Stability of an Allocation of Objects*

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Abstract

A central stability notion for allocation problems when there are private endowments is *core*: no coalition should be able to block the allocation. But, for an exchange economy of discrete resources, *core* can be empty. An alternative stability axiom is the *bargaining set* à la Aumann and Maschler (1964): a blocking by a coalition is justified only if there is no counter-objection to it and an allocation is in the *bargaining set* if there does not exist a justified blocking. Allowing for weak preferences, we prove that any allocation obtained by the well-known Top Trading Cycles class is in the *bargaining set*, but not all allocations in the *bargaining set* can be obtained by this class.

Keywords : Assignment problem, Core, Bargaining Set, Top Trading Cycles

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1 Introduction

An exchange economy of discrete resources with private endowments is when each agent owns an indivisible good (an object) and these objects are to be allocated among agents via direct mechanisms without monetary transfers. A central notion when there are private endowments is *individual rationality*, which requires that the assignment should be such that no agent is worse off than her endowment. Another important (stability) property of this problem is *core*: no coalition of agents should be able to block the assignment; that is, they should not prefer reallocating their endowments among themselves (and leaving the economy) over the assignment. But, *core* is in general empty in the weak preferences domain. An alternative (and weaker) notion is the *bargaining set* by Aumann and Maschler (1964): a blocking is justified only if there is no counter-objection to it and an allocation is in the *bargaining set* if there does not exist a justified blocking. We prove that any allocation obtained by the well-known Top Trading Cycles class is in the *bargaining set*, but not all allocations in the *bargaining set* can be obtained by this class.

If preferences are strict, *core* is a singleton and it is the only solution which satisfies *individual rationality*, *Pareto efficiency* and *strategy-proofness* (Ma (1994), Sönmez (1999)). Also, core is equivalent to the outcome of the well-known Top Trading Cycles (TTC) algorithm (Shapley and Scarf, 1974), which works as follows: Each agent points to her most preferred available object (all objects are available at the beginning) and each object points to its owner. Since all agents and objects point, there is at least one cycle. The algorithm assigns to each agent in the cycle her most preferred available object (that is, the object she points at) and removes her with her assigned object. This continues until no one is left. The resulting mechanism is *group strategy-proof* and *Pareto efficient* (Roth, 1982). When an agent may be endowed with multiple objects or no object, the top trading cycles rule is generalized to the *hierarchical change* rule, which is characterized by *Pareto efficiency*, *group strategy-proofness* and *reallocation-proofness* (Pápai, 2000). A more general trading mechanism is *trading-cycles* and it is characterized by *group strategy-proofness* and *Pareto efficiency* (Pycia and Ünver, 2016).

While the extension of the TTC algorithm to the weak preferences domain is not trivial, such extensions satisfying *individual rationality*, *Pareto efficiency* and *strategy-proofness* are shown to exist (Jaramillo and Manjunath (2012), Alcalde-Unzu and Molis (2011), Saban and Sethuraman (2013)). *Strategy-proofness* characterizes a subclass of these generalized TTC class satisfying *Pareto efficiency*

(Saban and Sethuraman, 2013).

When the restrictive strict preferences assumption is removed, *core* can be empty (Shapley and Scarf, 1974). Actually, *core* is non-empty only for a very special preference and endowment structure (Quint and Wako, 2004). A weakening of *core* is *weak core*: blocking is allowed only if each agent in the blocking coalition is strictly better off than the assignment. The extensions of the TTC (to the weak preferences domain) are in the *weak core*. Our focus is on another notion, the *bargaining set*, which incorporates an important consideration into the process of blocking an assignment: when blocking, coalitions should consider possible counter-blockings of other coalitions. More precisely, an assignment is in the *bargaining set* if blocking by a coalition implies that there is another coalition blocking the assignment resulting from the initial blocking (Definition 2). This notion is formulated by Aumann and Maschler (1964) and later analyzed for different economies. In the context of a market game with a continuum of players, the *bargaining set* is equivalent to the set of Walrasian allocations (Mas-Colell, 1989). For non-transferable utility games, the *bargaining set* is non-empty under certain conditions (Vohra, 1991).[1] For an exchange economy with differential information and a continuum of traders, the *bargaining set* and the set of Radner competitive equilibrium allocations are equivalent (Einy, Moreno, and Shitovitz, 2001). While the *bargaining set* notion in these works takes into account only one step of counter-objection to a blocking coalition, the consideration of a chain of counter-objections implies a more refined axiom (Dutt, Ray, Sengupta, and Vohra, 1989).

