π_{c_2}
$\{i, j\}$	$\{i, j\}$
$\{i_1, i_2\}$	$\{j_1, j_2\}$
$\{k, l\}$	$\{k, l\}$

First, note that agents i and j qualify for both categories and this implies a surplus for agents i_1 and i_2 under c_1, and for j_1 and j_2 under c_2. Thus, the claims of these two groups of agents over the surplus should be treated equally. Second, it turns out that an egalitarian allocation does not treat equals equally. To

[16]The impossibility still holds even if we restrict the domain of priority orders (see Appendix C).

see this, let us first characterize the set of egalitarian random allocations for the case where the initial reservation profile is obtained by treating categories separately (see Footnote 14). Then, (i) agents i and j are assigned a unit each with probability one, (ii) agents i_1, i_2, j_1 and j_2 are assigned a unit each with probability at least half. Thus, there are four units remaining with the constraint (ii). Non-wastefulness and respecting priorities imply that there are three alternatives for these units:

1. i_1, i_2, j_1, j_2 *(each with probability one)*
2. i_1, i_2 *(each with probability one) and* j_1, j_2, k, l *(*j_1, j_2 *each with probability at least half)*
3. j_1, j_2 *(each with probability one) and* i_1, i_2, k, l *(*i_1, i_2 *each with probability at least half)*

The second and third alternatives provide access to a higher number of agents than the first alternative. Thus, (it is straightforward to check that) an acceptable random allocation is egalitarian if and only if it generates one of the following utility profiles:

$$u = (u_i, u_j, u_{i_1}, u_{i_2}, u_{j_1}, u_{j_2}, u_k, u_l) = (1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}),$$

$$u' = (u'_i, u'_j, u'_{i_1}, u'_{i_2}, u'_{j_1}, u'_{j_2}, u'_k, u'_l) = (1, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}).$$

But note that, each of them favors either agents i_1 *and* i_2 *over agents* j_1 *and* j_2*, or vice versa. The reason is simple: by granting, say agents* i_1 *and* i_2*, a unit each, the remaining two units can be (probabilistically) allocated to agents* j_1, j_2, k *and* l*, instead of allocating four units equally among agents* i_1, i_2, j_1 *and* j_2*. Thus, neither of them treats* $\{i_1, i_2\}$ *and* $\{j_1, j_2\}$ *equally, and we conclude that an egalitarian allocation does not necessarily treat equals as equal.*

6.2 Reserve systems under a baseline priority order: Category-fairness

In many applications of reserve systems, there is an all-inclusive category with a baseline priority order. The priority ordering of preferential treatment categories are derived by prioritizing their beneficiaries over others and preserving their relative ranking in the baseline priority order. Typical examples of a baseline priority order are (i) the merit category in school choice, the ranking for which is determined by the merit scores of students, and (ii) the general community category in allocating medical units, where all eligible patients are ordered based on estimated mortality risk measured by the SOFA score (see Section 8 for both of these applications).[17] Note that in these models, the baseline priority ordering itself represents a category.

In this section, we assume, without loss of generality[18] and for notational simplicity, a strict ordering

[17] See Section 2.1 in Pathak, Sönmez, Ünver, and Yenmez (2021) for examples of pandemic resource allocation and more on the model with a baseline priority order.

[18] The definitions in this section and the related result in the next section extend for weak priority orderings.

of agents under the baseline priority order. For ease of convenience, we use the language of affirmative action in schools' seat assignment problem: there is a merit category, where all students are strictly ordered with respect to their merit scores (exam scores, composite scores, etc.), and there are preferential treatment categories (minorities, disadvantaged (or all) socioeconomic groups, etc.).

A rationing problem $R = (\mathcal{I}, C \cup \{c_M\}, \succ, (q_c)_{c \in C \cup \{c_M\}}, (B_c)_{c \in C})$ is a **reserve problem with a baseline priority order**, where c_M is the merit category, $\succ$ is the strict (baseline) priority ordering via merit scores, C is the set of preferential treatment categories, and for each $c \in C$, B_c is the set of beneficiaries of c.

We propose and formulate a compelling fairness notion: *category-fairness*. *Category fairness* essentially requires that if there is no justification for uneven treatment of a category (besides *respecting priorities* and *non-wastefulness*), then this unfairness should be eliminated.

We first define *unfairness for the merit category*. Let Z be an allocation. Let $i_M(Z)$ be the student who has a utility less than one such that each student j, who has a higher merit score than $i_M(Z)$, has a utility one.[19] Now, suppose that for each such student j, there exists a preferential treatment category c such that more than $|q_c|$ of its beneficiaries are assigned a seat, among which there is at least one student with a lower score than $i_M(Z)$. Then, we say that Z is *unfair for the merit category*. While $i_M(Z)$ is either not assigned a seat at all or assigned with probability less than one, for each relevant preferential treatment category c, there exists a beneficiary with utility one but with a lower score than $i_M(Z)$. This is not justified by the reserve requirements since more than $|q_c|$ of its beneficiaries are assigned a seat.

Definition 7. *A random allocation Z is **unfair for the merit category** (c_M) if for each j such that $j \succ i_M(Z)$, there exists $c \in C$ such that $j \in B_c$, and $|\{i' \in B_c : u_Z(i') = 1\}| > q_c$ and the set $\{i' \in B_c : u_Z(i') = 1\}$ contains at least one student with a lower merit score than $i_M(Z)$.*

To illustrate the notion, we present the following example.

Example 7. *(Unfairness for the merit category)*
Let us consider a school's seat assignment problem with affirmative action where $\mathcal{I} = \{i, j, k, l, i_1, i_2\}$ is the set of students and $C = \{c_M, c_1, c_2\}$ is the set of categories such that two seats are reserved for c_M (merit category), two seats are reserved for socioeconomic category c_1 and one seat is reserved for socioeconomic category c_2. Consider the following category priorities:

π_{c_M}	π_{c_1}	π_{c_2}
$\{j\}$	$\{j\}$	$\{k\}$
$\{k\}$	$\{l\}$	$\{i_2\}$
$\{l\}$	$\{i_1\}$	
$\{i\}$		

[19] Note that this student is uniquely well-defined. Also, there could be students with lower merit score than i and assigned to a seat with probability one.

Let Z be the allocation where $z_{jc_M} = z_{kc_M} = z_{lc_1} = z_{i_1c_1} = z_{i_2c_2} = 1$, and all other probabilities are zero. Then, $u_Z(i) = 0$ and $u_Z(j) = u_Z(k) = u_Z(l) = u_Z(i_1) = u_Z(i_2) = 1$. Note that Z is unfair for c_M.[20] *This unfairness is not justified since there is another acceptable (random) allocation Z' obtained from Z with the following probability transfers: $z'_{i_1c_1} = z_{i_1c_1} - \lambda$, $z'_{jc_1} = z_{jc_1} + \lambda$, $z'_{jc_M} = z_{jc_M} - \lambda$, and $z'_{ic_M} = z_{ic_M} + \lambda$. Note that this is essentially a utility transfer from i_1 to i, and mitigates the unfair treatment of c_M.*

We now define *unfairness for a preferential treatment category*. Let Z be a random allocation. Let $i_{\min}(Z)$ be the student with the lowest merit score, who is assigned a seat from the merit category with positive probability. Let c be a preferential treatment category such that each student, who is a beneficiary of c and assigned a seat with positive probability, has a higher merit score than $i_{\min}(Z)$. Then, we say Z is *unfair for c*. All the beneficiaries of c, who are assigned a seat, have higher merit scores than $i_{\min}(Z)$ and they should have been assigned a seat because of their merit anyways. Thus, the beneficiary of c, who is not assigned a seat and has the highest score among such beneficiaries of c objects to Z for this unfulfilled preferential treatment of c.

Definition 8. *A random allocation Z is **unfair for the preferential category** $c \in C$ if for each $j \in B_c$ with $u_Z(j) > 0$, $j \succ i_{min}(Z)$.*

We next revisit the structure in Example 1 to illustrate this notion.

Example 1 revisited. *(Unfairness for a preferential treatment category)* There are three student i, j, and k. There are two categories, the merit category and the preferential treatment category c_1. Students j and k are beneficiaries of c_1. One unit is reserved for each category. The priority orders are as follows:

π_{c_M}	π_{c_1}
$\{j\}$	$\{j\}$
$\{i\}$	$\{k\}$
$\{k\}$	

Note that any *acceptable* allocation assigns student j a seat with probability one. Let us consider the allocation where i is also assigned a seat with probability one. Now, i is the student with the lowest merit score, who is assigned a seat from the merit category with positive probability (with probability one actually). The only preferential treatment category student who is assigned a seat is j and she has a higher score than i. Thus, this allocation is *unfair for c_1*.[21]

We next define *category-fairness* based on the notions of *unfairness for the merit category* and *unfairness for the preferential treatment categories.*

[20] Also, note that this allocation is the outcome of the solution when merit category is processed first.
[21] Also, note that this allocation is the outcome of the solution when preferential treatment category c_1 being processed first.

Definition 9. *Let* $R = (\mathcal{I}, C \cup \{c_M\}, \succ, (q_c)_{c \in C \cup \{c_M\}}, (B_c)_{c \in C})$ *be a reserve problem under a baseline priority order. A random allocation* $Z \in \mathcal{Z}(R)$ *satisfies* ***category-fairness****, if for each* $c \in C \cup \{c_M\}$*, it is not unfair for* c*. A rule* φ *satisfies* ***category-fairness*** *if for each problem* R*,* $\varphi(R)$ *satisfies category-fairness.*

The question is whether there always exists an *acceptable* random allocation, which satisfies *category-fairness*.[22] The existence is not clear particularly because eliminating *unfairness for merit* and *preferential treatment categories* seem like conflicting tasks. Surprisingly, the answer is affirmative and existence follows from Theorem 2 above, and Proposition 1 and Theorem 4 below.

When/why is *category-fairness* relevant? Let us consider the model where each student belongs to a preferential treatment category (e.g. when students are divided into socioeconomic tiers). For each random allocation Z, since any student with a higher merit score than $i_M(Z)$ is a beneficiary of a preferential treatment category, *category-fairness* imposes constraints on the utilities of the beneficiaries of the associated preferential treatment categories, and is never vacuous. Thus, for an important class of reserve systems, *category-fairness* is relevant for any random allocation. For other models of reserve systems, *category-fairness* is relatively weaker since a higher priority agent at a given category might not be a beneficiary of another category. Motivated by this observation, we next propose a stronger fairness axiom, which is relevant for any model of reserve system and non-vacuous for any random allocation.

6.3 Procedural fairness

As we argued in Section 5, **each *acceptable* random allocation can be described as the outcome of a sequential (welfare) improvement solution** (Theorem 2). This is a step-by-step 'partial allocation' procedure, where at each step, units become available **only** for certain agents (i.e. *claimants* in Definition 4), and can be allocated **only** to these agents. Since we allow for random allocations, at each step, there are infinitely many *feasible* 'partial allocations' of available units to this uniquely defined set of *claimants*. Thus, a natural equity concept requires *claimants*' equitable access to their available units, which is defined as equalizing agents' utilities as much as possible.

