Equity In Allocating Identical Objects Through Reserve Categories*

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Abstract

The units of an object are to be distributed among a set of agents through reserve categories. For example, vaccines, ICU's or other medical units are reserved for certain patients based on their occupations, preexisting conditions and disadvantaged status, or school seats are allocated to students through tiers based on the socioeconomic status to eliminate segregation. A widespread mechanism is processing these categories in a precedence order. Since there are multiple categories through which an agent can be assigned a unit, any choice of precedence order has distributional consequences. To mitigate uneven treatment of agents, we consider processing reserve categories simultaneously. We propose a procedure to enhance equity and characterize the class of equitable random allocation rules.

Keywords: Market design, Matching, Reserve Systems, Covid-19, Egalitarianism

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1 Introduction

The recent pandemic has highlighted the importance of well-designed rationing mechanisms when resources are in short supply. How should we design and implement such mechanisms for the allocation of vaccines, ventilators, ICU's or other crucial medical units? A simple solution is a *priority system*, which allocates units to patients with respect to a single order of priority. These systems could fail to recognize certain ethical values and inherently embed discriminatory practices against disadvantaged groups (Pathak, Sönmez, Ünver, and Yenmez, 2021). A more flexible alternative is a *reserve system*, in which units are divided into multiple reserve categories with each having a distinct priority order of patients depending on their characteristics. Typically, these categories are processed in a precedence order. Since patients are in general beneficiaries of multiple categories, this procedure too has distributional consequences. This problem persists in other reserve settings such as school choice with affirmative action and immigration visa allocation. We consider a general *reserve system* framework with processing reserve categories simultaneously rather than sequentially. We propose and characterize random allocation rules to enhance equity by mitigating uneven treatment of agents due to precedence orders.

For the class of rationing problems with reserves, certain properties are indispensable (see Section 3). First, priorities of agents under categories should be respected (*respecting priorities*). Second, no unit should be wasted (*efficiency*). Third, an agent should be assigned at least as the share of a unit guaranteed to her by a single category at which she qualifies to receive a positive share (*individual rationality*). We call a (random) allocation *acceptable* if it satisfies all these standard axioms. Our first theorem is a characterization of the set of *acceptable* (random) allocation rules (Theorem 2).

The deterministic procedures within the *acceptable* ones imply uneven treatment of agents, which implies randomization as a significant alternative. There are basically two standard notions of fairness and equity in the randomization context. The first notion is a basic axiom of fairness, *no justified-envy*: if an agent prefers being in the position of another agent, then there is a justification (based on priorities) for assigning a higher utility to that other agent. The second notion is an axiom of equity, *egalitarianism*, which requires equating agents' utilities as much as possible (within the constraints of *acceptability*) through the criterion of *Lorenz dominance*. We show that *egalitarianism* is impossible in the current context (see Theorem 3). We then introduce and formalize a new equity property, *sequential egalitarianism*, which basically requires equating agents' utilities procedurally (as much as possible) throughout the simultaneous processing of reserves (Section 5.3). Interestingly, while an *egalitarian* allocation, if it exists for a problem, does not even satisfy *equal treatment of equals* in general, *sequential egalitarianism* is stronger than *no justified-envy* (Proposition 1). Thus, we argue that *sequential egalitarianism* is the plausible equity criterion for the reserves setting.

Our second theorem is the characterization of the *sequentially egalitarian* rules, the *Priority-Based Rawlsian (PBR)* (Theorem 4). The *PBR* rules constitute an intuitive class within the set of *acceptable* rules. Basically, they are based on defining a guaranteed utility for each agent and then increasing these utilities sequentially subject to the constraints of *acceptability* and also the Rawlsian principle of prioritizing the most disadvantaged agents. We show how to design this procedure (Section 5.4) with the help of ideas from graph theory (Appendix A).

We discuss relevant applications, rationing health care units, vaccines etc. (Section 6.1) and affirmative action in school choice (Section 6.2). We argue that the class of rules we propose could help policymakers to mitigate the inequalities due to uneven treatment of agents in deterministic allocation rules based on processing reserves sequentially.

Related Literature

Reserve systems with sequential processing has been proposed for affirmative action in school choice (Kominers and Sönmez, 2016).[1] When there are only two types of slots, reserve and open slots, both increasing the reserve quota and raising the precedence order positions of open seats will (weakly) increase the number of reserve-eligible students who are accepted (Dur, Kominers, Pathak, and Sönmez, 2018). For the case of multiple socioeconomic tiers along with the merit tier, the precedence orders for maximizing the number of the most disadvantaged students assigned a seat are character-

[1] Affirmative action in school choice has been widely studied. Controlled choice models provide choice to parents while maintaining the racial and ethnic balance at schools through type-specific reserves and quotas (Abdulkadiroğlu and Sönmez, 2003; Ehlers, Hafalir, Yenmez, and Yıldırım, 2014), or through adjusted priorities under minority reserves (Hafalir, Yenmez, and Yıldırım, 2013). A recent work studies how to minimize priority violations for a setting when there is only one ordering of students and there are type-specific reserves and quotas. A particular choice rule, where all applicants are first considered for units reserved for their own types, uniquely minimizes priority violations in this class (Abdulkadiroğlu and Grigoryan, 2021).

ized as follows: the slots of other tiers precede the merit slots which are succeeded by the slots of the tier for the most disadvantaged students (Dur, Pathak, and Sönmez, 2020).

A model closer to the current setting is when a student is in general a beneficiary at multiple reserve categories, the case of *overlapping reserves*, and the goal is to guarantee maximal compliance with reservations (as many of the reserved positions as possible are to be allocated to the candidates from target groups) (Sönmez and Yenmez, 2020). Equity under maximal compliance is studied in a similar setting when random allocations are allowed (Doğan and Yılmaz, 2022).

Reserve systems have been also relevant in various other contexts: medical rationing (Pathak, Sönmez, Ünver, and Yenmez, 2021), the H-1B visa program (Pathak, Rees-Jones, and Sönmez, 2022), university admissions in India (Sönmez and Yenmez, 2020; Aygün and Turhan, 2020a,b) and Brazil (Aygün and Bo, 2021).

Another strand of literature, to which the current work belongs as well, is the approach of processing reserves simultaneously. A recently proposed axiom in this setting is *category neutrality*: An allocation is *category neutral* if an agent who qualifies for multiple categories receives the same amount of capacity from all of them (Delacrétaz, 2021). In the context of hard reserves (only the beneficiaries of a given reserve category are eligible for the units under that category), every random allocation satisfying *efficiency*, *respecting priorities* and *category neutrality* assigns to each agent the same amount of probability of receiving a unit in aggregate, and a polynomial-time algorithm exists to compute these allocations (Delacrétaz, 2021). The difference between our approach and this work can be summarized as follows: while *category neutrality* requires that for an agent, the probability of being assigned a unit is the same across all categories for which she is eligible, *sequential egalitarianism* requires equating utilities across agents (procedurally and subject to the constraints of *acceptability*). Clearly, these two axioms and ideas are not only independent but also fundamentally different.[2]

An alternative approach is to apply a *Probabilistic Serial* (PS) mechanism, the *Rationing Eating (RE)* rule, to the current setting: Categories are treated as pseudo-agents and the agents as pseudo-items, as if categories are 'consuming' agents. The pseudo-agents categories now have preferences over the pseudo-items that are derived from the priorities of the corresponding categories. Then, the *PS*

[2] See the extended version of our work (Yılmaz, 2022) for an example clarifying the difference between the two works.

rule is implemented on this pseudo-market (Aziz, 2021). The *RE* rule retains the fairness property (*sd-envy-freeness*) in this pseudo-market: *category sd-envy-freeness*. This work and the current one are also substantially different: we do not reverse the roles of agents and categories. Thus, we cannot apply the *PS* rule directly for the categories (pseudo-agents) over the agents (pseudo-items). This leads to an analytical challenge: We should keep track of who can be assigned to units from which categories at a given instance of the random allocation rule. We explain this technical challenge, and propose a methodology for overcoming this difficulty (Section 4). More importantly, this fundamental difference implies another important distinction between these works: while the *RE* rule is designed to satisfy a property based on the comparison of categories in terms of the agents' probabilities assigned to units under these categories, our solution, the *PBR* rule (see Section 5.4), satisfies a property based purely on agents' welfare (see Definition 9 in Section 5).

The idea of *egalitarianism* and the principle of maximizing the minimum welfare are studied in several other contexts of discrete allocation models.[3] Recently, another such work analyzes the incentive schemes designed for plasma donation (Kominers, Pathak, Sönmez, and Ünver, 2020). Plasma donors are given priorities for prospective plasma therapies of their loved ones (*pay-it-backward*), and patients receive priority access for plasma therapy in exchange for a pledge to donate her own plasma in the near future (*pay-it-forward*).[4] The authors also design a mechanism, *plasma pooling procedure*, which guarantees an *egalitarian* distribution of plasma therapy by making non-prioritized patients' welfare as equal as possible across different blood types within *efficiency* constraint.[5]

2 Model

There is a set of **agents** $\mathcal{I}$ and a set of **reserve categories** $\mathcal{C}$. For each $c \in \mathcal{C}$, q_c identical units are reserved, and there is a **weak priority order** π_c over $\mathcal{I}$.[6] The strict and indifference parts of π_c

[3]See Bogolomania and Moulin (2004), Roth, Sönmez, and Ünver (2005) and Yılmaz (2011).

[4]A different health care setting where similar incentive schemes are analyzed is a kidney exchange model where compatible pairs are incentivized to participate in kidney exchange by insuring their patients against future renal failure via increased priority in the deceased-donor queue (Sönmez, Ünver, and Yenmez, 2020).

[5]This method is also based on graph theoretical ideas and in particular, on parametric flows (see also Katta and Sethuraman (2006)).

[6]The contribution of our model is **not** the assumption of a general domain of priority orders. Even under strict priority orders, the main analytical and conceptual challenges remain and our analysis is still a novel approach.

are denoted by π_c^P and π_c^I, respectively. For each c, the set of agents in the $k-$th indifference class of π_c is $\mathcal{I}_{\pi_c}(k)$ such that for $k' > k''$, $i \in \mathcal{I}_{\pi_c}(k')$ and $j \in \mathcal{I}_{\pi_c}(k'')$ imply $j \; \pi_c^P \; i$. The set of agents in the first k indifference classes is denoted by $UCS_{\pi_c}(k)$, thus, $UCS_{\pi_c}(k) = \bigcup_{k'=1}^{k} \mathcal{I}_{\pi_c}(k')$.

A **(rationing) problem** is a tuple $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$. Let $\mathcal{R}$ denote the set of all problems. We consider a setting where units are assigned to agents probabilistically such that for each $R \in \mathcal{R}$, the probability with which an agent is assigned a unit is at most one and for each $c \in \mathcal{C}$, at most q_c units are assigned to agents.

Definition 1. *Given a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, a **random allocation** is a stochastic $|\mathcal{I}| \times |\mathcal{C}|$ matrix Z where for each i and c, z_{ic} is the probability with which agent i is assigned one unit from category c such that*

i. for each $i \in \mathcal{I}$, $\sum_{c\in\mathcal{C}} z_{ic} \leq 1$,

ii. for each $c \in \mathcal{C}$, $\sum_{i\in\mathcal{I}} z_{ic} \leq q_c$.

Let $\mathcal{Z}(R)$ denote the set of all random allocations for a problem R, and $\mathcal{Z} = \bigcup_{R\in\mathcal{R}} \mathcal{Z}(R)$ the set of all random allocations. A **rule** is a mapping $\varphi : \mathcal{R} \to \mathcal{Z}$ such that for each problem R, $\varphi(R) \in \mathcal{Z}(R)$.

Since all units are identical, only the probability of receiving a unit is relevant for agents, not the specific reserve categories through which they are (randomly) assigned a unit. Let $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$ be a problem and $Z \in \mathcal{Z}(R)$ a random allocation. The **utility of agent** i is given by $u_Z(i) = \sum_{c\in\mathcal{C}} z_{ic}$. The vector $u_Z = (u_Z(i))_{i\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ is the **utility profile**. We also say that a utility profile u is **generated** by a random allocation Z if $u = u_Z$. Random allocations Z and Z' are **welfare equivalent** if $u_Z = u_{Z'}$. Similarly, rules φ and φ' are *welfare equivalent* if for each problem R, random allocations $\varphi(R)$ and $\varphi'(R)$ are *welfare equivalent*.