The idea of *bargaining set* also inspires some works on allocation of discrete resources in school choice context in terms of relaxing *stability* notion, which is central to matching theory: if a student has an objection to an allocation because she claims an empty slot at a school, then there will be a counter-objection once she is assigned to that school since the priority of some other student will be violated at that school. An outcome is in the *bargaining set* if and only if for each objection to the outcome, there exists a counter-objection (Ehlers, Hafalir, Yenmez, and Yildirim, 2014).[2] Some other works refer to *bargaining set* in similar ways (see Ehlers (2010), Kesten (2010), Alcade and Romero-Medina (2015)).

The paper is organized as follows: Section 2 introduces the model and the graph theoretical frame-

[1] There are slight differences in the formulation of the bargaining set defined by Aumann and Maschler (1964) and Mas-Colell (1989). See Vohra (1991) for the differences between these two formulations and also other variants of the notion.

[2] Ehlers, Hafalir, Yenmez, and Yildirim (2014) refer to this property as constrained non-wastefulness in the school choice context.

work, on which the mechanisms and some of the proofs are built. Section 3.1 defines *core* and *bargaining set* notions. We state our main result in Section 4. All proofs are in the Appendix.

2 Model

2.1 Assignment Problem

Let N be a set of agents and O a set of objects such that each agent is endowed with one object. An *assignment problem* is when the objects in O are to be allocated to the agents in N in such a way that each agent receives exactly one object. We fix O and N.

An **endowment profile** is a bijection $e : N \to O$. We denote the set of endowments of a set of agents $N' \subseteq N$ by $e(N')$. Each agent i has a complete and transitive preference relation R_i on O; that is, we allow for indifferences. For each i, let P_i and I_i denote the strict and indifferences parts of R_i, respectively. Let $R = (R_i)_{i \in N}$ be a preference profile. We denote an *assignment problem* (or a *market*) by a pair (e, R).

An **assignment** μ is a bijection $\mu : N \to O$. An *assignment* μ is **individually rational** if for each $i \in N$, $\mu(i)\ R_i\ e(i)$. An *assignment* μ is **Pareto efficient** if there does not exist another assignment ν such that for each $i \in N$, $\nu(i)\ R_i\ \mu(i)$ and for some $j \in N$, $\nu(j)\ P_j\ \mu(j)$.

2.2 Graph theoretical framework

Let $G = (V, E)$ be a directed graph, where V is the set of *vertices* and E is the set of *directed edges*, that is a family of ordered pairs from V. For each $U \subset V$, let $\delta^{in}(U)$ be the set of edges $(u, v) \in E$ such that $u \in V \setminus U$ and $v \in U$ (i.e. the set of edges **entering** U) and $\delta^{out}(U)$ be the set of edges $(u, v) \in E$ such that $u \in U$ and $v \in V \setminus U$ (i.e. the set of edges **leaving** U). If U is a singleton, say $U = \{v\}$, then we use $\delta^{in}(v)$ (and $\delta^{out}(v)$) instead of $\delta^{in}(U)$ (and $\delta^{out}(U)$). A **subgraph** of G is any directed graph $G' = (V', E')$ with $\emptyset \neq V' \subseteq V$ and $E' \subseteq E$ and each edge in E' consisting of vertices in V'. For a set of vertices $T \subseteq V$, the **subgraph of G induced by T** is the subgraph (T, E') such that $E' = \{(u, v) \in E : u, v \in T\}$. A sequence of vertices $\{v_1, \ldots\ v_m\}$ is a **path from v_1 to v_m** if (i) $m \geq 1$, (ii) $v_1, \ldots, v_m$ are distinct (except for possibly $v_1 = v_m$), and (iii) for each $k = 1, \ldots, m - 1$, $(v_k, v_{k+1}) \in E$. A **cycle** is a path $\{v_1, \ldots\ v_m\}$ if $m \geq 2$ and $v_1 = v_m$.