Definition 10. *A sequential improvement solution is procedurally fair if at each step, the selected reservation profile Lorenz dominates any other feasible reservation profile that can be selected at that step.*

Discussion of *procedural fairness.* If, at an instance of the sequential welfare improvement process, each agent is the beneficiary of only one category, then intuitively, they should be assigned the available unit under that category (this is the case in Example 4). This is actually implied by *procedural*

[22] *Category-fairness* is easily satisfied if *non-wastefulness* or *respecting priorities* is not imposed: If *non-wastefulness* is not imposed, then the allocation, where each agent receives each object with probability zero, is *category-fair*. If *respecting priorities* is not imposed, then any allocation of equal share (all agents have the same utility) is *category-fair*. Also, as implied by the last two examples, any solution via sequential processing of categories does not satisfy *category-fairness*.

fairness. If, on the other hand, they are the beneficiaries of multiple categories, then, since they can be assigned to a unit under multiple categories, there is a surplus (this is the case in all the examples above, except Example 4). In this case, *procedural fairness* requires that this surplus is allocated to the beneficiaries in an egalitarian way (in the sense of *Lorenz dominance*). It is conceptually the same notion as *'procedural equality'* underlying the *Probabilistic Serial (PS)* solution.[23] The difference is that in the current context, granting equal probabilities is not the right concept for equity: Since agents are ranked differently under categories, typically, a different set of agents (*claimants*) is assigned probabilities at each step. Thus, in general, agents have different levels of utilities at a given instance of the process. To restore equity, *procedural fairness* takes the *reservation profile* in the previous step into account to obtain an equal allocation of available units in the sense of *Lorenz dominance.* Thus, at each step, we replace *'procedural equality'* with an assignment of probabilities which *Lorenz dominates* any other feasible assignment. Nevertheless, conceptually, these two approaches are strongly similar.[24]

A fair question is the following: what is the point of stating *procedural fairness* for reserve systems *per se*, while there is no such explicit formulation of it in the context of assignment problems? In the assignment problem context, *procedural equality* directly specifies the corresponding allocation rule. But, in the current context, to specify the rules equivalent to *procedural fairness* is technically challenging and far from being an obvious exercise. Our second theorem establishes this characterization of *procedurally fair* rules (see Theorem 4 below).[25]

Normative justification for *procedural fairness.* *Procedural fairness* provides agents with equitable access to their available units throughout the welfare improvement process,[26] and clearly, it is a notion about the procedure. But, this is not enough to justify it as a fairness property. Is there a normative justification, without any reference to the procedure? Does *procedural fairness* imply a 'fair' outcome? The answers are both affirmative. **First**, in the context of a baseline priority order, the outcome of a *procedurally fair* solution always satisfies *category-fairness.*

[23] For the assignment problem, each *(ordinally) efficient* random allocation can be described by a member of the class of *simultaneous eating* algorithms (Theorem 1 by Bogomolnaia and Moulin (2001)). By this result, it is natural to formulate *procedural equality* within this class as assigning the same amount of probability for each agent (through a profile of uniform 'eating' speeds among all possible 'eating' speed functions).

Our approach is exactly the same: Each *acceptable* random allocation can be described by a member of the class of *PBSWI* algorithms (Theorem 2), and we formulate *procedural fairness* as an equitable assignment of probabilities at each step (among all possible assignments of probabilities).

[24] Also, although there is a procedural similarity (in terms of fairness) with the *PS* solution, the current context is very different, and the *PS* rule is not applicable here, basically because, as opposed to agents ranking the objects, here the object categories rank the agents. We discuss this difference in the Related Literature in the Introduction.

[25] A technically similar exercise is the characterization of the *Lorenz dominant* allocation rule in the context of bilateral matching under dichotomous preferences (see Theorem 1 by Bogolomania and Moulin (2004)).

[26] As we have shown in Section 6.1, an *egalitarian* random allocation may not exist. Also, when it exists, it does not imply *procedural fairness* (see the first example in Section 6.3 of the online version). On the other hand, a *procedurally fair* allocation always exists (see Section 6.4). Thus, *procedural fairness* does not imply *egalitarianism.* Thus, *egalitarianism* and *procedural fairness* are independent properties.

Interestingly, while (as mentioned in Section 6.1) *egalitarianism* does not imply even *equal-treatments-of-equals*, in the domain of reserve systems under a baseline priority order, *procedural fairness* implies *category-fairness* (see Proposition 1 below), which is stronger than *equal-treatments-of-equals.*

Proposition 1. *For the reserve systems under a baseline priority order, procedural fairness implies category-fairness.*

Proof. See Appendix E. □

Second, *procedural fairness* implies *equal treatment of equals*. To formally define this notion, let $R = (\mathcal{I}, C, (\pi_c)_{c\in C}, (q_c)_{c\in C})$ be a rationing problem, and $c, c' \in C$ with $q_c = q_{c'}$ be such that

1. there is a set of agents I such that for each *acceptable* random allocation Z, units are assigned to agents in I only from c and c', and the units under c and c' are assigned only to agents in I,
2. for each $k \geq 1$, kth indifference class of both π_c and $\pi_{c'}$ contains the same number of agents, and
3. if i is not in the kth indifference class of both π_c and $\pi_{c'}$, then $z_{ic} > 0$ ($z_{ic'} > 0$) for some *acceptable* random allocation Z implies that $z'_{ic'} = 0$ ($z'_{ic} = 0$) for each *acceptable* random allocation Z'.

We call c and c' as *symmetric categories*. These conditions are about two categories' priority orders being symmetrical so that we can refer to 'equals' in this domain. Note that we could have defined symmetric categories as two categories with the same priority ordering and the same number of reserved units. But, this would be a very restrictive definitions of 'equals'. The conditions above are much weaker as a definition of 'symmetric categories'.

Given *symmetric categories* c and c', agents i and j are **equals** if they are in the kth indifference class of π_c and $\pi_{c'}$, respectively.[27]

Definition 11. *A random allocation Z satisfies* ***equal treatment of equals*** *if it generates the same utility for equals i and j. A rule φ satisfies* ***equal treatment of equals*** *if for each problem R, $\varphi(R)$ satisfies equal treatment of equals.*

As in other models, *equal treatment of equals* is relevant only for the cases when there are *equals*. We argue that having *equals* (by our definition) is quite likely, thus, the property has a scope in the current context.[28] Thus, consideration of *symmetric categories* is not vacuous and therefore, *equal treatment*

[27] This is similar to the assignment problem context where two agents are *equals* if they have the same exact preferences. Here, the notion of *equals* is broader.

[28] The first condition is actually satisfied in many real-life applications where the same group of agents are beneficiaries of at most two categories c and c'. Clearly, this is always satisfied when there are only two categories in the problem (e.g. SVI and non-SVI categories, or essential workers and disadvantaged community members), and mostly satisfied when the number of categories is low. The second condition is clearly needed so that we can talk about *equal* agents. Also, note that this is always satisfied when the priority orders are strict. Finally, the third condition is about creating symmetrical priority orders under both c and c': either the rank of i is the same under both priority orders, or if the ranks are not the same, then they are very different such that, if she might be assigned a unit from a category, then she cannot be assigned a unit from the other category under no *acceptable* random allocation. The latter part of this condition holds in many real-life applications where i is the beneficiary of only one category: commonly, reserve units are in short supply, and therefore, an agent not being a beneficiary of the other category implies that her being assigned a unit from that category is impossible, even under soft reserves. Thus, in these model with beneficiaries, the third condition can also be formulated as follows: if an agent is the beneficiary of two categories, then her rank is the same under both categories.

of equals is relevant for many applications of reserve systems.

The following result provides the second normative justification for *procedural fairness*.

Proposition 2. *For any rationing problem, procedural fairness implies equal treatment of equals.*

Proof. See Appendix F. □

Before we state our second theorem, we revisit two examples to demonstrate the idea of *procedural fairness*.

Example 2 revisited. Let $\mathcal{I} = \{i, j, k, l\}$ and $\mathcal{C} = \{c_1, c_2\}$ such that one unit is reserved c_1 and two units for c_2. Consider the following category priorities:

π_{c_1}	π_{c_2}
$\{i\}$	$\{k\}$
$\{j\}$	$\{i\}$
	$\{l\}$

Agents i and k are *claimants* for categories c_1 and c_2, respectively. Initially, there is one unit available under c_1 and two units under c_2. Clearly, the allocation such that agents i and k each receiving one unit *Lorenz dominates* any other partial allocation at this step. In the second step, all agents are *claimants*: agents i, j are *claimants* for c_1 and agents i, k, l are *claimants* for c_2. Among all possible allocations, the allocation such that agent j and l each receiving one unit with probability half *Lorenz dominates* any other partial allocation. Thus, *procedural fairness* implies that agents i, k are assigned one unit each, and agents j, l are assigned a unit each with probability half. Remember that in Example 2, we showed that under the PS rule, agent l is assigned one unit, whereas agent j is assigned zero probability of receiving a unit.[29]

Example 6 revisited. At the initial step, only i and j are *claimants* for c_1 and c_2. *Procedural fairness* requires a *Lorenz dominant* partial allocation for this step, which implies that each is assigned one unit. Now, agents i, j, i_1, i_2, j_1 and j_2 are *claimants* for c_1 and c_2. Again, the *Lorenz dominant* partial allocation in this step is that each is assigned one unit. Thus, each agent in $\{i, j, i_1, i_2, j_1, j_2\}$ is assigned one unit. Note that the idea of equal access to units at each step implies treating i_1 (i_2) equally as j_1 (j_2), and *procedural fairness* implies *equal treatment of equals*, which was not the case under *egalitarianism* as demonstrated in Example 6.

[29]Note that this is the same allocation obtained when the reserve system is implemented with the precedence order of c_1 being processed first.

6.4 The Priority-Based Rawlsian (PBR) rule

Our second goal is to incorporate equity (Section 6.3) into the *acceptable* class and characterize *procedurally fair* solutions. The design of our solution, the **Priority-Based Rawlsian (PBR)**, relies on the Rawlsian principle of maximizing the minimum welfare. Basically, the utilities of the most disadvantaged agents are increased continuously as long as the constraints embedded through *claimants* and *reservation profile* are not binding. By Theorem 2, specifying this Rawlsian improvement process as the *welfare improvement* selection rule is sufficient to define the *PBR*.[30]

Step n.3 *(Welfare improvement selection rule of the PBR)*

The agents with the minimum reservation value are *selected* among agents, who are *claimants* for at least one *available* category. Their reservation values are increased equally up to the minimum of the following two, while other agents' reservation values do not change:

- The reservation value of a *non-selected* agent, who is a *claimant* for at least one *available* category.
- The level at which a set of categories is *exclusively reserved* for a subset of *claimants* for at least one *available* category.