Nevertheless, the assumption of weak priority orders adds another challenging analytical component to our work. Also, weak priority orders become relevant for some of the applications we consider in Section 6.

3 Axioms

There are three indispensable requirements: (1) Resources should not be wasted (*efficiency*), (2) each agent's utility should be at least as the utility guaranteed to them by a single category (*individual rationality*), and (3) an agent can be assigned a unit under a category only if each agent with a strictly higher priority for that category is assigned a unit with probability one (*respecting priorities*).

3.1 Efficiency

The first axiom states that no unit should be wasted. If there are agents demanding a unit and that unit is available, then it should not remain as unassigned.

Definition 2. *For a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, a random allocation $Z \in \mathcal{Z}(R)$ is* ***non-wasteful****, if for any $c \in \mathcal{C}$,*

$$\sum_{i\in\mathcal{I}} z_{ic} < q_c \implies \text{for each } i \in \mathcal{I},\ \sum_{c'\in\mathcal{C}} z_{ic'} = 1.$$

A rule φ is ***non-wasteful*** *if for any problem R, random allocation $\varphi(R)$ is non-wasteful.*

The only case for a unit remaining (partially) unassigned under *non-wastefulness* is when each agent is assigned a unit with probability one. For expositional simplicity, we exclude these cases: A problem is **non-trivial**, if it is not possible to assign each agent a unit. We assume that each problem in $\mathcal{R}$ is *non-trivial.*[7] Clearly, *non-wastefulness* and *non-triviality* together imply that Condition *(ii)* of Definiton 1 holds with equality.

3.2 Individual rationality

Given a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, let $R^c = (\mathcal{I}, \{c\}, \pi_c, q_c)$ denote the associated **single-category rationing problem**. Let k_c be such that there are sufficient number of units under

[7] There is no loss of generality in assuming *non-triviality*: the definitions and results hold also for *trivial* problems.

category c for the agents in the first k_c indifferent classes but not for the agents in the first $k_c + 1$ indifference classes.

The **priority rule** ρ maps each single-category rationing problem R^c into an $|\mathcal{I}|$-vector and for each $i \in \mathcal{I}$, it specifies the share of i at category c, denoted by $\rho_i(R^c)$. The priority rule allocates the units under category c sequentially by respecting the priority order π_c: the first $|\mathcal{I}_{\pi_c}(1)|$ units are assigned to the agents in $\mathcal{I}_{\pi_c}(1)$, the next $|\mathcal{I}_{\pi_c}(2)|$ units are assigned to the agents in $\mathcal{I}_{\pi_c}(2)$, so on, until all units are assigned such that in Step $k_c + 1$, the number of remaining units (if any), that is, $q_c - \sum_{k=1}^{k_c} |\mathcal{I}_{\pi_c}(k)|$, is assigned to the agents in $\mathcal{I}_{\pi_c}(k_c + 1)$ with equal probability. Thus, each agent in the first k_c priority classes is assigned a unit with probability one; agents in the $(k_c + 1)-$st priority class share the remaining units[8] equally among themselves and the remaining agents are not assigned a positive share. The *priority rule* ρ is formally defined as follows:

$$\rho_i(R^c) = \begin{cases} 1 & \text{if } \ i \in \bigcup_{k=1}^{k_c} \mathcal{I}_{\pi_c}(k) \\ \frac{q_c - \sum_{k=1}^{k_c} |\mathcal{I}_{\pi_c}(k)|}{|\mathcal{I}_{\pi_c}(k_c+1)|} & \text{if } i \in \mathcal{I}_{\pi_c}(k_c + 1) \\ 0 & \text{otherwise} \end{cases}$$

We consider an agents' share at category c given by the *priority rule* as their minimum utility.

Definition 3. *For a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, a random allocation $Z \in \mathcal{Z}(R)$ is* ***individually rational****, if for any c, and $i \in \mathcal{I}_{\pi_c}$, $u_Z(i) \geq \rho_i(R^c)$. A rule φ is* ***individually rational*** *if for any problem R, random allocation $\varphi(R)$ is individually rational.*

Individidual rationality implies that the utility of agent i is greater than or equal to

$$\max_{c\in C} \rho_i(R^c). \tag{1}$$

We call this value as the **initial reservation value of agent** i and denote it by v_i^0. The *initial reservation profile* is $v^0 = (v_i^0)_{i\in\mathcal{I}}$.

Remark 1. *While it is natural to consider agents' share at a single category as the least that they*

[8]Note that the number of remaining units may be zero, in which case the agents in this class are not assigned a positive share.

should receive (and that is what we assume in our theoretical analysis in Sections 4 and 5), some contexts may require defining these rights differently.[9] *Since, in this work, individual rationality is relevant only for its implication on the initial reservation profile, our solutions and theorems hold for any alternative definition of individual rationality.*

Our analysis throughout the paper is based on the simple idea of sequentially updating the initial reservation profile. Thus, we refer to **reservation profiles** in general, generically denoted by $v = (v_i)_{i\in\mathcal{I}}$. Although the *utility profile* u_Z under a random allocation Z and the *reservation profile* $v = (v_i)_{i\in\mathcal{I}}$ are mathematically the same type of objects, there is an important difference between them: While a *utility profile* represents agents' utilities induced by a random allocation, the interpretation of a *reservation profile* $v = (v_i)_{i\in\mathcal{I}}$ is that agent i is guaranteed a utility level at least as much as v_i, without any implication of a specific random allocation and agents' utilities. A reservation profile v is **feasible** if there exists a random allocation Z such that $v = u_Z$.

3.3 Respecting priorities

The third axiom is about priorities: an agent cannot be (probabilistically) assigned a unit from a category if there is another agent with a strictly higher priority and a utility less than one.

Definition 4. *For a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, a random allocation $Z \in \mathcal{Z}(R)$* ***respects priorities****, if for any $i \in \mathcal{I}$, and $c \in \mathcal{C}$,*

$$i \; \pi_c^P \; j \text{ and } u_Z(i) < 1 \implies z_{jc} = 0$$

A rule φ ***respects priorities*** *if for any problem R, random allocation $\varphi(R)$ respects priorities.*

4 Acceptable random allocations

We deem the axioms stated in Section 3 as indispensable and consider only the rules, which satisfy these axioms: For any problem R, a random allocation $Z \in \mathcal{Z}(R)$ is **acceptable** if it satisfies

[9] See Section 6.1 for such a case.

non-wastefulness, *individually rationality* and *respects priorities*. We denote the set of *acceptable* random allocations by $\mathcal{Z}^a(R)$. A rule φ is **acceptable** if for each problem R, $\varphi(R) \in \mathcal{Z}^a(R)$.

Our main goal is (1) to formalize a sensible equity notion in the current context, and (2) to design and characterize *acceptable* rule(s) satisfying this notion. Towards this goal, we first characterize *acceptable* rules. These rules are based on a procedure of starting with the *initial reservation profile*, and then improving (certain) agents' utilities by (probabilistically) assigning units simultaneously. This procedure is based on a simple idea but its design is not straightforward for mainly three difficulties.

First, since agents can receive units from different categories, it is not clear which agents should have access to a given category at a given instance of improving utilities.

Example 1. *(Determination of agents' access to reserve categories)*
Let $\mathcal{I}=\{i,j,k\}$ and $\mathcal{C}=\{c_1,c_2\}$ such that one unit is reserved for each category. The priority orders for categories are given below with each set in the table being a priority class (we use the same type of representation for a problem in all the remaining examples):

π_{c_1}	π_{c_2}
$\{i\}$	$\{i,j\}$
$\{k\}$	$\{k\}$

Individual rationality implies that i is assigned one unit, and j is assigned at least half units. A plausible argument is that since j has a higher priority than k, the remaining half units should be assigned to j, which is an acceptable allocation. On the other hand, for each $\lambda \in [0, \frac{1}{2}]$, the following random allocation is also acceptable:

$$Z = \begin{array}{c|cc} & c_1 & c_2 \\ \hline i & \frac{1}{2}+\lambda & \frac{1}{2}-\lambda \\ j & 0 & \frac{1}{2}+\lambda \\ k & \frac{1}{2}-\lambda & 0 \end{array}$$

In Example 1, since the unit under c_2 can also be (probabilistically) assigned to i (along with the unit under c_1), there is room for the unit under c_1 to be (probabilistically) assigned to k, and there is no reason why k should be excluded from the list of candidates for c_1. This simple example brings about the solution of the first difficulty: in characterizing *acceptable* allocation rules, agents' access to categories should be set as broad as possible and restricted only by the axioms in Section 3.

Definition 5. *Let $v = (v_i)_{i\in\mathcal{I}}$ be a reservation profile. Agent i is* ***eligible*** *for category c under v if $i \in \mathcal{I}_{\pi_c}(k)$ and for each $i \in UCS_{\pi_c}(k-1)$, $v_i = 1$. The set of eligible agents for category c under v is denoted by $\Gamma_c(v)$.*

Whenever the first $k-1$ priority classes consist of only agents with reservation value one, all these agents and the agents in the kth priority class are *eligible* for the corresponding reserve category.[10]

Second, there is an exception to *eligibility*: as the following example demonstrates, *eligibility* does not always imply that units under a category can be assigned to all of its *eligible* agents.

Example 2. *(Eligibility does not always imply a positive share.)*
Let $\mathcal{I}$= $\{i, j, k\}$ and $\mathcal{C}$= $\{c_1, c_2\}$ such that one unit is reserved for each category. Consider the following problem:

π_{c_1}	π_{c_2}
$\{i\}$	$\{j\}$
$\{k\}$	$\{k\}$

For the reservation profile $v = (v_i, v_j, v_k) = (1, 1, 0)$, all agents are eligible for all categories. But, any random allocation such that a unit is (probabilistically) assigned to k does not respect priorities.

In Example 2, *individual rationality* implies that agents i and j are assigned all units, one unit each. Thus, they should have 'exclusive rights' over the units under c_1 and c_2.

Let $v = (v_i)_{i\in\mathcal{I}}$ be a reservation profile. For each i with $v_i > 0$, let $C(i, v)$ denote the set of reserve categories, for which agent i is *eligible* under the reservation profile v. Let $C(I, v) = \bigcup\limits_{i\in I} C(i, v)$.

Definition 6. *Given a reservation profile $v = (v_i)_{i\in\mathcal{I}}$, agents in I have* ***exclusive rights*** *over the set of reserve categories $C(I, v)$ if $\sum\limits_{i\in I} v_i = \sum\limits_{c\in C(I,v)} q_c$.*

[10]In Example 1, i and k are *eligible* for c_1, while i and j are *eligible* for c_2 (note that k is not *eligible* for c_2).

For a given reservation profile, *exclusive rights* correspond to binding feasibility constraints for an underlying random allocation. The following characterization theorem implies that *exclusive rights* are the only exceptions to *eligibility*. We utilize this important insight to prove the characterization of the *acceptable* rules (Theorem 2).

Theorem 1. *(The Supply-Demand Theorem (Gale, 1957))*[11]
Let $v = (v_i)_{i \in \mathcal{I}}$ *be a reservation profile. There is a random allocation* Z *such that (i) for each* $i \in \mathcal{I}$, $u_z(i) \geq v_i$, *and (ii)* $z_{ic} > 0$ *implies* $i \in \Gamma_c(v)$, *if and only if, for each subset* I *of agents*

$$\sum_{i \in I} v_i \;\leq \sum_{c \,\in\, C(I,v)} q_c. \tag{2}$$

Third, the approach of sequentially updating the reservation values requires keeping track of changes in *eligibility*: while an agent may not be *eligible* for a category at a given *reservation profile*, as the agents' reservation values possibly go up, she might be *eligible* for it at a different one.