A set of vertices $T \subseteq V$ is **strongly connected** if the subgraph induced by T is such that for any $u, v \in T$, there is a path from u to v. A **minimal self-mapped set** is a set of vertices $S \subseteq V$ that satisfies two conditions: (i) $S = \underset{v \in S}{\cup} \delta^{out}(v)$ and (ii) $\nexists S'$ with $\emptyset \neq S' \subset S$ such that $S' = \underset{v \in S'}{\cup} \delta^{out}(v)$.[3] The following remark is by Quint and Wako (2004).[4]

Remark 1 *Let $G = (V, E)$ be a directed graph. A set of vertices $S \subseteq V$ is non-empty and strongly connected such that $\delta^{out}(S) = \emptyset$ if and only if S is a minimal self-mapped set.*

Whenever convenient, we refer to this equivalence result and say that a set of vertices S is a *minimal self-mapped set* if (i) for any two vertices in S, there is a path from one to the other, and (ii) there is no path from any vertex $u \in S$ to any vertex $v \notin S$. The next remark follows directly from Remark 1 and the *Minimal Self-Mapped Set* ($\mathcal{MSMS}$) algorithm introduced by Quint and Wako (2004).

Remark 2 *Let $G = (V, E)$ be a directed graph. If for each $v \in V$, $\delta^{out}(v) \neq \emptyset$, then a minimal self-mapped set exists.*

Let $w : E \to \Re$ be a function. We denote $\sum_{e \in F \subseteq E} w(e)$ by $w(F)$. A function $f : E \to \Re$ is called a **circulation** if for each $v \in V$, $f(\delta^{in}(v)) = f(\delta^{out}(v))$. Let $d, c : E \to \Re$ with $d \leq c$. A *circulation* f **respects d and c** if for each edge e, $c(e) \geq f(e) \geq d(e)$. A *minimal self-mapped set* S is **covered** if there exists an integer-valued *circulation* f such that for each $v \in S$, $f(e) = 1$ for some edge e entering v.

3 Stability

A central concept in exchange economies is stability. It prevents agents from forming coalitions and reallocating their endowments among themselves so that they are better off than the proposed assignment.

Let S be a group of agents. An assignment μ is **strictly blocked by** S if the agents in S can reallocate their endowments in a way that makes each of them better off than at μ; that is, there exists μ' such that $\mu'(S) = e(S)$ and for each $i \in S$, $\mu'(i)\ P_i\ \mu(i)$.[5] An assignment μ is **blocked by** S if the agents in S can reallocate their endowments in a way that makes no agent worse off

[3]Note that $\underset{v \in S}{\cup} \delta^{out}(v)$ and $\delta^{out}(S)$ are different sets in general.

[4]It follows directly from Proposition 2.2 in Quint and Wako (2004).

[5]We assume that agents in S **reallocate their endowments** in an efficient way: there does not exist another reallocation of these endowments such that no agent is worse off and at least one agent is better off than the original reallocation.

and at least one agent better off than at μ; that is, there exists μ' such that $\mu'(S) = e(S)$ and for each $i \in S$, $\mu'(i)\ R_i\ \mu(i)$, and for some $j \in S$, $\mu'(j)\ P_j\ \mu(j)$. An assignment μ is **weakly blocked by** S if the agents in S can reallocate their endowments in a way that makes no agent worse off; that is, there exists μ' such that $\mu'(S) = e(S)$ and for each $i \in S$, $\mu'(i)\ R_i\ \mu(i)$. For an assignment problem (e, R), the **weak core** (denoted by $WC(e, R)$) is the set of assignments that are not strictly blocked by any coalition. The **core** (denoted by $C(e, R)$) is the set of assignments that are not blocked by any coalition.

While *core* is non-empty and a singleton set for the special case of strict preferences, it is empty in general. Under weak preferences, *core* is non-empty only for a very special and restrictive structure (Quint and Wako, 2004). On the other hand, *weak core* is always non-empty. The problem with *weak core* is that it is "too" weak, as the following example demonstrates.

Example 1 *Let $N = \{i_1, i_2, i_3, i_4, i_5, i_6\}$ and $O = \{o_1, o_2, o_3, o_4, o_5, o_6\}$ where $e(i_k) = o_k$. The preferences are given below with each set in the table being an indifference set:*

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}	R_{i_5}	R_{i_6}
$\{o_2\}$	$\{o_3, o_4\}$	$\{o_1\}$	$\{o_2, o_6\}$	$\{o_6\}$	$\{o_5\}$
$\{o_1\}$	$\{o_2\}$	$\{o_3\}$	$\{o_4\}$	$\{o_4\}$	$\{o_6\}$
				$\{o_5\}$	

The assignment $\mu = (o_2, o_3, o_1, o_6, o_4, o_5)$ is in the weak core *and the* core *is empty. One can argue that agents i_5 and i_6 are endowed with each other's unique best objects and thus, they should be able to exchange their objects. Note that the coalition $S = \{i_5, i_6\}$ weakly dominates μ. Also, any assignment, at which agent i_5 and i_6 are assigned objects o_6 and o_5, respectively, is not weakly blocked by a coalition including agent i_5 or i_6.*

The intuition in Example 1 is a simple requirement for trade. This can be captured via the following property which guarantees trades of the best objects between the owners.