While this selection rule is intuitive, the difficulty is to analytically characterize the execution of its steps. First, when agents are allowed to receive a unit (probabilistically) at some step of the *PBSWI*, in general, they can receive it from multiple categories. Thus, the implication of increasing utilities on feasibility is not clear. Second, at any step, there are multiple constraints due to (1) *claimants* (a set of constraints on who can be assigned from which categories) and (2) the *reservation profile* of that step (a set of constraints in the form of guaranteed probabilities to agents). At some point, some constraints become binding, and the challenge is to track these instances. Thus, we need to analytically specify the *welfare improvement selection* rule described above to complete the definition of the *PBR*.

Step n.3 *(Welfare improvement selection rule of the PBR)*

Agent $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ is **prioritized** if, for each $j \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, $v_i^{n-1} \leq v_j^{n-1}$. Let $v^{n-1,1}$ be the reservation value of *prioritized* agents. If all agents in $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ are *prioritized*, then let $v^{n-1,2} = 1$, otherwise let $v^{n-1,2}$ be the lowest reservation value among *non-prioritized* agents in $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$. Let $\mathcal{B}_n$ be the set of all subsets of $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ with at least one *prioritized* agent. Let

$$\lambda_n^\star = \min_{I \in \mathcal{B}_n} \frac{\sum\limits_{c \in C(I, v^{n-1})} q_c - \sum\limits_{i \in I} v_i^{n-1}}{|\{i \in I : i \text{ is } prioritized\}|}.$$

[30] Each *acceptable* random allocation rule can be described via a *welfare improvement* selection rule in Step n.3 of the *PBSWI* algorithm (see also the discussion in Section 5).

For each $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, let

$$v_i^n = \begin{cases} \min\{v^{n-1,1} + \lambda_n^\star, v^{n-1,2}\} & \text{if } i \text{ is } \textit{prioritized} \\ v_i^{n-1} & \text{otherwise} \end{cases}$$

We are now ready to present our main theorem, which states that this specific improvement process characterizes *procedurally fair* rules.

Theorem 4. *A solution φ is procedurally fair if and only if φ is welfare-equivalent to the PBR.*

Proof. See Section D. □

The proof of this characterization result relies highly on exploiting parametric networks and an extension of the *Max-Flow Min-Cut Theorem* (Ford and Fulkerson, 1956) (see Appendix A).

We next present an example to demonstrate how the *PBR* solution works.

Example 8. *Let $\mathcal{I} = \{i, j, k, m\}$ and $\mathcal{C} = \{c_1, c_2, c_3\}$ such that one unit is reserved for each category. Consider the following category priorities:*

π_{c_1}	π_{c_2}	π_{c_3}
$\{i, j, k\}$	$\{m\}$	$\{m\}$
	$\{j, k\}$	$\{j, k\}$
	$\{i\}$	$\{i\}$

Step 1. *Agents i, j and k are claimants of c_1 and agent m is a claimant of c_2 and c_3. Since the reservation value is zero for each agent, (i) there are no exclusively reserved categories, and (ii) each agent is prioritized. The reservation value of each agent is increased up to $\lambda_1^\star = \frac{1}{3}$, the level at which c_1 is exclusively reserved for agents i, j and k. At this point, the reservation value is increased to $\frac{1}{3}$ for each agent.*

Step 2. *Category c_1 is not available, thus $A_2 = \{c_2, c_3\}$. Agent m is the only claimant of both c_2 and c_3, and also the only prioritized agent at this step (note that by definition, $v^{1,1} = v_m^1 = \frac{1}{3}$). This implies that all agents in $\bigcup_{c \in A_2} \Gamma_c(v^1)$ are prioritized. Thus, we let $v^{1,2} = 1$. Since $v^{1,1} = \frac{1}{3}$ and $q_{c_1} + q_{c_2} = 2$, and $\lambda_2^\star = \frac{5}{3}$, we have that $v_m^2 = \min\{v^{1,1} + \lambda_2^\star, 1\} = 1$. Since, in the previous step, c_1 is reserved for agents i, j, and k, these agents' reservation values do not change in Step 2.*

Step 3. *Since $v_m^2 = 1$, by definition of claimant, agents j and k become claimants of c_2 and c_3. Thus, i is a claimant of c_2, j and k are claimants of all categories and m is a claimant of both c_2 and c_3. Note that no feasibility constraint is binding at these reservation values of claimants and this implies that no category is exclusively reserved. Then, agents i, j, k are prioritized, and note that since their reservation values are not increased in Step 2, we have that $v_i^2 = v_j^2 = v_k^2 = \frac{1}{3}$. Then, it is easy to check that the set $\{i, j, k, m\}$*

minimizes the λ*-value of Step 3.3. Thus,* $\lambda_3^\star = \frac{q_{c_1}+q_{c_2}+q_{c_3}-1-(\frac{1}{3}+\frac{1}{3}+\frac{1}{3})}{3} = \frac{1}{3}$*. This means that the reservation value of agents* i, j *and* k *can be increased by* $\frac{1}{3}$ *such that the reservation values become* $v_i^3 = v_j^3 = v_k^3 = \frac{2}{3}$ *and* $v_m^3 = 1$*. Note that at this reservation profile, all categories are exclusively reserved for all agents. Thus, the solution of the PBR gives a random allocation with this utility profile. For example, the following random allocation* Z *gives this utility profile:* $z_{ic_1} = \frac{2}{3}$, $z_{jc_1} = z_{kc_1} = \frac{1}{6}$*, and* $z_{mc_2} = \frac{1}{2}$, $z_{jc_2} = \frac{1}{2}$*, and* $z_{mc_3} = \frac{1}{2}$, $z_{kc_3} = \frac{1}{2}$*. Note that* $u_Z(i) = u_Z(j) = u_Z(k) = \frac{2}{3}$ *and* $u_Z(m) = 1$*.*

Remark 2. *The example above illustrates an important component of our solution concept: at a given reservation profile, the reserve requirements might imply binding feasibility constraints (i.e. exclusively reserved categories), but this does not imply that the reservation values of the agents, for whom a set of categories is exclusively reserved, are final. As shown in the example, at the end of the first step,* c_1 *is exclusively reserved for agents* $i, j,$ *and* k*. But, later in the algorithm, at Step 3, when agents* j *and* k *become claimants of* c_2 *and* c_3*, the feasibility constraints become non-binding, therefore, there are no exclusively reserved categories anymore. Then, the reservation value of not only agents* j *and* k*, but also of agent* i *can be increased in Step 3. This is an important feature of the PBR capturing the sense of equity in procedural fairness.*

7 Conclusion

The current work is an analysis of the problem of rationing identical units. Our model builds on reserve systems: there are reserve categories with (weak) priority orders over their beneficiaries and certain amount of units is reserved for each of them. This design is for creating (better) access for disadvantaged communities or essential/health workers (e.g. allocating medical units) or better distribution of units among different socioeconomics groups (e.g. affirmative action in school choice). But, existing real-life mechanisms favor some categories over others. We propose a methodology of processing reserve categories simultaneously, which facilitates fair treatment of categories. This methodology relies on a simple idea of improving agents' welfare by assigning them probabilities of receiving units in such a way that priorities of categories are respected. We analyze the challenges about this idea, and propose and characterize a class of allocation rules. We characterize (within this class) an allocation rule by *procedural fairness*, which basically requires that the solution treats categories in an egalitarian way throughout the steps of the welfare improvement process. This property about the procedure has strong normative justifications: First, it implies fair treatment of categories (unless there is a justification for unfairness due to priorities). Second, it satisfies equal treatment of equals. We argue, given that, in real-life applications of reserve systems, categories have weak priority orderings, these fairness considerations (and thus our methodology) are relevant and have a scope.

8 Applications

A weighted lottery policy. We consider the lottery system implemented by the Department of Health, Pennsylvania (Section 2). Their goal of ensuring access to all patients by randomization is consistent with the motivation behind the *PBR*. By designing these categories (as specified in Table 1) and the **weak** priority orders appropriately, we can apply the *PBR* rule (1) to obtain a *procedurally fair* allocation (Theorem 4), and (2) to remove analytical inconsistencies explained in Section 2 such that **each** patient is assigned a unit with a positive probability (as stated in the Pandemic Guideline in Section 2).

Alternatively, a different rule in the *PBSWI* class can be specified for this setting to achieve the central authority's *targeted ratios between the weights* in Table 1 in an analytically consistent way. To achieve this, first, reserve categories are modeled with dichotomous indifference classes: for each category, the first indifference class is the set of all patients belonging to that category and the second one is the rest of the patients. Since our model allows for weak priority orders, this construction is clearly within our framework.

Second, *targeted ratios between the weights* are specified: The weights defined in Table 1 in Section 2 suggest that (1) each disadvantaged community member who is an essential worker should have a higher utility than each utility value obtained by the *priority rule* applied to these single-category problems, (2) each disadvantaged community member or essential worker with death likely within one year should have a lower (higher) utility than the utility of a disadvantaged community member or essential worker alone. A (weighted) average of the utilities applies to patients belonging to these multiple categories. Thus, there is a target for relative utilities of patients belonging to two groups.[31] Let u^k and $u^{k,l}$ represent the utility of a patient belonging to group c_k only, and to groups c_k and c_l, respectively. Given that $u^1 = u^2 > u^3$, the target utility ratios are defined such that $u^{1,2} = \alpha u^1$ and $u^{1,3} = u^{2,3} = w(u^1, u^3)$, where $\alpha > 1$ and $w(u^1, u^3)$ is a convex combination of u^1 and u^3.

Step 0. Let $C = \{c_1, c_2, c_3\}$ be the patient groups (i.e. categories) in Table 1. For each $c \in C$, and for each patient i belonging to group c, the weak order π_c is constructed such that $i \in \mathcal{I}_{\pi_c}(1)$. For each patient group, a certain number of units is reserved such that $\frac{q_{c_1}}{|\mathcal{I}_{\pi_{c_1}}(1)|} = \frac{q_{c_2}}{|\mathcal{I}_{\pi_{c_2}}(1)|} > \frac{q_{c_3}}{|\mathcal{I}_{\pi_{c_3}}(1)|}$.[32]

Step 1 For each patient i, let the initial reservation profile $v_i^0 = \min_{c \in \{c' : i \in \mathcal{I}_{\pi_{c'}}(1)\}} \frac{q_c}{|\mathcal{I}_{\pi_c}(1)|}$.

Step 2 The units are allocated by the *PBSWI* algorithm with the following welfare improvement selection rule: If there are *claimants* for at least one *available* category, who belong to two groups and have a reservation value lower than the targeted ratio, then these patients are *selected*; otherwise,

[31] We assume that there is no patient belonging to all three groups (see Table 1).

[32] Since the units and the number of patients are integers, we can only impose $\frac{q_{c_1}}{|\mathcal{I}_{\pi_{c_1}}(1)|} \approx \frac{q_{c_2}}{|\mathcal{I}_{\pi_{c_2}}(1)|}$. But, for the ease of notation, we assume that it is possible to reserve units such that this approximation holds with equality. Also, these numbers of units reserved for each group can be determined with respect to some target ratio between u^1 and u^3.

all patients who are *claimants* for at least one *available* category are *selected*. The reservation value of *selected* patients are increased equally up to the minimum of the following two:

- The level at which a subset of *claimants* for at least one *available* reserve category has *exclusive rights* over the categories for which they are *claimant*.
- The level at which the targeted ratio is achieved for a patient who had a reservation value lower than the targeted ratio.