Example 3. *(Sequential improvement of agents' access to reserve categories)*
Let $\mathcal{I} = \{i, j, k, l, i_1, i_2, i_3, i_4\}$ *and* $\mathcal{C} = \{c_1, c_2, c_3, c_4\}$ *such that for each* c_1 *and* c_3*, two units are reserved, and for each* c_2 *and* c_4*, one unit is reserved. Consider the following problem:*

π_{c_1}	π_{c_2}	π_{c_3}	π_{c_4}
$\{i, j, k\}$	$\{i, k, i_1\}$	$\{l\}$	$\{j, i_2\}$
$\{i_3, i_4\}$	$\{i_3, i_4\}$	$\{i, j, i_1\}$	$\{i_4\}$

For the initial reservation profile $v^0 = (v^0_i, v^0_j, v^0_k, v^0_l, v^0_{i_1}, v^0_{i_2}, v^0_{i_3}, v^0_{i_4}) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1, \frac{1}{3}, \frac{1}{2}, 0, 0)$*, agents* i_3 *and* i_4 *are not eligible for any category. On the other hand, for the profile* $v = (1, 1, 1, 1, 1, \frac{1}{2}, 0, 0)$*, agents* i_3 *and* i_4 *are eligible for* c_1 *and* c_2*.*

We characterize the set of *acceptable* random allocations by a sequential allocation procedure: the **Priority-Based Sequential Welfare Improvement (PBSWI) Algorithm**. The design relies on careful treatment of the difficulties discussed above. The idea is to sequentially update agents' access to reserve categories through the *eligibility* criterion by keeping track of *exclusive rights*.

[11] This is a generalization of Hall's Set Representation Theorem (Hall, 1935), which holds only for integers.

The PBSWI Class:

Step 0. Let the reservation profile be $v^0 = (v_i^0)_{i \in \mathcal{I}}$.[12]

For each $n \geq 1$ and the reservation profile v^{n-1}, the following steps are executed.

Step n.1 For each set of agents I with *exclusive rights* over $C(I, v^{n-1})$,

i. for each $i \in I$, let $v_i^n = v_i^{n-1}$, and

ii. mark each reserve category in the set $C(I, v^{n-1})$ as *unavailable*.

Let A_n denote the set of *available* reserve categories.

Step n.2 If $A_n = \emptyset$, then let $Z^\star$ with $u_{Z^\star} = v^{n-1}$ be the outcome. Otherwise, proceed to Step $n.3$.

Step n.3 *(Welfare improvement)* Select a *feasible* reservation profile $v^n \neq v^{n-1}$ such that for each i, $v_i^n = v_i^{n-1} + \lambda_i^n$ where $\lambda_i^n \in [0,1]$, and for each $i \notin \bigcup_{c \,\in\, A_n} \Gamma_c(v^{n-1})$, $\lambda_i^n = 0$.

The *PBSWI* selects a welfare improvement at each step, and it is a class of rules since each sequence of these selections implies a different random allocation. To define a rule in the *PBSWI* class, it is sufficient to specify the selection rule of welfare improvement at Step $n.3$. (We define such a rule in Section 5.4.) For each problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c \in \mathcal{C}}, (q_c)_{c \in \mathcal{C}})$, let $PBSWI(R)$ denote the set of all random allocations obtained by the class $PBSWI$.

Theorem 2. *For a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c \in \mathcal{C}}, (q_c)_{c \in \mathcal{C}})$, a random allocation Z is acceptable if and only if $Z \in PBSWI(R)$.*

Proof. See Appendix B □

This result provides an insight on how to describe an *acceptable* random allocation by means of a sequence of welfare improvement profiles. We use this insight later when we characterize the set of 'equitable' (see Section 5) allocations by means of a unique random allocation rule (see Section 5.4).

[12] By definition of reservation value, there exists a random allocation Z such that v^0 is the utility profile under Z.

5 Enhancing equity

Processing reserves sequentially implies uneven treatment of agents in general and two important questions follow this observation: First, is eliminating uneven treatment of agents a plausible consideration in the current context? Second, if so, how should it be formulated?

There are two possibly sensible ideas for the current setting. (1) *Equity*: Agents' utilities are equalized as much as possible through the criterion of *Lorenz dominance (egalitarianism)*. (2) *Fairness*: If an agent prefers being in the position of another agent, then there is a justification (based on priorities) for assigning a higher utility to that other agent in that position *(no justified-envy)*.

Our first observation is that an *egalitarian* rule does not exist in the current context (Theorem 3). Moreover, if an *egalitarian* random allocation exists for some problem, it may not even treat equals as equal, the most fundamental principle of fairness (Section 5.1). Thus, *egalitarianism* is not sensible in the current setting. On the other hand, the sense of *fairness* is severely restricted by the constraints of *respecting priorities*, and *no justified-envy* could be vacuous in some situations and quite weak in general (Section 5.2). Given these negative findings, we formulate and propose a new notion, *sequential egalitarianism*. This equity axiom is very much in the spirit of *egalitarianism*. Yet, it is independent from it, and interestingly, it implies the core axiom of fairness, *no justified-envy* (Section 5.3). Finally, we characterize the rules satisfying *sequential egalitarianism* in the class of *acceptable* rules (Section 5.4).

5.1 Egalitarianism

While *respecting priorities* captures some sort of fairness by emphasizing priorities, an independent attribute is equitable access to resources, the standard formulation of which is *Lorenz dominance*. For any vector $u \in \mathbb{R}^{|\mathcal{I}|}$, let $u^\star$ be the vector obtained upon rearranging the coordinates of u increasingly. Given a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$ and $Z,\ Z' \in \mathcal{Z}(R)$, Z **Lorenz dominates** Z' if

$$\text{for each } l = 1, \ldots, |\mathcal{I}|: \ \sum_{m=1}^{l}((u^\star_Z)_m - (u^\star_{Z'})_m) \geq 0.$$

The question of defining equitable access is entangled with the indispensability of the axioms in Section 3. Fortunately, it can easily be adapted to the current context in the form of 'equating utilities as much as possible'.

Definition 7. *A random allocation $Z \in \mathcal{Z}^a(R)$ is* ***egalitarian*** *if it is Lorenz dominant in the set $\mathcal{Z}^a(R)$. A random allocation $Z \in \mathcal{Z}^a(R)$ is* ***weakly egalitarian*** *if it is not Lorenz dominated by another allocation in the set $\mathcal{Z}^a(R)$. A rule φ is* ***(weakly) egalitarian*** *if for any problem R, random allocation $\varphi(R)$ is (weakly) egalitarian.*

There are two important issues regarding an *egalitarian* random allocation: First, it turns out that a *Lorenz dominant* allocation may not exist in the set of *acceptable* random allocations.[13]

Theorem 3. *No rule is egalitarian.*

Proof. See Appendix C. □

Second, even if an *egalitarian* random allocation exists for a problem, it does not necessarily '*treat equals as equal*', as the following example demonstrates. (Clearly, this observation holds also for *weakly egalitarian* allocations.)

Example 4. *(An egalitarian random allocation does not necessarily 'treat equals as equal'.)* *Let $\mathcal{I}= \{i, j, i_1, i_2, j_1, j_2, k, l\}$ and $\mathcal{C}= \{c_1, c_2\}$ such that three units are reserved for each category. Consider the following problem:*

π_{c_1}	π_{c_2}
$\{i, j\}$	$\{i, j\}$
$\{i_1, i_2\}$	$\{j_1, j_2\}$
$\{k, l\}$	$\{k, l\}$

Let us first characterize the set of egalitarian random allocations. By individual rationality, (i) agents i and j are assigned a unit each with probability one, (ii) agents i_1, i_2, j_1 and j_2 are assigned a unit each with probability at least half. Thus, there are four units remaining with the constraint (ii). Non-wastefulness and respecting priorities imply that there are three alternatives for these units:

[13] The impossibility still holds even if we restrict the domain of priority orders (see Appendix C).

1. i_1, i_2, j_1, j_2 (each with probability one)

2. i_1, i_2 (each with probability one) and j_1, j_2, k, l (j_1, j_2 each with probability at least half)

3. j_1, j_2 (each with probability one) and i_1, i_2, k, l (i_1, i_2 each with probability at least half)

The second and third alternatives provide access to a higher number of agents than the first alternative. Thus, (it is straightforward to check that) an acceptable random allocation is egalitarian if and only if it generates one of the following utility profiles:

$$u = (u_i, u_j, u_{i_1}, u_{i_2}, u_{j_1}, u_{j_2}, u_k, u_l) = (1, 1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}),$$

$$u' = (u'_i, u'_j, u'_{i_1}, u'_{i_2}, u'_{j_1}, u'_{j_2}, u'_k, u'_l) = (1, 1, \frac{1}{2}, \frac{1}{2}, 1, 1, \frac{1}{2}, \frac{1}{2}).$$

While it is in general not easy to define 'equals' in the current context, in some situations, it is. In Example 4, agents i and j qualify for both reserve categories and this implies a surplus for agents i_1 and i_2 under c_1, and for j_1 and j_2 under c_2. Thus, the claims of these two groups of agents over the surplus should be treated equally. But, any *egalitarian* random allocation, as characterized in Example 4, favours either agents i_1 and i_2 over agents j_1 and j_2, or vice versa. The reason is simple: by granting, say agents i_1 and i_2, a unit each, the remaining two units can be (probabilistically) allocated to agents j_1, j_2, k and l, instead of allocating four units equally among agents i_1, i_2, j_1 and j_2.

5.2 No justified-envy

Let $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$ be a problem and $Z \in \mathcal{Z}(R)$. For each c, let $k_c(Z)$ be such that for each $i \in UCS_{\pi_c}(k_c(Z))$, $u_Z(i) = 1$, and for some $j \in \mathcal{I}_{\pi_c}(k_c(Z) + 1)$, $u_Z(j) < 1$.

Suppose there exists an agent i with $u_Z(i) < 1$ and a reserve category c such that for each agent j with a higher priority under c,

1. $u_Z(j) \ = 1$, and

2. there exists another category c' such that $j \in U_{\pi_{c'}}(k_{c'}(Z) - 1)$.

Thus, Z assigns one unit to each agent, say j, with a higher priority than i under c, and also for some c', one unit to each agent in the next lower priority class of any such agent j. But then, although agent i is in a similar situation, that is, she is in the next lower priority class under c, she is assigned a lower utility than the agents under other categories. We argue that in this case, agent i has *justified-envy* for these agents. The next axiom eliminates this type of envy.

Definition 8. *For a problem $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$, a random allocation $Z \in \mathcal{Z}(R)$ satisfies* ***no justified-envy****, if for each $c \in \mathcal{C}$,*

$$UCS_{\pi_c}(k) \subseteq \bigcup_{c' \in C\setminus\{c\}} UCS_{\pi_{c'}}(k_{c'}(Z) - 1) \implies k_c(Z) > k.$$

A rule φ satisfies ***no justified-envy*** *if for any problem R, random allocation $\varphi(R)$ satisfies no justified-envy.*

Given that any random allocation we consider *respects priorities*, *no justified-envy* excludes only certain types of envy, while some other types of envy are not considered as justified due to the restrictions by *respecting priorities*.[14]

5.3 Sequential egalitarianism

As we argued in Section 4, each *acceptable* random allocation can be described as a simple procedure of sequential allocation of units to their *eligible* agents whenever these units become available. Our approach to equity is to provide equal access to these units for *eligible* agents.

Definition 9. *A random allocation rule in the class of the PBSWI is sequentially egalitarian if, at each step, the selected reservation profile Lorenz dominates any other feasible reservation profile that can be selected at that step.*

The units are allocated based on priorities, and as Theorem 2 demonstrates, at each step, there is a set of *eligible* agents who are the candidate receivers of the remaining units. *Sequential egalitarianism*

[14] We provide the associated examples in the Appendix of the extended version of our work (Yılmaz, 2022).

requires an equitable access of these units to these agents by equalizing their updated reservation values as much as possible (through *Lorenz dominance*) among all possible *acceptable* reservation profiles. That is, this principle requires applying the idea of *egalitarianism* sequentially.

To demonstrate the idea, let us revisit Example 4: At the initial step, only i and j are *eligible* for the units under c_1 and c_2. *Sequential egalitarianism* requires that their reservation values are increased equally, and each is assigned one unit. There are still four units available and agents i_1, i_2, j_1 and j_2 are now *eligible*. Again, their reservation values are increased equally, and we obtain an allocation such that each is assigned one unit. Thus, each agent in $\{i, j, i_1, i_2, j_1, j_2\}$ is assigned one unit. The idea of equal access to units at each step implies treating agents j_1 and j_2 equally as i_1 and i_2, which was not the case under *egalitarianism* as demonstrated in Example 4.[15]

This example also clarifies the logical relationship between the two notions. An *egalitarian* random allocation may not exist, but when it exists it does not imply *sequential egalitarianism*. On the other hand, a *sequentially egalitarian* allocation always exists (see Section 5.4). Thus, *sequential egalitarianism* does not imply *egalitarianism*.