Definition 1 *An assignment μ satisfies* **top-trade property** *if for any cycle $S = \{a_1, a_2, ..., a_K\}$ where $e(a_k)$ is agent a_{k-1}'s single best object for $k = 2, ..., K$ and $e(a_1)$ is agent a_K's single best object, $\mu(a_{k-1}) = e(a_k)$ for $k = 2, ..., K$ and $\mu(a_K) = e(a_1)$.*

The crucial insight from Example 1 is that even this simple *top-trade* property (e.g. agents 5 and 6 in Example 1 can trade their best objects) is not implied by the *weak core*. This raises the following question: is there any other stability concept that is weaker than *core* (weak enough to be non-empty) and stronger than *weak core* (strong enough to imply the *top-trade* property)?

3.1 The bargaining set

An alternative stability concept is *bargaining set*: an assignment is in the *bargaining set* if any blocking by a coalition S is deterred by a counter-objection of another coalition. This is clearly a weaker notion than *core*: By definition, *core* does not allow any such deterrence (or counter-blocking). As any blocking is possible, *core* is a strong axiom. On the other hand, in the *bargaining set*, as counter-blocking is possible, blocking is more difficult and the axiom is weaker.

Formulating the *bargaining set* notion in the current context requires careful consideration for three reasons. First, in the definition of *(weak) core*, we did not need to refer to what happens to agents and their assigned objects after blocking, mainly because blocking is a one-step process. *Bargaining set* is different: it is a two-step process, blocking and counter-blocking, and it matters how we define the assignment after blocking. We assume the least when an assignment is blocked: An assignment μ can be considered as a set of cycles, where each agent in a cycle is assigned to the object she points to in that cycle. When an assignment μ is blocked by a coalition S, we assume that (i) each agent in the coalition S is assigned to the endowment of another agent in S, (ii) each agent in a cycle, which has an empty intersection with S, is assigned to the same object which she is assigned under μ, and (iii) every other agent is assigned to her endowment. Thus, a blocking coalition's effect is only through the cycles it intersects. Thus, if coalition S **blocks** μ **via** η, then assignment η satisfies (i)-(iii).

Second, since the intuition behind counter-blocking is counter-objection to a blocking coalition, it cannot be defined by means of any arbitrary coalition. Clearly, a counter-blocking coalition has a deterrence power only if it contains agents from the blocking coalition. The definition of counter-blocking should embed at least this very minimal consistency requirement implicit in the very idea of counter-objection.

Third, the idea behind the *bargaining set* is to reduce the amount of blockings by allowing only the ones which are *counter-blocked*. But, if it is the case that for almost any assignment, any blocking is *counter-blocked*, then the notion of bargaining set is not strong enough to eliminate certain blockings.

The following result demonstrates that the restriction 'any blocking by a coalition is *counter-blocked* by another coalition' may not lead to any other elimination of blockings than the *weak core*.

Proposition 1 *There exists an assignment problem* (e, R) *and an assignment* $\mu \in WC(e, R)$ *such that, whenever* μ *is blocked by a coalition* S *via* η*,* η *is blocked.*

Proof. See Appendix A. ■

The problem with the stability notion that 'any blocking by a coalition is *counter-blocked* by another coalition' is an impossibility due to indifferences in preferences: whenever blocking coalitions intersect, and it is not possible to assign a best object to each agent in these coalitions, then each of these coalitions is a potential *counter-blocking* coalition. Thus, since the only requirement in this notion is the existence of a *counter-blocking* coalition, and the definition of blocking does not refer to the welfare of the agents in this coalition after *counter-blocking*, it does not really restrict the set of assignments from the *stability* point of view. Thus, the definition of *counter-blocking* cannot be the same as *blocking* and additional conditions should be embedded.

We impose the most natural restriction on *counter-blocking*: at an assignment μ, whenever a *counter-blocking* coalition forms against the blocking coalition, these agents counter-object to the blocking coalition by claiming their welfare at μ. Let S *block* an assignment μ via η. A coalition $C(S)$ **counter-blocks** η if (i) $S \cap C(S) \neq \emptyset$, (ii) $C(S)$ *blocks* η via an assignment μ' such that for each $i \in C(S)$, $\mu'(i)\ I_i\ \mu(i)$.