The above rule selects a random allocation with target utility ratios within the set of *acceptable* random allocations, whenever it is feasible. We do not claim that our rule is the only one: there are other ways to achieve target utility ratios for this very special case. Our point here is that, by Theorem 2, our approach is robust in delivering the desired properties for different settings.

Soft reserves. Reserve systems[33] are generally such that for each *preferential treatment* category $c \in C$, a *beneficiary group* is designated. A particular approach in this setting is *hard reserves*: A patient is qualified to receive a medical unit from a category if and only if they are in the *beneficiary group* of that category. *Hard reserves* are in general incompatible with efficiency (see Example 2 in Pathak, Sönmez, Ünver, and Yenmez (2021)). A more flexible interpretation is a *soft reserve system*, where all agents are qualified for all categories. In particular, a *soft reserve* system is obtained by applying the following to each *preferential treatment* category c: (1) If there is an *unreserved* category as well, π_c is obtained by ranking each non-beneficiary patient strictly below the *beneficiary group* and by preserving the ranking of the *unreserved* category. (2) Otherwise, all the non-beneficiary patients are ranked as an indifference class just below the last beneficiary patient. While our model applies to both cases, the second case necessarily implies weak priority orders, the generality of which is provided by our work.

Affirmative action in school choice. Affirmative action schemes are widespread in school admissions around the world. Typically, a fraction of slots is reserved for disadvantaged students and the rest is assigned based on merit. A compelling example is Chicago's place-based affirmative action at the K-12 level: Schools fill 40% of their slots with the applicants having the highest composite scores[34] and the remaining 60% by dividing the slots equally across four tiers based on the socioeconomic characteristics. For each socioeconomic tier, beneficiary students are prioritized over others such that students both inside and outside the group are ordered by composite score. For the merit tier, all students are ordered by composite score. Our model fits this setting, and our results apply directly. One of the themes in this affirmative action scheme is to eliminate explicit targeting of applicants by differentiating across tiers, that is *tier-blindness* (Dur, Pathak, and Sönmez, 2020). Let C be the socioeconomic tiers and c_M the merit tier. Also, any two socioeconomic tiers $c, c' \in C$, $q_c = q_{c'}$. Also, for each $c \in C$, we fix π_c, and

[33] After the circulation of Pathak, Sönmez, Ünver, and Yenmez (2021) and the authors' interaction with public health officials, the National Academies of Sciences, Engineering, and Medicine (NASEM) started to formulate recommendations on the fair allocation of COVID-19 vaccines. Later, Tennessee, Massachusetts and New Hampshire announced their plans to adopt a reserve system (Tennessee DH, 2020; Massachusetts DPH, 2020; New Hampshire DHHS, 2021).

[34] The composite score is the equally-weighted combination of the admission test score, the applicant's 7th grade GPA, and the standardized test score.

use $\succcurlyeq$ to represent the ordering of c_M. A **merit-preserving bijection** $\theta : C \cup \{c_M\} \to C \cup \{c_M\}$ is a one-to-one and onto function where $\theta(c_M) = c_M$. A random allocation rule φ is **tier-blind** if for each set of students $\mathcal{I}$, for each set $C \cup \{c_M\}$ and for each *merit-preserving bijection* θ, the random allocations $Z = \varphi(\mathcal{I}, C \cup \{c_M\}, (\pi_c)_{c \in C}, \succcurlyeq, (q_c)_{c \in C \cup \{c_M\}})$ and $Z' = \varphi(\mathcal{I}, C \cup \{c_M\}, (\pi_{\theta(c)})_{c \in C}, \succcurlyeq, (q_{\theta(c)})_{c \in C \cup \{c_M\}})$ are such that $u_Z = u_{Z'}$. *Tier-blindness* implies that relabeling tiers does not change the probability with which a student is assigned a seat. Since the *PBR* rule is based on the set of *claimants* at each step, and that structure is independent from the tiers' labels, the next observation follows immediately.

Observation 1. *The PBR rule is tier-blind.*

Appendix A Maximum flow problem: Preliminaries

A **directed graph,** or **digraph** is a pair $G = (V, A)$, consisting of a set of **vertices** V and a set of *ordered* pairs of vertices, A, called **arcs.** For a set of vertices $V' \subseteq V$, the set $\delta^{\text{out}}(V')$ is the set of all outgoing arcs; that is, the arcs (x, y) such that $x \in V'$ and $y \notin V'$. Similarly, the set $\delta^{\text{in}}(V')$ is the set of all incoming arcs; that is, the arcs (x, y) such that $x \notin V'$ and $y \in V'$. Let $l, k : A \to \Re_+$ be two functions, which associate each arc $a = (x, y)$ of G with non-negative real numbers $l(x, y)$ and $k(x, y)$ called the **lower-bound** and **capacity** of the arc (x, y), respectively, such that for each arc (x, y), $l(x, y) \leq k(x, y)$. For a set of arcs $A' \subseteq A$, $l(A') = \sum_{a \in A'} l(a)$ and $k(A') = \sum_{a \in A'} k(a)$.

A **network** (V, A, l, k) is a digraph with lower-bound and capacity functions. A **supply-demand network** is a *network* (V, A, l, k) with $V = V_1 \cup V_2 \cup \{s, t\}$, where V_1 and V_2 are the set of demand and supply vertices, respectively, s the **source vertex**, and t the **sink vertex** such that there is an arc from the source vertex into each demand vertex, an arc from each supply vertex into the sink vertex, and all the other arcs are from demand vertices into supply vertices. (An arc from a demand vertex $x \in V_1$ into a supply vertex $y \in V_2$ is interpreted that x demands units from y.)

A **flow** in a supply-demand network (V, A, l, k) is a function $f : A \to \Re_+$, satisfying the following properties:

(i) $\sum_x f(x, y) = \sum_z f(y, z)$ for each y in $V_1 \cup V_2$ and,

(ii) $l(x, y) \leq f(x, y) \leq k(x, y)$ for each (x, y) in A.

The **value of** f, denoted by $v(f)$ is defined as $\sum_x f(s, x)$. Given a supply-demand network (V, A, l, k), the **maximum flow problem** is to find the maximum value of flow. The solution for this problem is characterized by the following theorem (Schrijver, 2003):

Theorem 5. *Let (V, A, l, k) be a supply-demand network such that there exists a flow f. Then, the maximum value of a flow is equal to the minimum value of*

$$k(\delta^{out}(V')) - l(\delta^{in}(V'))$$

taken over $V' \subseteq V$ with $s \in V'$ and $t \notin V'$.[35]

[35]This theorem is an extension of the well-known Max-flow Min-cut Theorem (Ford and Fulkerson, 1956).

Appendix B Proof of Theorem 2

We prove that (1) each random allocation given by the *PBSWI* class is *acceptable* (Lemma 1) and (2) each *acceptable* random allocation Z can be obtained by a sequence of selections of *reservation profiles* in the *PBSWI* (Lemma 3).

Lemma 1. *Let $R = (\mathcal{I}, C, (\pi_c)_{c\in C}, (q_c)_{c\in C})$ be a problem. If a random allocation Z is an outcome of the PBSWI(R), then it is acceptable.*

Proof. Let N be the last step of the *PBSWI*. By definition of the *PBSWI*, the algorithm ends at the end of Step N.2, and the reservation values are not updated at Step N. Thus, the outcome of the algorithm is v^{N-1}. Let $Z^\star$ be a random allocation such that $u_{Z^\star} = v^{N-1}$.

$Z^\star$ is non-wasteful. Suppose $Z^\star$ is not *non-wasteful*. Then, by Definition 2, there exists a category c and an agent i, such that

$$\sum_{j\in\mathcal{I}} z^\star_{jc} < q_c \text{ and } v_i^{N-1} = u_{Z^\star}(i) = \sum_{c'\in C} z^\star_{ic'} < 1. \tag{2}$$

Since $u_{Z^\star}(i) < 1$, by definition of Step N.2, Category c is *unavailable* at this step. Thus, there exists a set of agents I such that categories in $C(I, v^{N-1})$ with $c \in C(I, v^{N-1})$ are *exclusively reserved* for I (note that agent i is not necessarily in the set $\Gamma_c(v^{N-1})$) and

$$\sum_{j\in I} v_j^{N-1} = \sum_{c'\in C(I,v^{N-1})} q'_c \tag{3}$$

By definition of the set $C(I, v^{N-1})$, each $j \in I$ with $v_j^{N-1} > 0$ is not a *claimant* for categories out of $C(I, v^{N-1})$. Thus, for each $j \in I$, $v_j^{N-1} = \sum_{c'\in C(I,v^{N-1})} z^\star_{jc'}$. By rewriting Condition (3), we obtain

$$\sum_{c'\in C(I,v^{N-1})} q_{c'} = \sum_{j\in I} \sum_{c'\in C(I,v^{N-1})} z^\star_{jc'} = \sum_{c'\in C(I,v^{N-1})} \sum_{j\in I} z^\star_{jc'} \tag{4}$$

Since, by definition of the *PBSWI*, $Z^\star$ is a random allocation, by Property *(ii)* of a random allocation (Definition 1), for each $c' \in C(I, v^{N-1})$,

$$\sum_{j\in I} z^\star_{jc'} \leq q_{c'}. \tag{5}$$

Thus, Conditions (4) and (5) together imply that the weak inequality in Condition (5) holds with equality. By definition of a random allocation, this also implies, for each $c' \in C(I, v^{N-1})$, $\sum_{j\in\mathcal{I}} z^\star_{jc'} = q_{c'}$. Since $c \in C(I, v^{N-1})$, this contradicts with (2).

$Z^\star$ respects priorities. Let $i \in \mathcal{I}$, and $c \in C$ such that $i \ \pi_c^P \ j$ and $u_{Z^\star}(i) < 1$. At Step N, since there exists at least one agent with a utility less than one, by definition of Step N.2, Category c must be *unavailable* at the end of the algorithm. Since $u_{Z^\star}(i) < 1$, and $i \ \pi_c^P \ j$, by definition of a *claimant*, agent j is not a *claimant* for c. Moreover, by definition of a *claimant*, for each c and each $n \geq 1$, $\Gamma_c(v^n) \supseteq \Gamma_c(v^{n-1})$. This implies that j has not been a *claimant* at any step before N. Thus, as $Z^\star$ underlies v^{N-1} and v_j^{N-1} is the sum of agent j's shares at categories for which she is a *claimant*, $z_{jc}^\star = 0$. □

Lemma 2. *Let v^{n-1} be a reservation profile with an underlying random allocation Z^{n-1}. If categories in $C(I_1, v^{n-1})$ and $C(I_2, v^{n-1})$ are exclusively reserved for I_1 and I_2, respectively, then categories in $C(I_1, v^{n-1}) \cup C(I_2, v^{n-1})$ are exclusively reserved for $I_1 \cup I_2$.*

Proof. Let categories in $C(I_1, v^{n-1})$ and $C(I_2, v^{n-1})$ be *exclusively reserved* for I_1 and I_2, respectively. There are two cases.