Remark 2. *Egalitarianism and sequential egalitarianism are independent properties.*

Interestingly, while (as discussed in Section 5.1) *egalitarianism* does not imply even treating equal agents equally, *sequential egalitarianism* implies *no justified-envy* (the central fairness concept in the current context).

Proposition 1. *Sequential egalitarianism implies no justified-envy.*

Proof. See Appendix E. □

This logical relationship does not hold between *egalitarianism* and *no justified-envy*, and *egalitarianism* does not imply *no justified-envy*:[16] Let us reconsider Example 4. Let Z be an *egalitarian* random

[15] *Sequential egalitarianism* is not compatible with rules based on a precedence order of reserve categories, or a priority ordering of agents (see the Appendix of the extended version of the current work (Yılmaz, 2022)).

[16] This observation intuitively follows from the insight in Example 4 and the fact treating equals equally is a weaker notion than *no justified-envy*. Although it is straightforward to formally define equal treatment of equals in the current context, we have not provided the definition for brevity. Thus, we need to argue for why *egalitarianism* does not imply *no justified-envy* over an example.

allocation. Then, it generates the utility profile u in that example. Thus, $k_{c_1}(Z) = 2$ and $k_{c_1}(Z) = 1$. For category c_2, $UCS_{\pi_{c_2}}(1) \subseteq UCS_{\pi_{c_1}}(k_{c_1}(Z) - 1) = UCS_{\pi_{c_1}}(1)$. But, since $u_Z(j_1) < 1$, $k_{c_1}(Z) = 1$. Thus, agent j_1 has *justified-envy* (similarly, agent j_2 also has *justified-envy*). Thus, Z dos not satisfy *no justified-envy*.

5.4 The Priority-Based Rawlsian (PBR) rule

Our goal is to incorporate equity (Section 5.3) into the *acceptable* class and characterize *sequentially egalitarian* rules. The design of our solution, the **Priority-Based Rawlsian (PBR) rule**, relies on the Rawlsian principle of maximizing the minimum welfare. Basically, the utilities of the most disadvantaged agents are increased continuously as long as the constraints embedded in *eligibility* and *reservation profile* are not binding. By Theorem 2, specifying this Rawlsian improvement process as the *welfare improvement* selection rule is sufficient to define the *PBR*.[17]

Step n.3 *(Welfare improvement selection rule of the PBR)*

The agents with the minimum reservation value are *selected* among agents, who are *eligible* for at least one *available* category. Their reservation values are increased equally up to the minimum of the following two, while other agents' reservation values do not change:

- The reservation value of a *non-selected* agent, who is *eligible* for at least one *available* category.
- The level at which a subset of agents *eligible* for at least one *available* category has *exclusive rights* over the categories for which they are *eligible*.

While this selection rule is quite intuitive, the difficulty is to analytically characterize the execution of its steps. First, when agents are allowed to receive a unit (probabilistically) at some step of the *PBSWI*, in general, they can receive it from multiple categories. Thus, the implication of increasing utilities on feasibility is not clear. Second, at any step, there are multiple constraints due to (1) *eligibility* (a set of constraints on who can be assigned from which categories) and (2) the

[17] Each *acceptable* random allocation rule can be described via a *welfare improvement* selection rule in Step $n.3$ of the *PBSWI* algorithm (see also the discussion in Section 4).

reservation profile of that step (a set of constraints in the form of guaranteed probabilities to agents). At some point, some constraints become binding, and the challenge is to track these instances. Thus, we need to analytically specify the *welfare improvement selection* rule described above to complete the definition of the *PBR*.

Step n.3 *(Welfare improvement selection rule of the PBR)*

Agent $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ is **prioritized** if, for each $j \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, $v_i^{n-1} \leq v_j^{n-1}$. Let $v^{n-1,1}$ be the reservation value of *prioritized* agents. If all agents in $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ are *prioritized*, then let $v^{n-1,2} = 1$, otherwise let $v^{n-1,2}$ be the lowest reservation value among *non-prioritized* agents in $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$. Let $\mathcal{B}_n$ be the set of all subsets of $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ with at least one *prioritized* agent. Let

$$\lambda^\star = \min_{I \in \mathcal{B}_n} \frac{\sum_{c \in C(I, v^{n-1})} q_c - \sum_{i \in I} v_i^{n-1}}{|\{i \in I : i \text{ is } \textit{prioritized}\}|}.$$

For each $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, let

$$v_i^n = \begin{cases} \min\{v^{n-1,1} + \lambda^\star, v^{n-1,2}\} & \text{if } i \text{ is } \textit{prioritized} \\ v_i^{n-1} & \text{otherwise} \end{cases}$$

We are now ready to present our main theorem, which states that this specific improvement process characterizes *sequentially egalitarian* rules.

Theorem 4. *A rule φ is sequentially egalitarian if and only if φ is welfare-equivalent to the PBR.*

Proof. See Section D. □

The proof of this characterization result relies highly on exploiting parametric networks and an extension of the *Max-Flow Min-Cut Theorem* (Ford and Fulkerson, 1956) (see Appendix A).

6 Applications

6.1 Rationing health care units

6.1.1 A weighted lottery policy

The Department of Health, Pennsylvania has been recently implementing a weighted lottery mechanism for the allocation of medications to treat COVID-19 (Pennsylvania DH, 2020). As outlined in the "Pandemic Guidelines for the Interim Pennsylvania Crisis Standards of Care", this framework is designed such that "all patients who meet clinical eligibility criteria should have a chance to receive treatment". In the preliminary step, the number of available courses of the COVID-19 therapy is determined and the number of eligible patients (for which the drug is allotted) is *estimated*. By dividing the first number by the second, the chances for each eligible "general community" patient to receive the drug is determined. In the second step, patients' characteristics relevant to the weighted lottery are determined to adjust the *general community chances* found in the preliminary step. These adjustments are done according to the formula in Table 1. Finally, each patient enters into the lottery constructed by the probability with which she receives treatment. Basically, a lottery number between 1 and 100 is randomly selected for each eligible patient. If the lottery chances for the patient is x out of 100 and the patient's randomly drawn lottery number is less than or equal to x, they should be offered the scarce drug. If the lottery number is greater than x, then they should not be offered the scarce drug.

There are two issues with this mechanism. First, the implementation of the lottery (i.e. single patient-single lottery) does not imply a probability distribution. Second, since these probabilities are fixed and do not depend on the number of patients in each group, *target ratios between the weights of each pair of patient groups* (Table 1) are not feasible in general.

The goal of creating meaningful access to patients by randomization is consistent with the motivation of *sequential egalitarianism* and the *PBR*. By designing these categories (as specified in Table 1) and the **weak** priority orders appropriately, we can apply the *PBR* rule (1) to create *sequentially egalitarian* access for patients (Theorem 4), and (2) to remove analytical inconsistencies explained

Group	Chances to receive treatment
Disadvantaged community member (c_1)	1.25 x (general community chances)
Essential worker (c_2)	1.25 x (general community chances)
Death likely within 1 year (c_3)	0.5 x (general community chances)
Disadvantaged community member + Essential worker	1.5 x (general community chances)
Disadvantaged community member + death likely within 1 year	0.75 x (general community chances)
Essential worker + death likely within 1 year	0.75 x (general community chances)

Table 1: Probabilities in the weighted lottery

above. For an appropriate design of the reserve structure, this rule implies that **each** patient is assigned a unit with a positive probability (as stated in the Pandemic Guideline above).

Alternatively, a different rule in the *PBSWI* class can be specified for this setting to achieve *targeted ratios between the weights* in an analytically consistent way. First, reserve categories are modeled with dichotomous indifference classes: for each category, the first indifference class is the set of all patients belonging to that category and the second one is the rest of the patients. Since our model allows for weak priority orders, this construction is clearly within our framework.

Second, *targeted ratios between the weights* are specified: The weights defined in Table 1 suggest that (1) each disadvantaged community member who is an essential worker should have a higher utility than each utility value obtained by the *priority rule* applied to these single-category problems, (2) each disadvantaged community member or essential worker with death likely within one year should have a lower (higher) utility than the utility value obtained by the *priority rule* applied to disadvantaged community member or essential worker category (death likely within one year category). A (weighted) average of the utilities applies to patients belonging to these multiple categories. Thus, there is a target for relative utilities of patients belonging to two groups.[18] Let u^k and $u^{k,l}$ represent the utility of a patient belonging to group c_k only, and to groups c_k and c_l, respectively. Given that $u^1 = u^2 > u^3$, the target utility ratios are defined such that $u^{1,2} = \alpha u^1$ and $u^{1,3} = u^{2,3} = w(u^1, u^3)$, where $\alpha > 1$ and $w(u^1, u^3)$ is a convex combination of u^1 and u^3.

[18] We assume that there is no patient belonging to all three groups (see Table 1).

Step 0. Let $C = \{c_1, c_2, c_3\}$ be the patient groups (i.e. categories) in Table 1. For each $c \in C$, and for each patient i belonging to group c, the weak order π_c is constructed such that $i \in \mathcal{I}_{\pi_c}(1)$. For each patient group, a certain number of units is reserved such that $\frac{q_{c_1}}{|\mathcal{I}_{\pi_{c_1}}(1)|} = \frac{q_{c_2}}{|\mathcal{I}_{\pi_{c_2}}(1)|} > \frac{q_{c_3}}{|\mathcal{I}_{\pi_{c_3}}(1)|}$.[19]

Step 1 For each patient i, let the initial reservation profile $v_i^0 = \min_{c \in \{c' : i \in \mathcal{I}_{\pi_{c'}}(1)\}} \rho_i(R^c)$.[20]

Step 2 The units are allocated by the *PBSWI* algorithm with the following welfare improvement selection rule: If there are *eligible* patients for at least one *available* category, who belong to two groups and have a reservation value lower than the targeted ratio, then these patients are *selected*; otherwise, all patients who are *eligible* for at least one *available* category are *selected*. The reservation value of *selected* patients are increased equally up to the minimum of the following two:

- The level at which a subset of patients *eligible* for at least one *available* reserve category has *exclusive rights* over the categories for which they are *eligible*.
- The level at which the targeted ratio is achieved for a patient who had a reservation value lower than the targeted ratio.

The above rule selects a random allocation with target utility ratios within the set of *acceptable* random allocations, whenever it is feasible. We do not claim that our rule is the only one: there are other ways to achieve target utility ratios for this very special case. Our point here is that, by Theorem 2, our approach is robust in delivering the desired properties for different settings.

6.1.2 Soft reserves

Reserve systems have been adopted in several settings with rationing of medical resources.[21] These systems are generally such that for each category $c \in \mathcal{C}$, a **beneficiary group** is designated. When

[19]Since the units and the number of patients are integers, we can only impose $\frac{q_{c_1}}{|\mathcal{I}_{\pi_{c_1}}(1)|} \approx \frac{q_{c_2}}{|\mathcal{I}_{\pi_{c_2}}(1)|}$. But, for the ease of notation, we assume that it is possible to reserve units such that this approximation holds with equality. Also, these numbers of units reserved for each group can be determined with respect to some target ratio between u^1 and u^3.

[20]This specification of initial reservation profile is due to the fact that *individual rationality* in the current setting implies that each patient should receive a share at least as the minimum of their shares given by the priority rule applied to all category groups they belong to. Note that this is different than the *individual rationality* constraint given by Equation 1 in Section 3.2. Also, see Remark 1 in Section 3.2 for the generalization of Definition 3.

[21]After the circulation of Pathak, Sönmez, Ünver, and Yenmez (2021) and the authors' interaction with public health officials, the National Academies of Sciences, Engineering, and Medicine (NASEM) started to formulate rec-

the *beneficiary group* is exclusive and a strict subset of patients, the associated category is referred to as a **preferential treatment** category. There is also an **unreserved** category such that its *beneficiary group* is the set of all patients. A particular approach in this setting is **hard reserves**: A patient is qualified to receive a medical unit from a category if and only if they are in the *beneficiary group* of that category. *Hard reserves* are in general incompatible with efficiency (see Example 2 in Pathak, Sönmez, Ünver, and Yenmez (2021)). A more flexible interpretation of reserve categories is a **soft reserve system**, where all individuals are qualified for all categories, that is, for each $c \in \mathcal{C}$, all individuals are ranked under priority order π_c. In particular, a *soft reserve* system is obtained by applying the following to each *preferential treatment* category c: (1) If there is an *unreserved* category as well, π_c is obtained by ranking each non-beneficiary patient strictly below the *beneficiary group* and by preserving the ranking of non-beneficiary patients under the *unreserved* category. (2) If there does not exist an *unreserved* category, then all the non-beneficiary patients are ranked as an indifference class just below the last beneficiary patient in the associated category. While our model applies clearly to both cases, we emphasize that the second case necessarily implies a weak order of priorities under reserve categories, the generality of which is provided by our work.

A plausible requirement for the *soft reserves* setting is to maximally allocate the reserves to target beneficiaries: **maximal in beneficiary** (Pathak, Sönmez, Ünver, and Yenmez, 2021). It is straightforward to see that the *PBR* rule is not *maximal in beneficiary*: At each step, since the rule treats all *eligible* patients equally and *soft reserves* setting is such that all patients are ranked under a category and the set of *eligible* patients may contain both beneficiary and non-beneficiary patients, it may not prioritize target beneficiaries over the others. Actually, instead of maximizing the number of beneficiaries, the *PBR* rule maximizes (at each step of the *PBR*) the number of *eligible* patients receiving treatment (see Remark 3 in Appendix D). We analyze *equitable* allocation rules within the set of *maximal in beneficiary* allocations in a separate work (Doğan and Yılmaz, 2022).

ommendations on the fair allocation of COVID-19 vaccines. Later, Tennessee, Massachusetts and New Hampshire announced their plans to adopt a reserve system (Tennessee DH, 2020; Massachusetts DPH, 2020; New Hampshire DHHS, 2021).

6.2 Affirmative action in school choice

Affirmative action schemes are widespread in school admissions around the world. Typically, a fraction of slots is reserved for disadvantaged students and the rest is assigned based on merit. A compelling example is Chicago's place-based affirmative action at the K-12 level: Schools fill 40% of their slots with the applicants having the highest composite scores[22] and the remaining 60% of slots by dividing the slots equally across four tiers based on the socioeconomic characteristics of applicants' neighborhoods. For each socioeconomic tier, students in that corresponding group are prioritized over all other students such that students both inside and outside the group are ordered by composite score. For the merit tier, all students are ordered by composite score. This setting fits perfectly into our model, and our results apply directly.[23]

One of the themes in this affirmative action scheme is to eliminate explicit targeting of applicants by differentiating across tiers, that is *tier-blindness* (Dur, Pathak, and Sönmez, 2020). Let T denote the socioeconomic tiers and m the merit tier. Thus, $\mathcal{C} = T \cup \{m\}$. Also, any two socioeconomic tiers $t, t' \in T$, $q_t = q_{t'}$. Also, for each $c \in \mathcal{C}$, we fix π_c. A **merit-preserving bijection** $\theta : \mathcal{C} \to \mathcal{C}$ is a one-to-one and onto function where $\theta(m) = m$.

Definition 10. *A random allocation rule φ is* ***tier-blind*** *if for each set of students $\mathcal{I}$, for each set of tiers $\mathcal{C}$ and for each merit-preserving bijection θ, the random allocations $Z = \varphi(\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$ and $Z' = \varphi(\mathcal{I}, \mathcal{C}, (\pi_{\theta(c)})_{c\in\mathcal{C}}, (q_{\theta(c)})_{c\in\mathcal{C}})$ are such that*

$$u_Z = u_{Z'}. \tag{3}$$

Tier-blindness implies that relabeling tiers does not change the probability with which a student is assigned a seat. Since the *PBR* rule is based on the set of *eligible* patients at each step, and that structure is independent from the labels of the tiers, the following observation follows immediately.

Observation 1. *The PBR rule is tier-blind.*

[22] The composite score is the equally-weighted combination of the admission test score, the applicant's 7th grade GPA, and the standardized test score.

[23] Under the assumption that for each socioeconomic tier, the number of students in that tier is more than the sum of all slots, this setting becomes a special case of our model since the students in socioeconomic tiers are mutually exclusive. It is easy to show that the independence of *egalitarianism* and *sequential egalitarianism* prevails and the *PBR* is not *egalitarian*, even for this special case.

Appendix A Maximum flow problem: Preliminaries

A **directed graph,** or **digraph** is a pair $G = (V, A)$, consisting of a set of **vertices** V and a set of *ordered* pairs of vertices, A, called **arcs.** For a set of vertices $V' \subseteq V$, the set $\delta^{\text{out}}(V')$ is the set of all outgoing arcs; that is, the arcs (x, y) such that $x \in V'$ and $y \notin V'$. Similarly, the set $\delta^{\text{in}}(V')$ is the set of all incoming arcs; that is, the arcs (x, y) such that $x \notin V'$ and $y \in V'$. Let $l, k : A \to \Re_+$ be two functions, which associate each arc $a = (x, y)$ of G with non-negative real numbers $l(x, y)$ and $k(x, y)$ called the **lower-bound** and **capacity** of the arc (x, y), respectively, such that for each arc (x, y), $l(x, y) \leq k(x, y)$. For a set of arcs $A' \subseteq A$, $l(A') = \sum_{a\in A'} l(a)$ and $k(A') = \sum_{a\in A'} k(a)$.

A **network** (V, A, l, k) is a digraph with lower-bound and capacity functions. A **supply-demand network** is a *network* (V, A, l, k) with $V = V_1 \cup V_2 \cup \{s, t\}$, where V_1 and V_2 are the set of demand and supply vertices, respectively, s the **source vertex**, and t the **sink vertex** such that there is an arc from the source vertex into each demand vertex, an arc from each supply vertex into the sink vertex, and all the other arcs are from demand vertices into supply vertices. (An arc from a demand vertex $x \in V_1$ into a supply vertex $y \in V_2$ is interpreted that x demands units from y.)

A **flow** in a supply-demand network (V, A, l, k) is a function $f : A \to \Re_+$, satisfying the following properties:

(i) $\sum_x f(x, y) = \sum_z f(y, z)$ for each y in $V_1 \cup V_2$ and,

(ii) $l(x, y) \leq f(x, y) \leq k(x, y)$ for each (x, y) in A.

The **value of** f, denoted by $v(f)$ is defined as $\sum_x f(s, x)$. Given a supply-demand network (V, A, l, k), the **maximum flow problem** is to find the maximum value of flow. The solution for this problem is characterized by the following theorem (Schrijver, 2003):

Theorem 5. *Let (V, A, l, k) be a supply-demand network such that there exists a flow f. Then, the maximum value of a flow is equal to the minimum value of*

$$k(\delta^{out}(V')) - l(\delta^{in}(V'))$$

taken over $V' \subseteq V$ with $s \in V'$ and $t \notin V'$.[24]

[24]This theorem is an extension of the well-known Max-flow Min-cut Theorem (Ford and Fulkerson, 1956).

Appendix B Proof of Theorem 2

We prove that (1) each random allocation given by the *PBSWI* class is *acceptable* (Lemma 1) and (2) each *acceptable* random allocation Z can be obtained by a sequence of selections of reservation values in the *PBSWI* (Lemma 3).

Lemma 1. *Let $R = (\mathcal{I}, \mathcal{C}, (\pi_c)_{c\in\mathcal{C}}, (q_c)_{c\in\mathcal{C}})$ be a problem. If a random allocation Z is an outcome of the PBSWI(R), then it is acceptable.*

Proof. Let N be the last step of the *PBSWI*. By definition of the *PBSWI*, the algorithm ends at the end of Step N.2, and the reservation values are not updated at Step N. Thus, the outcome of the algorithm is v^{N-1}. Let $Z^\star$ be a random allocation such that $u_{Z^\star} = v^{N-1}$.

$Z^\star$ is non-wasteful. Suppose $Z^\star$ is not *non-wasteful.* Then, by Definition 2, there exists a reserve category c and an agent i, such that

$$\sum_{j\in\mathcal{I}} z^\star_{jc} < q_c \text{ and } v_i^{N-1} = u_{Z^\star}(i) = \sum_{c'\in\mathcal{C}} z^\star_{ic'} < 1. \tag{4}$$

Since $u_{Z^\star}(i) < 1$, by definition of Step N.2, Category c is *unavailable* at this step. Thus, there exists a set of agents I with *exclusive rights* over the set $C(I, v^{N-1})$ with $c \in C(I, v^{N-1})$ (note that agent i is not necessarily in the set $\Gamma_c(v^{N-1})$) such that

$$\sum_{j\in I} v_j^{N-1} = \sum_{c'\in C(I,v^{N-1})} q'_c \tag{5}$$

By definition of the set $C(I, v^{N-1})$, each $j \in I$ with $v_j^{N-1} > 0$ is not *eligible* for categories out of $C(I, v^{N-1})$. Thus, for each $j \in I$, $v_j^{N-1} = \sum_{c'\in C(I,v^{N-1})} z^\star_{jc'}$. By rewriting Condition (5), we obtain

$$\sum_{c'\in C(I,v^{N-1})} q_{c'} = \sum_{j\in I} \sum_{c'\in C(I,v^{N-1})} z^\star_{jc'} = \sum_{c'\in C(I,v^{N-1})} \sum_{j\in I} z^\star_{jc'} \tag{6}$$

Since, by definition of the *PBSWI*, $Z^\star$ is a random allocation, by Property *(ii)* of a random allocation

(Definition 1), for each $c' \in C(I, v^{N-1})$,

$$\sum_{j \in I} z^\star_{jc'} \leq q_{c'}. \tag{7}$$

Thus, Conditions (6) and (7) together imply that the weak inequality in Condition (7) holds with equality. By definition of a random allocation, this also implies that for each $c' \in C(I, v^{N-1})$, $\sum_{j \in \mathcal{I}} z^\star_{jc'} = q_{c'}$. Since $c \in C(I, v^{N-1})$, this contradicts with (4).

$Z^\star$ is individually rational. By construction of the initial reservation profile v^0, Z^0 is *individually rational*. Since, at each step, utility increases non-negatively for each agent, for each $n \geq 1$, $v^n \geq v^{n-1}$. Finally, for each $0 \leq n \leq N-1$, by definition of Step $n.3$, there exists an underlying random allocation Z^n such that that $u_{Z^n} = v^n$. Thus, $Z^\star = Z^{N-1}$ is *individually rational*.

$Z^\star$ respects priorities. Let $i \in \mathcal{I}$, and $c \in C$ such that $i \;\; \pi^P_c \; j$ and $u_{Z^\star}(i) \; < 1$. At Step N, since there exists at least one agent with a utility less than one, by definition of Step $N.2$, Category c must be *unavailable* at the end of the algorithm. Since $u_{Z^\star}(i) < 1$, and $i \;\; \pi^P_c \;\; j$, by definition of *eligibility*, agent j is not *eligible* for c. Moreover, by definition of *eligibility*, for each c and each $n \geq 1$, $\Gamma_c(v^n) \supseteq \Gamma_c(v^{n-1})$. This implies that j has not been *eligible* at any step before N. Thus, as $Z^\star$ underlies v^{N-1} and v^{N-1}_j is the sum of agent j's shares at categories for which she is *eligible*, $z^\star_{jc} = 0$. □

Lemma 2. *Let v^{n-1} be a reservation profile with an underlying random allocation Z^{n-1}. If I_1 and I_2 have exclusive rights over $C(I_1, v^{n-1})$ and $C(I_2, v^{n-1})$, respectively, then $I_1 \cup I_2$ has exclusive rights over $C(I_1, v^{n-1}) \cup C(I_2, v^{n-1})$.*

Proof. Let I_1 and I_2 have *exclusive rights* over $C(I_1, v^{n-1})$ and $C(I_2, v^{n-1})$, respectively. There are two cases.

Case 1: $C(I_1, v^{n-1}) \cap C(I_2, v^{n-1}) = \emptyset$.