Case 1: $C(I_1, v^{n-1}) \cap C(I_2, v^{n-1}) = \emptyset$.
By definition of a *claimant*, we have $I_1 \cap I_2 = \emptyset$. By definition of *exclusively reserved* categories,

$$\sum_{i \in I_1} v_i = \sum_{c \in C(I_1, v^{n-1})} q_c \text{ and } \sum_{i \in I_2} v_i = \sum_{c \in C(I_2, v^{n-1})} q_c. \tag{6}$$

Since $I_1 \cap I_2 = \emptyset$, these two equalities together imply, $\sum_{i \in I_1 \cup I_2} v_i = \sum_{c \in C(I_1 \cup I_2, v^{n-1})} q_c$. Thus, categories in $C(I_1 \cup I_2, v^{n-1})$ are *exclusively reserved* for $I_1 \cup I_2$.

Case 2: $C(I_1, v^{n-1}) \cap C(I_2, v^{n-1}) \neq \emptyset$.
Suppose $I_1 \cap I_2 = \emptyset$. (Note that equalities in (6) hold in this case as well.) Clearly, $\sum_{i \in I_1 \cup I_2} v_i = \sum_{i \in I_1} v_i + \sum_{i \in I_2} v_i$. Moreover, by definition of a *claimant*, $C(I_1 \cup I_2, v^{n-1}) = C(I_1, v^{n-1}) \cup C(I_2, v^{n-1})$. Since $C(I_1, v^{n-1}) \cap C(I_2, v^{n-1}) \neq \emptyset$, this implies that

$$\sum_{c \in C(I_1 \cup I_2, v^{n-1})} q_c < \sum_{c \in C(I_1, v^{n-1})} q_c + \sum_{c \in C(I_2, v^{n-1})} q_c.$$

This, together with equalities in (6), imply

$$\sum_{c \in C(I_1 \cup I_2, v^{n-1})} q_c < \sum_{i \in I_1 \cup I_2} v_i.$$

Then, Condition 1 in Theorem 1 does not hold for $I_1 \cup I_2$. Thus, by Theorem 1, there does not exist a random allocation underlying v^{n-1}, a contradiction. Thus, $I_1 \cap I_2 \neq \emptyset$. Now consider the sets $I_1 \cup I_2$

and $C(I_1 \cup I_2, v^{n-1})$. Since there exists a random allocation Z^{n-1} underlying v^{n-1}, by Theorem 1,

$$\sum_{c \in C(I_1 \cup I_2, v^{n-1})} q_c \geq \sum_{i \in I_1 \cup I_2} v_i. \tag{7}$$

Suppose Inequality (7) is strict. First, note that

$$C(I_1 \cap I_2, v^{n-1}) \subseteq C(I_1, v^{n-1}) \cap C(I_2, v^{n-1}). \tag{8}$$

The inclusion is by definition of a *claimant*, and since different agents can be *claimants* for the same category, these two sets do not necessarily coincide. We rewrite Inequality (7) as follows:

$$\sum_{c \in C(I_1, v^{n-1}) \setminus C(I_2, v^{n-1})} q_c + \sum_{c \in C(I_2, v^{n-1}) \setminus C(I_1, v^{n-1})} q_c + \sum_{c \in C(I_1, v^{n-1}) \cap C(I_2, v^{n-1})} q_c > \sum_{i \in I_1 \setminus I_2} v_i + \sum_{i \in I_2 \setminus I_1} v_i + \sum_{i \in I_1 \cap I_2} v_i.$$

Together with equalities in (6), this implies that

$$\sum_{c \in C(I_1, v^{n-1}) \cap C(I_2, v^{n-1})} q_c < \sum_{i \in I_1 \cap I_2} v_i. \tag{9}$$

Inequality (9), together with Inclusion (8), implies

$$\sum_{c \in C(I_1 \cap I_2, v^{n-1})} q_c < \sum_{i \in I_1 \cap I_2} v_i. \tag{10}$$

This violates Condition 1 in Theorem 1. Thus, by Theorem 1, there does not exist a random allocation underlying v^{n-1}, a contradiction. Thus, Inequality (7) cannot be strict. Thus, by definition of *exclusively reserved* categories, categories in $C(I_1 \cup I_2, v^{n-1})$ are *exclusively reserved* for $I_1 \cup I_2$. □

Lemma 3. *Each acceptable random allocation is an outcome of a member of the PBSWI class.*

Proof. Let Z be an *acceptable* random allocation such that $v = u_Z$. We prove by induction that there is a sequence of reservation profiles v^n for $n = 1$ to N, each obtained by a welfare improvement from v^{n-1} as defined by Step n.3 of the *PBSWI*, and $v^{N-1} = v$. Let $n \geq 1$. Our inductive hypothesis is that there exists a random allocation Z^{n-1} underlying v^{n-1} where v^{n-1} is obtained through a sequence of welfare improvements and $v \geq v^{n-1}$. Since for each i, $v_i^0 = 0$, initial step holds trivially. Suppose that $v \neq v^{n-1}$. We prove that there exists a welfare improvement for some *claimants* to obtain a reservation profle v^n from v^{n-1} such that $v \geq v^n$. This completes the proof.

Suppose $A_n = \emptyset$. Since each category c is not *available*, there is a set of categories including c, which are *exclusively reserved* for a set of agents. Thus, there is a collection of categories $C(I_1, v^{n-1}), \ldots, C(I_m, v^{n-1})$ being *exclusively reserved* for sets of agents, $I_1, \ldots, I_m$, respectively, such that C is the union of the sets $C(I_1, v^{n-1}), \ldots, C(I_m, v^{n-1})$. By Lemma 2, categories in C are *exclusively reserved* for $\bigcup_{k=1}^{m} I_k$. But then, all units are assigned to agents for whom categories are *exclusively reserved* under v^{n-1}. Thus, $\sum_{i\in\mathcal{I}} v_i^{n-1} = \sum_{c\in C} q_c$. Since $v \geq v^{n-1}$ and $v_i > v_i^{n-1}$ for some agent i, this implies that there exists a category c with $\sum_{i\in\mathcal{I}} z_{ic} > q_c$. But, this violates Property *(ii)* of Definition 1 and contradicts Z being a random allocation.

Suppose $A_n \neq \emptyset$. Let I be the subset of agents such that $i \in I$ if and only if $v_i = v_i^{n-1}$. We claim that there exists a category $c \in A_n$ such that $\Gamma_c(v^{n-1}) \setminus I \neq \emptyset$. Suppose, on the contrary, that for each $c' \in A_n$, $\Gamma_{c'}(v^{n-1}) \subseteq I$. Let c be such a category. By *non-triviality* assumption (see Section 3), it is not possible that for each $j \in \mathcal{I}$, $v_j^{n-1} = 1$. By definition of a *claimant*, this implies that there exists an agent $i \in \Gamma_c(v^{n-1})$ with $v_i^{n-1} = v_i < 1$. Also, each agent in the indifference class including agent i is in the set $\Gamma_c(v^{n-1})$. Since $\Gamma_c(v^{n-1}) \subseteq I$, that is, for each $j \in \Gamma_c(v^{n-1})$, $v_j^{n-1} = v_j$, this implies that there exists an indifference class such that both v^{n-1} and v coincide for the agents in this and higher indifference classes. Then, since Z is *acceptable*, any agent with a positive utility in a lower indifference class cannot be assigned a unit from c with a positive probability, which would violate priorities. Moreover, v^{n-1} is obtained through a sequence of steps of the *PBSWI* algorithm. Thus, for each category c, there is an integer $k(c)$ such that each agent in the first $k(c)$ indifference classes has a reservation value one and there exists an agent in the next indifference class with a reservation value less than one under v^{n-1}. Also, $v \neq v^{n-1}$ and $v \geq v^{n-1}$. Thus, since Z is *acceptable*, for some $c \notin A_n$, it is possible to increase the utility of an agent in $\Gamma_c(v^{n-1})$. But, by definition of *exclusively reserved* categories, for each $c \notin A_n$, and $i \in \Gamma_c(v^{n-1})$, and for each $\lambda > 0$, there does not exist a random allocation generating the utility profile $(v_{-i}^{n-1}, v_i^{n-1} + \lambda)$, which is a contradiction.

Thus, there exists a category $c \in A_n$ such that $\Gamma_c(v^{n-1}) \setminus I \neq \emptyset$. By definition of a *claimant*, there exists an agent $i \in \Gamma_c(v^{n-1}) \setminus I$ such that $v_i^{n-1} < 1$ and $\lambda > 0$ with $v_i^n = v_i^{n-1} + \lambda$ underlying a random allocation. Thus, there is a welfare improvement to obtain v^n from v^{n-1} such that $v \geq v^n \geq v^{n-1}$. This completes the inductive step. □

Appendix C Proof of Theorem 3

Let $C = \{c_1, c_2, c_3, c_4, c_5\}$ each category with capacity one and $\mathcal{I} = \{i, j, k, i_1, i_2, j_1, j_2, j_3\}$. *Initial reservation value* for each agent is zero. The (strict) priority orders for categories are given below:

π_{c_1}	π_{c_2}	π_{c_3}	π_{c_4}	π_{c_5}
i	i	i	i	i
j	j	j	j	j
k	k	i_1	i_1	j_1
j_1	j_2	i_2	i_2	i_2
i_1	j_3	j_1	j_2	j_3
j_2	j_1	k	k	k
j_3	i_1	j_3	j_1	j_2
i_2	i_2	j_2	j_3	i_1

Let R be the problem above and $Z \in \mathcal{Z}^a(R)$. Also, let

$$u_Z = (u_Z(i), u_Z(j), u_Z(k), u_Z(i_1), u_Z(i_2), u_Z(j_1), u_Z(j_2), u_Z(j_3))$$

Respecting priorities implies, $u_Z(i) = 1$, and by *non-wastefulness* and *respecting priorities* together, $u_Z(j) = 1$. Thus, there are three units remaining for agents k, i_1, i_2, j_1, j_2 and j_3. Suppose

$$u_Z(k), u_Z(i_1), u_Z(j_1) < 1. \tag{11}$$

Then, by *respecting priorities*, only agents i, j, k are assigned positive probabilities for the units under c_1, c_2, only agents i, j, i_1 are assigned positive probabilities for the units under c_3, c_4, and only agents i, j, j_1 are assigned positive probabilities for the unit under c_5. But, then agents receive in total less than five units and this contradicts with *non-wastefulness*. Thus, at least one of the agents in $\{k, i_1, j_1\}$ receives one unit under Z. By considering all possible cases, we obtain the set $\mathcal{Z}^a(R)$.

Case 1: $u_Z(k) = u_Z(i_1) = u_Z(j_1) = 1$
There is only one utility profile satisfying this condition: agents i, j, k, i_1 and j_1 receive one unit and the other agents are not assigned a unit with positive probability. It is straightforward to check that there exists a random allocation, say Z^1, generating this utility profile. Thus, $u_{Z^1} = (1, 1, 1, 1, 0, 1, 0, 0)$.