By definition of *eligibility*, we have $I_1 \cap I_2 = \emptyset$. By definition of *exclusive rights*,

$$\sum_{i \in I_1} v_i = \sum_{c \in C(I_1, v^{n-1})} q_c \text{ and } \sum_{i \in I_2} v_i = \sum_{c \in C(I_2, v^{n-1})} q_c. \tag{8}$$

Since $I_1 \cap I_2 = \emptyset$, these two equalities together imply, $\sum\limits_{i \in I_1 \cup I_2} v_i = \sum\limits_{c \in C(I_1 \cup I_2, v^{n-1})} q_c$. Thus, the set $I_1 \cup I_2$ has *exclusive rights* over the set of reserve categories $C(I_1 \cup I_2, v^{n-1})$.

Case 2: $C(I_1, v^{n-1}) \cap C(I_2, v^{n-1}) \neq \emptyset$.

Suppose $I_1 \cap I_2 = \emptyset$. (Note that equalities in (8) hold in this case as well.) Clearly, $\sum\limits_{i \in I_1 \cup I_2} v_i = \sum\limits_{i \in I_1} v_i + \sum\limits_{i \in I_2} v_i$. Moreover, by definition of *eligibility*, $C(I_1 \cup I_2, v^{n-1}) = C(I_1, v^{n-1}) \cup C(I_2, v^{n-1})$. Since $C(I_1, v^{n-1}) \cap C(I_2, v^{n-1}) \neq \emptyset$, this implies that

$$\sum_{c \in C(I_1 \cup I_2, v^{n-1})} q_c < \sum_{c \in C(I_1, v^{n-1})} q_c + \sum_{c \in C(I_2, v^{n-1})} q_c.$$

This, together with equalities in (8), imply

$$\sum_{c \in C(I_1 \cup I_2, v^{n-1})} q_c < \sum_{i \in I_1 \cup I_2} v_i.$$

Then, Condition 2 in Theorem 1 does not hold for the set $I_1 \cup I_2$. Thus, by Theorem 1, there does not exist a random allocation underlying v^{n-1}, which is a contradiction. Thus, we have $I_1 \cap I_2 \neq \emptyset$. Now consider the sets $I_1 \cup I_2$ and $C(I_1 \cup I_2, v^{n-1})$. Since there exists a random allocation Z^{n-1} underlying v^{n-1}, by Theorem 1,

$$\sum_{c \in C(I_1 \cup I_2, v^{n-1})} q_c \geq \sum_{i \in I_1 \cup I_2} v_i. \tag{9}$$

Suppose Inequality (9) is strict. First, note that

$$C(I_1 \cap I_2, v^{n-1}) \subseteq C(I_1, v^{n-1}) \cap C(I_2, v^{n-1}). \tag{10}$$

The inclusion follows from the definition of *eligibility*, and since different agents can be *eligible* for the same reserve category, these two sets do not necessarily coincide. We can rewrite Inequality (9) as follows:

$$\sum_{c \in C(I_1, v^{n-1}) \setminus C(I_2, v^{n-1})} q_c + \sum_{c \in C(I_2, v^{n-1}) \setminus C(I_1, v^{n-1})} q_c + \sum_{c \in C(I_1, v^{n-1}) \cap C(I_2, v^{n-1})} q_c > \sum_{i \in I_1 \setminus I_2} v_i + \sum_{i \in I_2 \setminus I_1} v_i + \sum_{i \in I_1 \cap I_2} v_i.$$

Together with equalities in (8), this implies that

$$\sum_{c \in C(I_1, v^{n-1}) \cap C(I_2, v^{n-1})} q_c < \sum_{i \in I_1 \cap I_2} v_i. \tag{11}$$

Inequality (11), together with Inclusion (10), implies

$$\sum_{c \in C(I_1 \cap I_2, v^{n-1})} q_c < \sum_{i \in I_1 \cap I_2} v_i. \tag{12}$$

This violates Condition 2 in Theorem 1. Thus, by Theorem 1, there does not exist a random allocation underlying v^{n-1}, which is a contradiction. Thus, Inequality (9) cannot be strict. Thus, by definition of *exclusive rights*, the set of agents $I_1 \cup I_2$ has *exclusive rights* over the set of reserve categories $C(I_1 \cup I_2, v^{n-1})$. □

Lemma 3. *Each acceptable random allocation is obtained as an outcome by a member of the PBSWI class.*

Proof. Let Z be an *acceptable* random allocation and $v = u_Z$. Since Z is *acceptable* (and thus *individually rational*), $v \geq v^0$. First suppose $v = v^0$. This is possible only if the units under each category is allocated to that category's highest ranked agents with respect to the priority rule ρ (see Section 3.2) and each agent receives shares from at most one category. Since Z is *non-wasteful*, all units are allocated and Z is the unique *acceptable* random allocation. Thus, this case is trivial.

We prove by induction that there is a sequence of reservation profiles v^n for $n = 1$ to N, each obtained by a welfare improvement from v^{n-1} as defined by Step n.3 of the *PBSWI*, and $v^{N-1} = v$. Let $n \geq 1$. Our inductive hypothesis is that there exists a random allocation Z^{n-1} underlying v^{n-1}, the reservation profile v^{n-1} is obtained through a sequence of welfare improvements and $v \geq v^{n-1}$.

(We have already shown that the initial step holds since $v \geq v^0$.) Suppose that $v \neq v^{n-1}$. We prove that there exists a welfare improvement for some *eligible* agents to obtain a reservation profle v^n from v^{n-1} such that $v \geq v^n$. This completes the proof.

Suppose $A_n = \emptyset$. Since each reserve category c is not *available*, there is a set of agents having *exclusive rights* over a set of categories including c. Thus, there is a collection of sets of agents, $I_1, \dots, I_m$ having *exclusive rights* over $C(I_1, v^{n-1}), \dots, C(I_m, v^{n-1})$, respectively, such that the union of the sets $C(I_1, v^{n-1}), \dots, C(I_m, v^{n-1})$ is $\mathcal{C}$. By Lemma 2, $\bigcup_{k=1}^{m} I_k$ has *exclusive rights* over $\mathcal{C}$. But then, all units are assigned to agents with *exclusive rights* under v^{n-1}. Thus, $\sum_{i \in \mathcal{I}} v_i^{n-1} = \sum_{c \in \mathcal{C}} q_c$. Since $v \geq v^{n-1}$ and $v_i > v_i^{n-1}$ for some agent i, this implies that there exists a reserve category c with $\sum_{i \in \mathcal{I}} z_{ic} > q_c$. But, this violates Property *(ii)* of Definition 1 and contradicts Z being a random allocation.

Suppose $A_n \neq \emptyset$. Let I be the subset of agents such that $i \in I$ if and only if $v_i = v_i^{n-1}$. We claim that there exists a reserve category $c \in A_n$ such that $\Gamma_c(v^{n-1}) \setminus I \neq \emptyset$. Suppose, on the contrary, that for each $c' \in A_n$, $\Gamma_{c'}(v^{n-1}) \subseteq I$. Let c be such a reserve category. By *non-triviality* assumption (see Section 2), it is not possible that for each $j \in \mathcal{I}$, $v_j^{n-1} = 1$. By definition of *eligibility*, this implies that there exists an agent $i \in \Gamma_c(v^{n-1})$ with $v_i^{n-1} = v_i < 1$. Also, each agent in the indifference class including agent i is in the set $\Gamma_c(v^{n-1})$. Since $\Gamma_c(v^{n-1}) \subseteq I$, that is, for each $j \in \Gamma_c(v^{n-1})$, $v_j^{n-1} = v_j$, this implies that there exists an indifference class such that both v^{n-1} and v coincide for the agents in this and higher priority classes. Then, since Z is *acceptable*, any agent with a positive utility in a lower priority class cannot be assigned a unit from c with a positive probability, which would violate priorities. Moreover, v^{n-1} is obtained through a sequence of steps of the *PBSWI* algorithm. Thus, for each reserve category c, there is an integer $k(c)$ such that each agent in the first $k(c)$ priority classes has a reservation value one and there exists an agent in the next priority class with a reservation value less than one under v^{n-1}. Also, $v \neq v^{n-1}$ and $v \geq v^{n-1}$. Thus, since Z is *acceptable*, for some $c \notin A_n$, it is possible to increase the utility of an agent in $\Gamma_c(v^{n-1})$. But, by definition of *exclusive rights*, for each $c \notin A_n$, and $i \in \Gamma_c(v^{n-1})$, and for each $\lambda > 0$, there does not exist a random allocation generating the utility profile $(v_{-i}^{n-1}, v_i^{n-1} + \lambda)$, which is a contradiction.

Thus, there exists a reserve category $c \in A_n$ such that $\Gamma_c(v^{n-1}) \setminus I \neq \emptyset$. By definition of *eligibility*, there exists an agent $i \in \Gamma_c(v^{n-1}) \setminus I$ such that $v_i^{n-1} < 1$ and $\lambda > 0$ with $v_i^n = v_i^{n-1} + \lambda$ underlying a random allocation. Thus, there is a welfare improvement to obtain v^n from v^{n-1} such that $v \geq v^n \geq v^{n-1}$. This completes the inductive step. $\square$

Appendix C Proof of Theorem 3

Let $\mathcal{C}= \{c_1, c_2, c_3, c_4, c_5\}$ each category with capacity one and $\mathcal{I}= \{i, j, k, i_1, i_2, j_1, j_2, j_3\}$. The (strict) priority orders for categories are given below:

π_{c_1}	π_{c_2}	π_{c_3}	π_{c_4}	π_{c_5}
i	i	i	i	i
j	j	j	j	j
k	k	i_1	i_1	j_1
j_1	j_2	i_2	i_2	i_2
i_1	j_3	j_1	j_2	j_3
j_2	j_1	k	k	k
j_3	i_1	j_3	j_1	j_2
i_2	i_2	j_2	j_3	i_1

Let R be the problem above and $Z \in \mathcal{Z}^a(R)$. Also, let

$$u_Z = (u_Z(i), u_Z(j), u_Z(k), u_Z(i_1), u_Z(i_2), u_Z(j_1), u_Z(j_2), u_Z(j_3))$$

By *individual rationality*, $u_Z(i) = 1$, and by *non-wastefulness* and *respecting priorities* together, $u_Z(j) = 1$. Thus, there are three units remaining for agents k, i_1, i_2, j_1, j_2 and j_3. Suppose

$$u_Z(k), u_Z(i_1), u_Z(j_1) < 1. \tag{13}$$

Then, by *respecting priorities*, only agents i, j, k are assigned positive probabilities for the units under c_1, c_2, only agents i, j, i_1 are assigned positive probabilities for the units under c_3, c_4, and only agents i, j, j_1 are assigned positive probabilities for the unit under c_5. But, then agents receive in total less than five units and this contradicts with *non-wastefulness*. Thus, at least one of the agents in $\{k, i_1, j_1\}$ receives one unit under Z. By considering all possible cases, we obtain the set $\mathcal{Z}^a(R)$.

Case 1: $u_Z(k) = u_Z(i_1) = u_Z(j_1) = 1$

There is only one utility profile satisfying this condition: agents i, j, k, i_1 and j_1 receive one unit and the other agents are not assigned a unit with positive probability. It is straightforward to check that there exists a random allocation, say Z^1, generating this utility profile. Thus, $u_{Z^1} = (1,1,1,1,0,1,0,0)$.

Case 2: $u_Z(k) = 1$; $u_Z(i_1), u_Z(j_1) < 1$

Since agents i, j and k receive one unit each, there are two units to be assigned to the rest of the agents. Since $u_Z(i_1), u_Z(j_1) < 1$, and Z *respects priorities*, either (1) these two units are to be assigned to agents i_1, j_1 and j_2, or (2) j_2 is assigned one unit and the remaining one unit is assigned to agents i_1 and j_1, or (3) j_2 and j_3 are assigned one unit each. While there is no *acceptable* random allocation generating the utility profile in (3), there are random allocations generating the utility profiles in (1) and (2). Among all possible random allocations generating the utility profiles in (1), random allocation, say Z^2, such that $u_{Z^2} = (1,1,1,\frac{2}{3},0,\frac{2}{3},\frac{2}{3},0)$ is *Lorenz dominant.* Among all possible random allocations generating the utility profiles in (2), random allocation, say Z^3, such that $u_{Z^3} = (1,1,1,\frac{1}{2},0,\frac{1}{2},1,0)$ is *Lorenz dominant.*

Case 3: $u_Z(k) = u_Z(i_1) = 1$; $u_Z(j_1) < 1$

There is one unit remaining for agents j_1, j_2 and i_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^4, such that $u_{Z^4} = (1,1,1,1,\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$ is *Lorenz dominant.*

Case 4: $u_Z(k) = u_Z(j_1) = 1$; $u_Z(i_1) < 1$

There is one unit remaining for agents i_1, i_2 and j_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^5, such that $u_{Z^5} = (1,1,1,\frac{1}{3},\frac{1}{3},1,\frac{1}{3},0)$ is *Lorenz dominant.*

Case 5: $u_Z(k) < 1$; $u_Z(i_1) = u_Z(j_1) = 1$

There is one unit remaining for agents k and i_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^6, such that $u_{Z^6} = (1, 1, \frac{1}{2}, 1, \frac{1}{2}, 1, 0, 0)$ is *Lorenz dominant*.