Case 2: $u_Z(k) = 1; u_Z(i_1), u_Z(j_1) < 1$
Since agents i, j and k receive one unit each, there are two units to be assigned to the rest of the agents. Since $u_Z(i_1), u_Z(j_1) < 1$, and Z *respects priorities*, either (1) these two units are to be assigned to agents i_1, j_1 and j_2, or (2) j_2 is assigned one unit and the remaining one unit is assigned to agents i_1 and j_1, or (3) j_2 and j_3 are assigned one unit each. While there is no *acceptable* random allocation generat-

ing the utility profile in (3), there are random allocations generating the utility profiles in (1) and (2). Among all possible random allocations generating the utility profiles in (1), random allocation, say Z^2, such that $u_{Z^2} = (1,1,1,\frac{2}{3},0,\frac{2}{3},\frac{2}{3},0)$ is *Lorenz dominant*. Among all possible random allocations generating the utility profiles in (2), random allocation, say Z^3, such that $u_{Z^3} = (1,1,1,\frac{1}{2},0,\frac{1}{2},1,0)$ is *Lorenz dominant*.

Case 3: $u_Z(k) = u_Z(i_1) = 1$; $u_Z(j_1) < 1$

There is one unit remaining for agents j_1, j_2 and i_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^4, such that $u_{Z^4} = (1,1,1,1,\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$ is *Lorenz dominant*.

Case 4: $u_Z(k) = u_Z(j_1) = 1$; $u_Z(i_1) < 1$

There is one unit remaining for agents i_1, i_2 and j_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^5, such that $u_{Z^5} = (1,1,1,\frac{1}{3},\frac{1}{3},1,\frac{1}{3},0)$ is *Lorenz dominant*.

Case 5: $u_Z(k) < 1$; $u_Z(i_1) = u_Z(j_1) = 1$

There is one unit remaining for agents k and i_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^6, such that $u_{Z^6} = (1,1,\frac{1}{2},1,\frac{1}{2},1,0,0)$ is *Lorenz dominant*.

Case 6: $u_Z(k), u_Z(i_1) < 1$; $u_Z(j_1) = 1$

There are two units remaining for agents k, i_1 and i_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^7, such that $u_{Z^7} = (1,1,\frac{2}{3},\frac{2}{3},\frac{2}{3},1,0,0)$ is *Lorenz dominant*.

Case 7: $u_Z(k), u_Z(j_1) < 1$; $u_Z(i_1) = 1$

There are two units remaining for agents k, j_1 and i_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^8, such that $u_{Z^8} = (1,1,\frac{2}{3},1,\frac{2}{3},\frac{2}{3},0,0)$ is *Lorenz dominant*.

Since *Lorenz domination* is a transitive binary relation, it is enough to consider the random allocations Z^1 to Z^8 and find the random allocation *Lorenz dominating* others. Note that (i) Z^2, Z^7 and Z^8 are *Lorenz indifferent*, (ii) Z^3 and Z^6 are *Lorenz indifferent*, and (iii) Z^4 and Z^5 are *Lorenz indifferent*. Thus, it is enough to compare Z^1, Z^2, Z^3 and Z^4. But, while Z^2 *Lorenz dominates* Z^1 and Z^3, it does not *Lorenz dominate* Z^4. Also, Z^4 does not *Lorenz dominate* Z^2. Thus, there does not exist a *Lorenz dominant* random allocation in the set $\mathcal{Z}^a(R)$. Thus, there does not exist an *egalitarian* random allocation for this problem, and no rule is *egalitarian*.

Appendix D Proof of Theorem 4

Let N be the last step of the *PBR* algorithm and $Z^{\star}$ be one of its outcomes. Thus, $u_{Z^{\star}} = v^{N-1}$. We first show that the *PBR* is a rule in the *PBSWI* class (Lemma 4). This implies that $Z^{\star}$ is *acceptable*. Then, to complete the proof, we show that any *procedurally fair* random allocation generates $u_{Z^{\star}}$ (Lemma 5).

First, we show that for each *reservation profile* v^n obtained at the end of Step n.3, there exists a random allocation Z^n such that $v^n = u_{Z^n}$ (Lemma 4). Thus, the selection of the *reservation values* at Step n.3 of the *PBR* complies with Step n.3 of the *PBSWI*.

Lemma 4. *For each reservation profile v^n obtained at the end of Step n.3, there exists a random allocation Z^n such that $v^n = u_{Z^n}$.*

Proof. The algorithm starts with the initial reservation value zero for each agent. By setting each probability as zero under Z^0, this step is trivial.

By induction, we show that given an underlying Z^{n-1} for v^{n-1}, there exists a random allocation Z^n for the utility profile v^n obtained at the end of Step n. For each set of agents I, for which there are *exclusively reserved* categories, since the reservation value of each such agent is the same as in the previous step, by inductive hypothesis, there exists an assignment of probabilities of units under categories in the set $C(I, v^{n-1})$. Note that, by Definition 5, all units under these categories are assigned to these agents. Also, at Step n, these agents are not *claimants* for any category out of the set $C(I, v^{n-1})$. Thus, we can consider the set of *available* categories separately from the set of *unavailable* categories.

Let us consider the set A_n and $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$. By inductive hypothesis, there exists a random allocation Z^{n-1} inducing v_i^{n-1} for each $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$.[36] Thus, by Theorem 1, for each set of agents, the condition in Theorem 1 is satisfied at the end of Step $n-1$. Since for each agent i, for whom there are *exclusively reserved* categories, $v_i^n = v_i^{n-1}$, for any subset of such agents, the condition in Theorem 1 is also satisfied at the end of Step n. Thus, we need to check this condition only for agents for whom there are no *exclusively reserved* categories. By Step n.3, only *prioritized* agents' reservation values are updated. Thus, to complete the proof, it is enough to check the condition only for the subsets including *prioritized* agents. Suppose there exists such a set of agents I violating the condition in Theorem 1 at the end of Step n. Thus,

$$\sum_{i \in I} v_i^n > \sum_{c \in C(I, v^{n-1})} q_c. \tag{12}$$

[36] Note that there could be an agent $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, who is a *claimant* also for an *unavailable* category. By Step n.1, no category is *exclusively reserved* for her and, by the inductive hypothesis that there is a random allocation Z^{n-1} for the reservation profile v^{n-1}, she is assigned probabilities equivalent to v_i^{n-1} from categories in $C(i, v^{n-1}) \cap A_n$.

By Step n.3, for a *non-prioritized* agent i, $v_i^n = v_i^{n-1}$ and for a *prioritized* agent j,

$$v_j^n \leq v^{n-1,1} + \lambda^\star = v_j^{n-1} + \lambda^\star.$$

Let p be the number of *prioritized* agents in the set I. Thus, Inequality (12) can be rewritten as

$$\sum_{i \in I} v_i^{n-1} + p\lambda^\star \geq \sum_{i \in I} v_i^n > \sum_{c \in C(I, v^{n-1})} q_c.$$

Thus,

$$\lambda^\star > \frac{\sum_{c \in C(I,v^{n-1})} q_c - \sum_{i \in I} v_i^{n-1}}{p}.$$

Since $p = |\{i \in I : i \text{ is } \textit{prioritized}\}|$ and $I \in \mathcal{B}_k$, this inequality contradicts with the definition of $\lambda^\star$. Thus, no subset of $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ with at least one *prioritized* agent violates the condition in Theorem 1. Thus, by Theorem 1, there is a random allocation Z^n for the reservation profile v^n obtained at Step n.3.

□

While the *PBR* increases only the welfare of *prioritized* patients, the next observation demonstrates that eventually, the total reservation value of all *claimants* is maximized.

Remark 3. *The PBR rule maximizes the total reservation value of claimants at each step.*

The *PBR* increases the reservation value of only the *prioritized* agents up to a level such that either (i) their reservation value reaches to the level of the lowest reservation value of *non-prioritized* agents, or (ii) there are *exclusively reserved* categories for a set of agents, or (iii) the reservation value of each *claimant* is equal to one. The last one is possible only if all *claimants* are *prioritized* and each such agent's reservation value can be increased to one. Note that in this case, the total reservation value of *claimants* is maximized. Suppose (i) holds. At the updated reservation profile, the set of *claimant* is the same as the beginning of the step. Thus, at the next step, the reservation values of the *prioritized* agents of the current step, and also of the *claimants* with the second-lowest reservation value at the end of the current step are increased. Suppose (ii) holds. Then, all the units under the *exclusively reserved* categories are assigned to agents for whom they are *exclusively reserved*. Since only the reservation values of the *claimant* are increased, all the remaining units under these categories at the beginning of the current step are assigned to *claimants*. Moreover, since no other *claimant*'s reservation value is updated to one, by definition of a *claimant*, there are no new *claimants* at the beginning of the next step. Thus, under both (i) and (ii), the next step is such that only a subset (if not all) of the current *claimants*' reservation values are increased. By an inductive argument, this ends at a step where for all of these *claimants*, there are *exclusively reserved* categories, or their reservation value becomes one. In the former case, all the units available for the current *claimants* are assigned to these agents. In the latter case, each *claimant* has a reservation value one. Thus, in both cases the total reservation value of *claimants* is maximized. Note that this maximization holds in general in multiple steps. But, since there are no new *claimants* during these steps, the welfare improvements for these *claimants* can be also defined as being realized in only one step.

Lemma 5. *A random allocation Z is procedurally fair if and only if it is welfare equivalent to $Z^\star$.*

Proof. Let Z be a *procedurally fair* random allocation. We prove by induction that for each $n \geq 0$, and $i \in \mathcal{I}$, $u_Z(i) \geq v_i^n$. Since, by definition of the *PBR* algorithm, no agent's reservation value can be improved at the last step of the algorithm, this completes the proof.

Initial step: Since for each agent $i \in \mathcal{I}$, $v_i^0 = 0$, and $u_Z(i) \geq v_i^0$, this step is trivial.

Inductive step: By inductive hypothesis, for each $i \in \mathcal{I}$, $u_Z(i) \geq v_i^{n-1}$. We show that for each $i \in \mathcal{I}$, $u_Z(i) \geq v_i^n$. At the beginning of Step n, if there is a set of agents I, for whom there are *exclusively reserved* categories, then they are assigned the units under the categories $C(I, v^{n-1})$, and by definition of *exclusively reserved* categories, there are no units left under these categories, and these categories are not *available* for other agents.

Let us now consider agents for whom there are no *exclusively reserved* categories. For any subset of this group of agents, Condition 1 in Theorem 1 is not binding. Thus, their welfare can be improved. To make the reservation values as equal as possible among the set of *claimants*, we construct a supply-demand network (see Figure 1) by setting $V_1^n = \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ as the demand vertices and $V_2^n = A_n$ as the supply vertices.[37] Agent $i \in V_1^n$ points to $c \in V_2^n$ if and only if $i \in \Gamma_c(v^{n-1})$. For each of these arcs (i, c), $l(i, c) = 0$ and $k(i, c) = \infty$. For each *prioritized* agent $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, arc (s, i) has lower bound $l(s, i) = v_i^{n-1} + \lambda$ and capacity, $k(s, i) = v_i^{n-1} + \lambda$. For each *non-prioritized* agent $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, arc (s, i) has lower bound $l(s, i) = v_i^{n-1}$ and capacity, $k(s, i) = v_i^{n-1}$. Also, for each arc (c, t) from V_2^{n-1} into t, let $l(c, t) = 0$ and $k(c, t) = q_c$.

We set up this network as parametric in the following way: For the *prioritized* agents among all *claimants* under *available* categories, the parameter λ captures that their reservation values, and only their reservation values, at the relevant categories are improved equally and continuously as long as *(a)* the feasibility conditions in Definition 1 are not violated[38] and *(b)* there are no others joining the group of *prioritized* agents.[39]

Since Z *respects priorities*, a unit under a category is not (probabilistically) assigned to an agent in an indifference class of that category until the utility of each agent in the higher indifference classes is one.

[37] The subscripts in V_1^n and V_2^n stand for describing them either as the demand or supply vertices, and the superscripts for the number of the step of the algorithm.

[38] Part *(a)* is captured by the setting of arcs and their lower bounds and capacities: Condition (i) in Definition 1 by setting the capacity of the arcs from the source to agents by one, and Condition (ii) by setting the capacity of each arc from c to the sink by the capacity q_c.

[39] Whether an agent is *prioritized* or not depends on her relative reservation value at categories. Thus, as the agents' reservation values change, their status of being *prioritized* or *non-prioritized* might change as well.

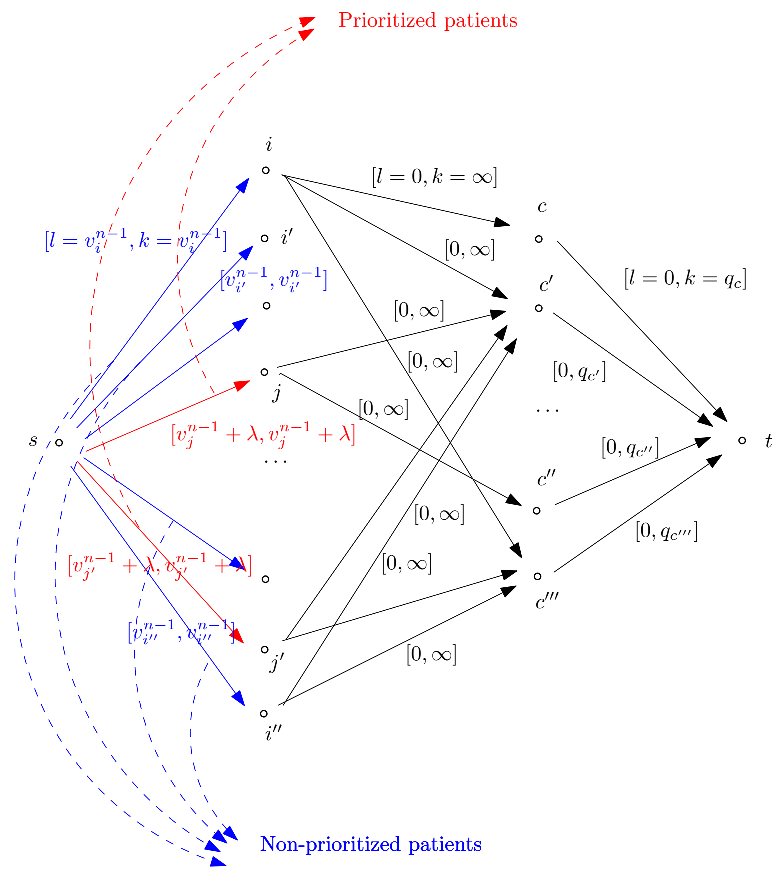

Figure 1:

Thus, when *prioritized* agents' reservation values are increased via λ, the agents in the next indifference class cannot be allowed to be assigned units under the same category. Also, while the reservation values of *prioritized* agents ranked under a category are increased, the reservation values of **other** *prioritized* agents ranked under **other** categories are also increased.

At the beginning of Step n.3, if it is possible to increase the reservation value of each *claimant* to one, then by *procedural fairness*, each such agent's value should be increased to one. By the argument for Remark 3, the *PBR* rule achieves it, in multiple consecutive steps in general. Suppose that it is not possible to increase the reservation value of each *claimant* to one. The idea is to use λ as a continuously increasing parameter until a breakpoint where *(a)* or *(b)* described above will be violated if increased further. Thus, the breakpoint is when: *(a)* Condition 1 in Theorem 1 becomes binding or *(b)* the reservation value of a *prioritized* agent (note that the reservation value of these agents is the minimum among the *claimants* under *available* categories) becomes equivalent to the level of the reservation value of a *non-prioritized claimant*. If the latter holds, then the reservation value of each *prioritized* agent can be increased to the level of the second-lowest reservation value among other *claimants*. Thus, by definition of *Lorenz dominance*, at Z, among the set of *claimants*, no agent's utility is lower than this updated reservation value. Moreover, since this value is lower than one, the set of *claimants* does not change. The only change is that, at this reservation profile, the set of agents with the minimum level of reservation value becomes larger. This process of increasing the reservation values of the *prioritized* continues until either the breakpoint is given by *(a)* or each *claimant*'s reservation value becomes one. If it is the latter, the definition of *Lorenz dominance* implies clearly that each *claimant*'s utility is one (since there exists such an underlying random allocation), which coincides with the outcome of the *PBR* for the *claimants*. Thus, the only case that remains is when the breakpoint is given by *(a)*.

Suppose the breakpoint is given by *(a)*. Since agents are *claimants* for multiple categories in general, to check whether Condition 1 in Theorem 1 becomes binding as λ is increased, we need to consider all subsets of agents. Also, at the beginning of each step when $\lambda = 0$, clearly the condition cannot be binding for a subset of agents for whom there are no *exclusively reserved* categories.

The *prioritized* agents have the lowest level of reservation value among all *claimants*. Thus, to equate reservation values, the parameter λ is increased continuously. Since Condition 1 in Theorem 1 is not binding for no subset of agents, a *flow* exists for some values of $\lambda > 0$. The question is to find the maximum possible value for this parameter. Since the breakpoint is due to Condition 1 becoming binding, there will not be a *flow* respecting the lower bounds of the arcs from s to the demand vertices of *prioritized* agents, if the reservation values of *prioritized* agents are increased above this breakpoint level. By Theorem 5, the value of this maximum *flow* is equal to the minimum value

$$k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V')) \tag{13}$$

taken over $V' \subseteq V$ with $s \in V'$ and $t \notin V'$. Since the *flow* is always maximum, the set $\{s\}$ gives this minimum value. Moreover, at the breakpoint, there exists another set of vertices with the minimum value of (13). We next find this bottleneck set of vertices $V' = \{s\} \cup I' \cup C'$.

The set V' satisfies that each $i \in I'$ points only to the categories in C' (because otherwise $k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V')) = \infty$). Thus, $C(I', v^{n-1}) \subseteq C'$. Also, there cannot be a category c such that $c \in C' \setminus C(I', v^{n-1})$, since then, by removing c from the set V', the value of (13) is decreased by the amount q_c due to the capacity of the outgoing arc from c to t. Thus, $C' = C(I', v^{n-1})$.[40] Thus,

$$k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V')) = \sum_{i \in \left(\bigcup\limits_{c \in A_n} \Gamma_c(v^{n-1})\right) \setminus I'} v_i^{n-1} + \lambda |\{i \notin I' : i \text{ is prioritized}\}| + \sum_{c \in C(I', v^{n-1})} q_c.$$

The first and second terms of the right-hand side in this equation is the total capacity of all the edges from s to the set of *claimants* excluding the set I'. Since $\{s\}$ minimizes (13) as well, and only the reser-

[40] Also, categories in C' cannot be pointed by an agent who is not in I' and a *claimant* only for categories in C' (because otherwise, by adding such an agent to the set I', the value $k(\delta^{\text{out}}(V' \cup \{i\})) - l(\delta^{\text{in}}(V' \cup \{i\}))$ is lower than the value $k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V'))$. Note that any such agent i provides an incoming edge to V' with a lower-bound zero, and an outgoing edge from V' with a capacity v_i^{n-1}.

vation values of *prioritized* agents are increased by λ, we also have

$$k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V')) = \sum_{i\in \bigcup_{c\in A_n} \Gamma_c(v^{n-1})} v_i^{n-1} + \lambda|\{i \in \bigcup_{c\in A_n} \Gamma_c(v^{n-1}) : i \text{ is prioritized}\}|$$

This implies

$$\sum_{i\in\left(\bigcup_{c\in A_n} \Gamma_c(v^{n-1})\right)\setminus I'} v_i^{n-1} + \sum_{c\in C(I',v^n-1)} q_c = \sum_{i\in \bigcup_{c\in A_n} \Gamma_c(v^{n-1})} v_i^{n-1} + \lambda|\{i \in I' : i \text{ is prioritized}\}| \quad (14)$$

Since Equality (14) is the necessary condition for V' to be a bottleneck set, the reservation values of the *prioritized* agents can be increased by the minimum of λ satisfying (14). Note that this minimum λ is equivalent to $\lambda^\star$ defined in Step n.3 of the *PBR*. By definition of *procedural fairness*, at Z, the lowest reservation value is maximized. Thus, since the reservation value of a *prioritized* agent, say i, is the lowest among all *claimants*, and it is feasible to increase their reservation value to $v_i^{n-1}+\lambda^\star$, their utility must be greater than or equal to this value. Moreover, the *PBR* is such that, for each *non-prioritized* agent, the reservation value does not change at Step n. Thus, for each *claimant* j, the updated reservation value is at least $v_j^{n-1} + \lambda^\star$. Thus, $u_Z(j) \geq v_i^n$. This completes the proof.

□

Appendix E Proof of Proposition 1

Let $R = (\mathcal{I}, C \cup \{c_M\}, \succ, (q_c)_{c \in C \cup \{c_M\}}, (B_c)_{c \in C})$ be a reserve problem under a baseline priority order. We show that the *PBR* rule satisfies *category-fairness*. The result then follows directly from Theorem 4.

Let Z be an allocation which is welfare-equivalent to the *PBR* outcome. Remember that $i_M(Z)$ is the student who has a utility less than one such that each student j with a higher merit score than $i_M(Z)$ has a utility one. Also, $i_{\min}(Z)$ is student with the lowest merit score, who is assigned a seat from the merit category with positive probability.

Since the *PBR* is in the class of *PBSWI*, by Theorem 2, Z is *acceptable*. Thus, by *respecting priorities*, each student with a higher score than $i_{\min}$ is assigned a seat with probability one. This implies that $i_{\min}(Z) \succcurlyeq i_M(Z)$.