Case 6: $u_Z(k), u_Z(i_1) < 1$; $u_Z(j_1) = 1$

There are two units remaining for agents k, i_1 and i_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^7, such that $u_{Z^7} = (1, 1, \frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 1, 0, 0)$ is *Lorenz dominant*.

Case 7: $u_Z(k), u_Z(j_1) < 1$; $u_Z(i_1) = 1$

There are two units remaining for agents k, j_1 and i_2. Among all possible random allocations generating these utility profiles, random allocation, say Z^8, such that $u_{Z^8} = (1, 1, \frac{2}{3}, 1, \frac{2}{3}, \frac{2}{3}, 0, 0)$ is *Lorenz dominant*.

Since *Lorenz domination* is a transitive binary relation, it is enough to consider the random allocations Z^1 to Z^8 and find the random allocation *Lorenz dominating* others. Note that (i) Z^2, Z^7 and Z^8 are *Lorenz indifferent*, (ii) Z^3 and Z^6 are *Lorenz indifferent*, and (iii) Z^4 and Z^5 are *Lorenz indifferent*. Thus, it is enough to compare Z^1, Z^2, Z^3 and Z^4. But, while Z^2 *Lorenz dominates* Z^1 and Z^3, it does not *Lorenz dominate* Z^4. Also, Z^4 does not *Lorenz dominate* Z^2. Thus, there does not exist a *Lorenz dominant* random allocation in the set $\mathcal{Z}^a(R)$. Thus, there does not exist an *egalitarian* random allocation for this problem, and no rule is *egalitarian*.

Appendix D Proof of Theorem 4

Let N be the last step of the *PBR* algorithm and $Z^\star$ be one of its outcomes. Thus, $u_{Z^\star} = v^{N-1}$. We first show that the *PBR* is a rule in the *PBSWI* class (Lemma 4). This implies that $Z^\star$ is *acceptable*. Then, we prove that any *sequentially egalitarian* random allocation generates $u_{Z^\star}$ (Lemma 5), which completes the proof of the theorem.

First, we show that for each reservation profile v^n obtained at the end of Step $n.3$, there exists a random allocation Z^n such that $v^n = u_{Z^n}$ (Lemma 4). Thus, the selection of the reservation values at Step $n.3$ of the *PBR* complies with Step $n.3$ of the *PBSWI*.

Lemma 4. *For each reservation profile v^n obtained at the end of Step $n.3$, there exists a random allocation Z^n such that $v^n = u_{Z^n}$.*

Proof. The algorithm starts with the initial reservation profile v^0. Since this profile corresponds to the outcomes of the priority rule applied to each category separately, it is straightforward to obtain the underlying Z^0. In particular, each i is entitled the probability share $\rho_i(R^c)$ of one unit at category c. If agent i is entitled at multiple categories, then we choose the category with the highest probability share and assign a unit at that category with this highest probability. (If there are multiple such categories, we choose one of them randomly.) Moreover, the probabilities assigned to agents are not greater than one under Z^0 (because the priority rule assigns probabilities less than or equal to one, and in case there are multiple such probabilities for an agent, then the highest such probability is chosen for her). For each category c, the priority rule $\rho(R^c)$ allocates to agents no more than q_c units. Thus, Properties *(i)* and *(ii)* of a random allocation are satisfied (Definition 1), and Z^0 is a random allocation.

By induction, we show that given an underlying Z^{n-1} for v^{n-1}, there exists a random allocation Z^n for the utility profile v^n obtained at the end of Step n. For each set of agents I with *exclusive rights*, since the reservation value of each such agent is the same as in the previous step, by inductive hypothesis, there exists an assignment of probabilities of units at reserve categories in the set $C(I, v^{n-1})$. Note that, by Definition 6, all units under these reserve categories are assigned to these agents. Also, at Step n, these agents are not *eligible* for any category out of the set $C(I, v^{n-1})$. Thus, we can consider the set of *available* categories separately from the set of *unavailable* categories.

Let us consider the set A_n and $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$. By inductive hypothesis, there exists a random allocation Z^{n-1} inducing the utility profile v^{n-1}, in particular the reservation value v_i^{n-1} for each $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$.[25] Thus, by Theorem 1, for each set of agents, the condition in Theorem 1 is satisfied at the end of Step $n-1$. Since for each agent i with *exclusive rights*, $v_i^n = v_i^{n-1}$, for any subset of agents with *exclusive rights*, the condition in Theorem 1 is also satisfied at the end of Step n. Thus, we need to check this condition only for the set of agents with no *exclusive rights*, that is for each subset of $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$. By Step $n.3$, only *prioritized* agents' reservation values are updated. Thus, to complete the proof, it is enough to check the condition only for the subsets including *prioritized* agents. Suppose there exists such a set of agents I violating the condition in Theorem 1 at the end of Step n. Thus,

$$\sum_{i \in I} v_i^n \; > \sum_{c \,\in\, C(I, v^{n-1})} q_c. \tag{14}$$

By Step $n.3$, for a *non-prioritized* agent i, $v_i^n = v_i^{n-1}$ and for a *prioritized* agent j,

$$v_j^n \leq v^{n-1,1} + \lambda^\star = v_j^{n-1} + \lambda^\star.$$

Let p be the number of *prioritized* agents in the set I. Thus, Inequality (14) can be rewritten as

$$\sum_{i \in I} v_i^{n-1} + p\lambda^\star \; \geq \sum_{i \in I} v_i^n \; > \sum_{c \,\in\, C(I, v^{n-1})} q_c.$$

Thus,

$$\lambda^\star > \frac{\sum\limits_{c \,\in\, C(I, v^{n-1})} q_c - \sum\limits_{i \in I} v_i^{n-1}}{p}.$$

Since $p = |\{i \in I : i \text{ is } \textit{prioritized}\}|$ and $I \in \mathcal{B}_k$, this inequality contradicts with the definition of $\lambda^\star$. Thus, no subset of $\bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ with at least one *prioritized* agent violates the condition in Theorem 1. Thus, by Theorem 1, there exists a random allocation Z^n for the reservation profile v^n

[25]Note that there could be an agent $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, who is *eligible* also for an *unavailable* reserve category. By Step $n.1$, she does not have any *exclusive rights* and, the inductive hypothesis that there is a random allocation Z^{n-1} for the reservation profile v^{n-1} implies that she is assigned probabilities equivalent to v_i^{n-1} from reserve categories in $C(i, v^{n-1}) \cap A_n$.

obtained at Step $n.3$. $\square$

While the *PBR* increases only the welfare of *prioritized* patients, the next observation demonstrates that eventually, the total reservation value of all *eligible* agents is maximized.

Remark 3. *The PBR rule maximizes the total reservation value of eligible agents at each step.*

The *PBR* increases the reservation value of only the *prioritized* agents up to a level such that either (i) their reservation value reaches to the level of the lowest reservation value of *non-prioritized* agents, or (ii) a set of agents have *exclusive rights*, or (iii) the reservation value of each *eligible* agent is equal to one. The last one is possible only if all *eligible* agents are *prioritized* and each such agent's reservation value can be increased to one. Note that in this case, the total reservation value of *eligible* agents is maximized. Suppose (i) holds. At the updated reservation profile, the set of *eligible* agents is the same as the beginning of the step. Thus, at the next step, the reservation values of the *prioritized* agents at the current step, and also of the *eligible* agents with the second-lowest reservation value at the end of the current step are increased. Suppose (ii) holds. Then, all the units under the reserve categories, for which there are now *exclusive rights*, are assigned to agents with these rights. Since only the reservation values of the *eligible* agents are increased, all the remaining units under these categories at the beginning of the current step are assigned to *eligible* agents. Moreover, since no other *eligible* agent's reservation value is updated to one, by definition of *eligibility*, there are no new *eligible* agents at the beginning of the next step. Thus, under both (i) and (ii), the next step is such that only a subset (if not all) of the current *eligible* agents' reservation values are increased. By an inductive argument, this ends at a step where all of these *eligible* agents have *exclusive rights* or their reservation value becomes one. In the former case, all the units available for the current *eligible* agents are assigned to these agents. In the latter case, each *eligible* agent has a reservation value one. Thus, in both cases the total reservation value of *eligible* agents is maximized. Note that this maximization holds in general in multiple steps. But, since these steps are such that there are no new *eligible* agents, the welfare improvements for these *eligible* agents can be also defined as being realized in only one step instead of multiple steps.

Lemma 5. *A random allocation Z is sequentially egalitarian if and only if it is welfare equivalent to $Z^\star$.*

Proof. Let Z be a *sequentially egalitarian* random allocation. We prove by induction that for each $n \geq 0$, and $i \in \mathcal{I}$, $u_Z(i) \geq v_i^n$. Since, by definition of the *PBR* algorithm, no agent's reservation value can be improved at the last step of the algorithm, this completes the proof.

Initial step: The initial reservation profile v^0 is determined by the *priority rule* for each single-category rationing problem (Section 3.2). Thus, *individual rationality* implies that, for each agent $i \in \mathcal{I}$, their utility is at least v_i^0. This implies that for each $i \in \mathcal{I}$, $u_Z(i) \geq v_i^0$.

Inductive step: By inductive hypothesis, for each $i \in \mathcal{I}$, $u_Z(i) \geq v_i^{n-1}$. We show that for each $i \in \mathcal{I}$, $u_Z(i) \geq v_i^n$. At the beginning of Step n, if there is a set of agents I with *exclusive rights*, then they are assigned the units under the reserve categories $C(I, v^{n-1})$, and by definition of *exclusive rights*, there are no units left under these categories, and these categories are not *available* for other agents.

Let us now consider agents without *exclusive rights*. For any subset of this group of agents, Condition 2 in Theorem 1 is not binding. Thus, their welfare can be improved. To make the reservation values as equal as possible among the set of *eligible* agents, we construct a supply-demand network (see Figure 1) by setting $V_1^n = \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$ as the demand vertices and $V_2^n = A_n$ as the supply vertices.[26] Agent $i \in V_1^n$ points to $c \in V_2^n$ if and only if $i \in \Gamma_c(v^{n-1})$. For each of these arcs (i, c), $l(i, c) = 0$ and $k(i, c) = \infty$. For each *prioritized* agent $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, arc (s, i) has lower bound $l(s, i) = v_i^{n-1} + \lambda$ and capacity, $k(s, i) = v_i^{n-1} + \lambda$. For each *non-prioritized* agent $i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})$, arc (s, i) has lower bound $l(s, i) = v_i^{n-1}$ and capacity, $k(s, i) = v_i^{n-1}$. Also, for each arc (c, t) from V_2^{n-1} into t, let $l(c, t) = 0$ and $k(c, t) = q_c$.

We set up this network as parametric in the following way: For the *prioritized* agents among all *eligible* agents under *available* reserve categories, the parameter λ captures that their reservation values, and only their reservation values, at the relevant categories are improved equally and continuously as long as *(a)* the feasibility conditions in Definition 1 are not violated[27] and *(b)* there are no others joining the group of *prioritized* agents.[28]

[26] The subscripts in V_1^n and V_2^n stand for describing them either as the demand or supply vertices, and the superscripts for the number of the step of the algorithm.

[27] Part *(a)* is captured by the setting of arcs and their lower bounds and capacities: Condition (i) in Definition 1 by setting the capacity of the arcs from the source to agents by one, and Condition (ii) by setting the capacity of each arc from c to the sink by the capacity q_c.