We first show that Z is not unfair for the merit category. Suppose that each student j with $j \succ i_M(Z)$ is a beneficiary of a preferential category and there exists at least one other beneficiary of the same category, who has a lower merit score than j and is assigned a seat with probability one. Now consider the set $\{j : j \succ i_M(Z)\}$ and the last step of the *PBR*, say Step k, at which the reservation value of a student in this set, say j', as a beneficiary of the preferential treatment category, say c', is increased to one. By construction of the priority ordering $\pi_{c'}$, each student with a lower merit score than j' is ranked below j'. Let j'' be the beneficiary of c' ranked just below j'. Note that by our supposition, $u_Z(j'') = 1$. Moreover, at the end of Step k, the reservation value of j'' is zero since (i) she is ranked below j under both c_M and c', and (ii) by assumption, reserved seats under preferential treatment categories are assigned only to their beneficiaries, and this implies that j' can be assigned a seat only from c_M or c' (‡). Since each student with a higher score than j' is assigned a seat with probability one at the end of Step k, j'' is a claimant for c'. Also, since, by definition Step k, each student in the set $\{j : j \succ i_M(Z)\}$ is assigned a seat with probability one, $i_M(Z)$ is a claimant for c_M. Now suppose that at the beginning of Step $k+1$, there is a group of students I' such that categories in $C(I', v^k)$ are *exclusively reserved* for I'.

Case 1: The set I' contains a beneficiary of c' with a higher score than j''. Since such a student is a claimant for c', by definition of *exclusively reserved* categories, $c' \in C(I', v^k)$. Since, under Z, i is the student who is assigned a seat with probability less than one such that each student with a higher merit score than $i_M(Z)$ is assigned a seat with probability one, the set of claimants for c_M does not change at any later step of the *PBR*. Also, for each preferential treatment category in $C(I', v^k)$ (including c'), the set of claimants changes only when one their beneficiaries is assigned a seat with probability one at a later step. Since each student is a beneficiary of at most two categories, a preferential treatment category and c_M, and the set of claimants of c_M does not change at a later step, this is not possible. Thus, under Z, all seats reserved for the categories in $C(I', v^k)$ are assigned to the students in I'. Since $v^k_{j''} = 0$ (‡), $u_Z(j'') = 0$. This contradicts with $u_Z(j'') = 1$.

Case 2: The set I' does not contain a beneficiary of c' with a higher score than j''. By definition of *exclusively reserved* categories, $c' \notin C(I', v^k)$. Note that it must be $c_M \notin C(I', v^k)$, since otherwise, each beneficiary of c' with a higher score than $i_M(Z)$ is assigned a seat from c', and since by assumption, the number of beneficiaries of c' with a higher score than j'' is at least $q_{c'}$, this would imply that the first $q_{c'}$ beneficiaries of c' belong to I' and $c' \in C(I', v^k)$. Since $c_M \notin C(I', v^k)$, at Step $k+1$, the reservation values of $i_M(Z)$ and j'' can be increased. But, this contradicts with $u_Z(i_M(Z)) < 1$ and $u_Z(j'') = 1$ since this is Lorenz dominated by another allocation at this step where their utilities are equivalent to each other.

Finally, if there are no *exclusively reserved* categories at the beginning of Step $k+1$, by applying the same argument in Case 2 above, we conclude that $u_Z(i_M(Z)) < 1$ and $u_Z(j'') = 1$ is a contradiction. Thus, Z is not unfair for c_M.

We now show that for each preferential treatment category c, Z is not unfair for c. Let c be a preferential treatment category such that each student who is a beneficiary of c and assigned a seat with positive probability has a higher merit score than $i_{\min}$. Let k' be the last step at which the reservation value of $i_{\min}(Z)$ is increased. By definition of the *PBR*, at Step k', c_M is available. Thus, if there are *exclusively reserved* categories for I' at this step, I' does not contain any student with a higher score than $i_{\min}(Z)$. Thus, since this set contains all the beneficiaries of c who are assigned a seat with probability one, c is also available. But then, since, by definition of the *PBR*, the reservation value chosen at this step *Lorenz dominates* any other, the student with the highest merit score among all beneficiaries of c with the reservation value zero must be increased. But, this contradicts that each student who is a beneficiary of c and assigned a seat with positive probability has a higher merit score than $i_{\min}(Z)$.

Appendix F Proof of Proposition 2

Let $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$ be a rationing problem. We show that the *PBR* rule satisfies *equal treatment of equals*. The result then follows directly from Theorem 4. Let Z be an allocation which is welfare-equivalent to the *PBR* outcome. By the first condition of symmetric categories, we can assume, without loss of generality, that $\mathcal{C} = \{c, c'\}$. Let us start with the first step of the *PBR* as the initial step of an inductive argument. Each agent in the first indifference classes of both categories are *prioritized*. Thus, by the second condition of *symmetric categories*, the reservation values of equals i and j are increased by the same amount. Suppose the step ends with *exclusively reserved* categories for agents in the first indifference classes. (Note that these indifference classes could have a non-empty intersection.) Since categories are symmetric, it must be that all categories in $\mathcal{C}$ are *exclusively reserved*. Then, by the first condition of *symmetric categories*, the algorithm terminates and since all probabilities are the same for all these agents, *equal treatment of equals* follow immediately. Suppose now that the reservation values of all agents are increased to one. Again, all agents have the same reservation value and this implies that and agent in the first indifference class of c and another one in the first indifference class of c' are treated equally. Let us now consider Step n as the inductive step. Since we assume that the argument for the first step holds for all steps up to $n-1$, and the algorithm did not terminate before Step n, we have that the agents in the nth indifference classes of c and c' are the *claimants* of c and c'. Also, by the third condition of *symmetric categories*, all these agents have not been *claimants* before. Thus, each of them has a reservation value of zero at the beginning of Step n. Thus, the same argument in the initial step applies for the current step as well. This concludes the proof.

References

ABDULKADIROĞLU, A., AND T. SÖNMEZ (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

ABDULKADIROĞLU, A., AND A. GRIGORYAN (2021): "Priority-based Assignment with Reserves and Quotas," working paper.

AYGÜN, O., AND I. BO (2021): "College Admission with Multidimensional Privileges: The Brazilian Affirmative Action Case," *American Economic Journal: Microeconomics*, 13, 1–28.

AYGÜN, O., AND B. TURHAN (2020a): "Designing Direct Matching Mechanisms for India with Comprehensive Affirmative Action," SSRN working paper.

——— (2020b): "Dynamic Reserves in Matching Markets," *Journal of Economic Theory*, 188, 105069.

AZIZ, H. (2021): "Probabilistic Rationing with Categorized Priorities: Processing Reserves Fairly and Efficiently," working paper.

BOGOLOMANIA, A., AND H. MOULIN (2004): "Random Matching Under Dichotomous Preferences," *Econometrica*, 72, 257–279.

BOGOMOLNAIA, A., AND H. MOULIN (2001): "A New Solution to the Random Assignment Problem," *Journal of Economic Theory*, 100, 295–328.

DELACRÉTAZ, D. (2021): "Processing Reserves Simultaneously," working paper.

DUR, U., S. KOMINERS, P. PATHAK, AND T. SÖNMEZ (2018): "Reserve Design: Unintended Consequences and The Demise of Boston's Walk Zones," *Journal of Political Economy*, 126(6), 2457–2479.

DUR, U., P. PATHAK, AND T. SÖNMEZ (2020): "Explicit vs. statistical targeting in affirmative action: Theory and evidence from Chicago's exam schools," *Journal of Economic Theory*, 187, 104996.

EHLERS, L., I. E. HAFALIR, M. B. YENMEZ, AND M. A. YILDIRIM (2014): "School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds," *Journal of Economic Theory*, 153, 648–683.

FORD, L. R., AND D. R. FULKERSON (1956): "Maximal flow through a network," *Canadian Journal of Mathematics*, 8, 399–404.

GALE, D. (1957): "A Theorem on Flows in Networks," *Pacific Journal of Mathematics*, 7, 1073–1082.

HAFALIR, I. E., M. B. YENMEZ, AND M. A. YILDIRIM (2013): "Effective Affirmative Action in School Choice," *Theoretical Economics*, 8, 325–363.

HALL, P. (1935): "On representatives of subsets," *Journal of London Mathematical Society*, 10, 26–30.

Katta, A.-K., and J. Sethuraman (2006): "A Solution to The Random Assignment Problem on The Full Preference Domain," *Journal of Economic Theory*, 131, 231–250.

Kominers, S., P. Pathak, T. Sönmez, and U. Ünver (2020): "Paying It Backward and Forward: Expanding Access to Convalescent Plasma Through Market Design," working paper.

Kominers, S. D., and T. Sönmez (2016): "Matching with slot-specific priorities: Theory," *Theoretical Economics*, 11, 683–710.

Massachusetts DPH (2020): "MA Covid-19 Vaccine Presentation," December 9, https://www.mass.gov/doc/ma-covid-19-vaccine-presentation-1292020/download.

New Hampshire DHHS (2021): "COVID-19 Vaccine Training for Community Leaders," January, https://www.dhhs.nh.gov/dphs/cdcs/covid19/documents/covid-vaccine-comm-leader-training.pdf.

Pathak, P., T. Sönmez, U. Ünver, and B. Yenmez (2021): "Fair Allocation of Vaccines, Ventilators and Antiviral Treatments: Leaving No Ethical Value Behind in Health Care Rationing," working paper.

Pathak, P. A., A. Rees-Jones, and T. Sönmez (2022): "Reversing Reserves," forthcoming, *Management Science*.

Pennsylvania DH (2020): "Ethical Allocation Framework for Emerging Treatments of COVID-19," https://www.health.pa.gov/topics/disease/coronavirus/Pages/Guidance/Ethical-Allocation-Framework.aspx.

Roth, A. E., T. Sönmez, and M. U. Ünver (2005): "Pairwise Kidney Exchange," *Journal of Economic Theory*, 125, 151–188.

Schrijver, A. (2003): *Combinatorial Optimization: Polyhedra and Efficiency*. Springer-Verlag.

Sönmez, T., M. U. Ünver, and M. B. Yenmez (2020): "Incentivized Kidney Exchange," *American Economic Review*, 110(7), 2198–2224.

Sönmez, T., and B. Yenmez (2020): "Affirmative Action with Overlapping Reserves," working paper.

Tennessee DH (2020): "Covid-19 Vaccination Plan," Version 3.0, December 30, https://www.tn.gov/content/dam/tn/health/documents/cedep/novel-coronavirus/COVID-19_Vaccination_Plan.pdf.

White, D. B., M. H. Katz, J. M. Luce, and B. Lo (2009): "Who Should Receive Life Support During a Public Health Emergency? Using Ethical Principles to Improve Allocation Decisions," *Annals of Internal Medicine*, 150(2), 132–138.

Yilmaz, O. (2011): "Kidney Exchange: An Egalitarian Mechanism," *Journal of Economic Theory*, 146, 592–618.

Zucker, H., K. Adler, D. Berens, R. Bleich, R. Brynner, K. Butler, et al. (2015): "Ventilator allocation guidelines," *Albany: New York State Department of Health Task Force on Life and the Law*.