[28] Whether an agent is *prioritized* or not depends on her relative reservation value at reserve categories. Thus, as the agents' reservation values change, their status of being *prioritized* or *non-prioritized* might change as well.

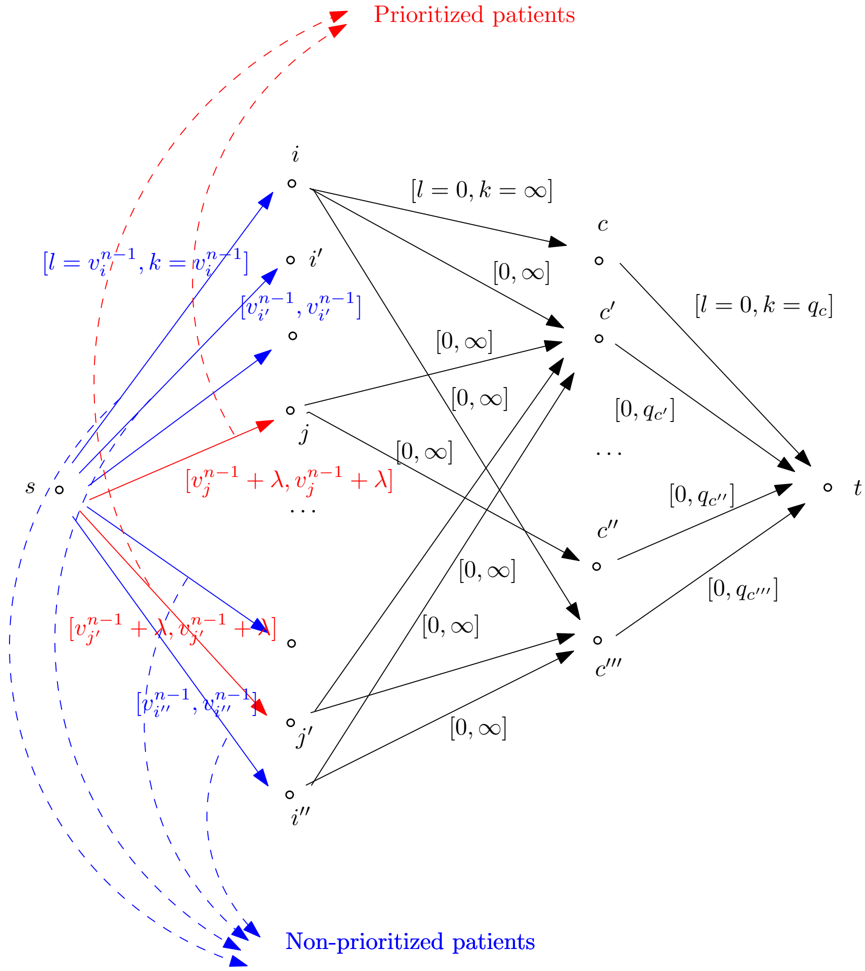

Figure 1

Since Z *respects priorities*, a unit under a category is not (probabilistically) assigned to an agent in a priority class of that category until the utility of each agent in the higher priority classes is one. Thus, when *prioritized* agents' reservation values are increased via λ, the agents in the next priority class cannot be allowed to be assigned units under the same reserve category. Also, while the reservation values of *prioritized* agents ranked under a reserve category are increased, the reservation values of **other** *prioritized* agents ranked under **other** categories are also increased.

At the beginning of Step $n.3$, if it is possible to increase the reservation value of each *eligible* agent to one, then by *sequential egalitarianism*, each such agent's value should be increased to one. By the argument for Remark 3, the *PBR* rule achieves it, in multiple consecutive steps in general. Suppose that it is not possible to increase the reservation value of each *eligible* agent to one. The idea is to use λ as a continuously increasing parameter until a breakpoint where Part *(a)* or *(b)* will be violated if increased further. Thus, there are two candidates for this breakpoint: Condition 2 in Theorem 1 becomes binding (a) or the reservation value of a *prioritized* agent (note that the reservation value of these agents is the minimum among the *eligible* agents under *available* reserve categories) becomes equivalent to the level of the reservation value of a *non-prioritized* and *eligible* agent (b). If the latter holds, then the reservation value of each *prioritized* agent can be increased to the level of of the second-lowest reservation value among other *eligible* agents. Thus, by definition of *Lorenz dominance*, at Z, among the set of *eligible* agents, no agent's utility is lower than this updated reservation value. Moreover, since this value is lower than one, the set of *eligible* agents does not change. The only change is that, at this reservation profile, the set of agents with the minimum level of reservation value becomes larger. This process of increasing the reservation values of the *prioritized* continues until either the breakpoint is given by (a) or each *eligible* agent' reservation value becomes one. If it is the latter, the definition of *Lorenz dominance* implies clearly that each *eligible* agent's utility is one (since there exists such an underlying random allocation), which coincides with the outcome of the *PBR* for the *eligible* agents. Thus, the only case that remains is when the breakpoint is given by (a).

Suppose the breakpoint is given by (a). Since agents are *eligible* for multiple categories in general, to check whether Condition 2 in Theorem 1 becomes binding as λ is increased, we need to consider all subsets of agents. Also, at the beginning of each step when $\lambda = 0$, clearly the condition cannot be binding for a subset of agents without *exclusive rights*.

The *prioritized* agents have the lowest level of reservation value among all *eligible* agents. Thus, to equate reservation values, the parameter λ is increased continuously. Since Condition 2 in Theorem 1 is not binding for no subset of agents, a *flow* exists for some values of $\lambda > 0$. The question is to find the maximum possible value for this parameter. Since the breakpoint is due to Condition 2 becoming binding, there will not be a *flow* respecting the lower bounds of the arcs from s to the demand vertices of *prioritized* agents, if the reservation values of *prioritized* agents are increased above this breakpoint level. By Theorem 5, the value of this maximum *flow* is equal to the minimum value

$$k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V')) \tag{15}$$

taken over $V' \subseteq V$ with $s \in V'$ and $t \notin V'$. Since the *flow* is always maximum, the set $\{s\}$ gives this minimum value. Moreover, as the breakpoint is reached, there exists another set of vertices with the minimum value of (15). We need to find this bottleneck set of vertices $V' = \{s\} \cup I' \cup C'$, which prevent λ to be increased further.

The set V' satisfies that each $i \in I'$ points only to the reserve categories in C' (because otherwise $k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V')) = \infty$). Thus, $C(I', v^{n-1}) \subseteq C'$. Also, there cannot be a reserve category c such that $c \in C' \setminus C(I', v^{n-1})$, since then, by removing c from the set V', the value of (15) is decreased by the amount q_c due to the capacity of the outgoing arc from c to t. Thus, $C' = C(I', v^{n-1})$.[29] Thus,

$$k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V')) = \sum_{i \in \left(\bigcup_{c \in A_n} \Gamma_c(v^{n-1})\right) \setminus I'} v_i^{n-1} + \lambda |\{i \notin I' : i \text{ is prioritized}\}| + \sum_{c \in C(I', v^{n-1})} q_c.$$

The first and second terms of the right-hand side in this equation is the total capacity of all the edges from s to the set of *eligible* agents excluding the set I'. Since $\{s\}$ minimizes (15) as well, and only the reservation values of *prioritized* agents are increased by λ, we also have

$$k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V')) = \sum_{i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})} v_i^{n-1} + \lambda |\{i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1}) : i \text{ is prioritized}\}|$$

This implies

$$\sum_{i \in \left(\bigcup_{c \in A_n} \Gamma_c(v^{n-1})\right) \setminus I'} v_i^{n-1} + \sum_{c \in C(I', v^n-1)} q_c = \sum_{i \in \bigcup_{c \in A_n} \Gamma_c(v^{n-1})} v_i^{n-1} + \lambda |\{i \in I' : i \text{ is prioritized}\}| \tag{16}$$

Since Equality (16) is the necessary condition for V' to be a bottleneck set, the reservation values of the *prioritized* agents can be increased by the minimum of λ satisfying (16). Note that this minimum λ is equivalent to $\lambda^\star$ defined in Step $n.3$ of the *PBR*. By definition of *sequential egalitarianism*, at Z, the lowest reservation value is maximized. Thus, since the reservation value of a *prioritized* agent, say i, is the lowest among all *eligible* agents, and it is feasible to increase their reservation value to $v_i^{n-1} + \lambda^\star$, their utility must be greater than or equal to this value. Moreover, the *PBR* is such

[29] Also, reserve categories in C' cannot be pointed by an agent who is not in I' and *eligible* only for categories in C' (because otherwise, by adding such an agent to the set I', the value $k(\delta^{\text{out}}(V' \cup \{i\})) - l(\delta^{\text{in}}(V' \cup \{i\}))$ is lower than the value $k(\delta^{\text{out}}(V')) - l(\delta^{\text{in}}(V'))$. Note that any such agent i provides an incoming edge to V' with a lower-bound zero, and an outgoing edge from V' with a capacity v_i^{n-1}.

that, for each *non-prioritized* agent, the reservation value does not change at Step n. Thus, for each *eligible* agent j, the updated reservation value is at least $v_j^{n-1} + \lambda^\star$. Thus, $u_Z(j) \geq v_i^n$. This completes the proof. $\square$

Appendix E Proof of Proposition 1

We first show that the *PBR* rule satisfies *no justified-envy*. The result then follows directly from Theorem 4. Let $c \in C$ and $k > 0$ such that

$$UCS_{\pi_c}(k) \subseteq \bigcup_{c' \in C \setminus \{c\}} UCS_{\pi_{c'}}(k_{c'}(Z^\star) - 1).$$

Let $C' \subseteq C \setminus \{c\}$ be the set of reserve categories such that for each $c' \in C'$, $UCS_{\pi_{c'}}(k_{c'}(Z^\star) - 1) \cap UCS_{\pi_c}(k) \neq \emptyset$ (†). Let n be the first step of the *PBR* such that for all agents in $\bigcup_{c' \in C'} UCS_{\pi_{c'}}(k_{c'}(Z^\star) - 1)$, reservation value is one. There are two cases:

Case 1: At Step $n+1$, for some category $c' \in C'$, agents in $\mathcal{I}_{\pi_{c'}}(k_{c'}(Z^\star))$ are *eligible* for c'. By (†), this implies that each agent in $\mathcal{I}_{\pi_c}(k+1)$ is *eligible* for c. By definition of $k_{c'}(Z^\star)$, at some step $n' > n$, for each $i' \in \mathcal{I}_{\pi_{c'}}(k_{c'}(Z^\star))$, $v_{i'}^{n'} = 1$. By (†), this implies that for each agent $j \in \mathcal{I}_{\pi_c}(k+1)$, $v_j^{n'} = 1$. Thus, $k_c(Z^\star) \geq k+1$.

Case 2: At Step $n+1$, for each category $c' \in C'$, no agent in $\mathcal{I}_{\pi_{c'}}(k_{c'}(Z^\star))$ is *eligible* for c'. Since the reservation value of each agent in $\bigcup_{c' \in C'} UCS_{\pi_{c'}}(k_{c'}(Z^\star) - 1)$ is one, by definition of *eligibility*, this case is possible only if there is the exception of *exclusive rights* to *eligibility*. By (†) and definition of *exclusive rights*, agents in $\mathcal{I}_{\pi_c}(k+1)$ are not *eligible* for c. By definition of $k_{c'}(Z^\star)$, at a step $n'' > n$, for categories in C', all agents in $\mathcal{I}_{\pi_{c'}}(k_{c'}(Z^\star))$ and for category c, all agents in $\mathcal{I}_{\pi_c}(k+1)$, become *eligible* for their corresponding categories.[30] But then, as, for some $c' \in C'$, the reservation value of each agent in $\mathcal{I}_{\pi_{c'}}(k_{c'}(Z^\star))$ reaches one (this has to be the case by definition of $k_{c'}(Z^\star)$), by (†), the reservation value of each agent in $\mathcal{I}_{\pi_c}(k+1)$ reaches one as well. Thus, $k_c(Z^\star) \geq k+1$.

[30] Note that this happens when an agent in $\bigcup_{c' \in C'} UCS_{\pi_{c'}}(k_{c'}(Z^\star) - 1)$ becomes *eligible* for a category out of $C' \cup \{c\}$.

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