Definition 2 *An assignment* μ *is in the* **bargaining set** *if and only if*

(i) *it is not strictly blocked by any coalition, and*

(ii) *if* S *blocks* μ *via* η*, then there exists a coalition* $C(S)$ *which counter-blocks* η*.*

For an assignment problem (e, R)*, we denote the bargaining set by* $B(e, R)$*.*

Bargaining set is clearly stronger than *weak core*. Moreover, it implies the *top-trade* property (see Example 1 and Definition 1).

Proposition 2 *Each assignment in the bargaining set satisfies the top-trade property.*

Proof. See Appendix B ■

Bargaining set is a plausible stability concept; it is non-empty valued (this follows as a corollary from our main result, Theorem 2), stronger than *weak core*, and satisfies the *top-trade property.*

4 The Top Trading Cycles (TTC) class and stability

The TTC class is a set of assignment rules as an extension of the well-known TTC mechanism defined on the strict domain. Since agents have indifference classes, they point to multiple objects during the execution of the TTC. Thus, the problem is to select a particular cycle among intersecting cycles. This is the crux in defining a particular mechanism.

Let F be a selection rule: for each *minimal self-mapped set* that is not *covered*, F selects one of the cycles in the *minimal self-mapped sets*. The TTC updates the endowment profile by assigning each agent in the cycle to the object that she points to in the same cycle. Let $e_1 = e$ and for $k \geq 1$, the steps below are repeated until all agents and objects are removed.

The TTC Algorithm:

Step k. Let each agent point to her best objects among the remaining objects[6] and each remaining object points to its owner according to the endowment profile e_k. Select a *minimal self-mapped set* T_k in this digraph.

- **(k.1)** If T_k is *covered*, then each agent in T_k is removed by assigning her one of the best objects in T_k.
- **(k.2)** Otherwise, select one of the cycles in the *minimal self-mapped set* using the selection rule F, and update the endowment profile in the cycle to obtain e_{k+1}.

Each outcome of the TTC class is *Pareto efficient* and a member of the *weak core* (Jaramillo and Manjunath (2012), Alcalde-Unzu and Molis (2011), Saban and Sethuraman (2013)).

For each problem (e, R), the set of outcomes for all selection rules is denoted by $TTC(e, R)$. While each outcome of the TTC is in the *weak core*, its relation to the *core* is very weak, mainly because *core* is mostly empty. The *core* is non-empty only for a very special market structure, the *top trading segmentation* (TTS) (Quint and Wako, 2004). The TTS structure is as follows: When each agent points to her best objects, there is a *covered minimal self-mapped set* in the market. Once the agents and endowments in this set is removed, and the remaining agents point to their best objects among the remaining objects, this remaining market has another *covered minimal self-mapped set* (note that for some agents in this remaining (smaller) market, their best objects might not be available anymore).

[6] At Step 1, the set of remaining objects is the set of all objects, O.

Once the second *covered minimal self-mapped set* is also removed, there is another one in the remaining market, and so on. Thus, a market has a TTS when the market can be partitioned into a sequence of mutually exclusive smaller markets (that is, into *covered minimal self-mapped sets*) in the specific way described. In this market, each 'segment' (a *covered minimal self-mapped set*) is 'self-sufficient' in the sense that each agent can be assigned an object from that segment where such an object is a best object in the remaining market once previous segments are removed. This is clearly a very restrictive structure.

Theorem 1 *(Quint and Wako, 2004) The core is non-empty if and only if the market has a TTS.*

The *core*, if non-empty, is such that each agent is assigned a best object in the segment she is in (Quint and Wako, 2004) and it is not necessarily single-valued: there could be multiple ways of assigning agents to their best objects in their segments, but, by definition of the TTS, each such assignment gives the same welfare level for each agent. Thus, each agent is indifferent between any two assignments in the *core*. Thus, the *core* might not be single-valued, but it is *essentially single-valued*. These findings imply an immediate characterization result.

Corollary 1 *In a market with a TTS, an assignment is in the core if and only if it is an outcome of the TTC.*

4.1 Main result

Our main result is that any allocation obtained by some mechanism in the TTC class is in the *bargaining set*, but not all allocations in the *bargaining set* can be obtained by some mechanism in the TTC class. The intuition for this result is as follows: at each step of a mechanism in the TTC class, a set of agents is picked so that the agents in this set are guaranteed a welfare level that is best after the removal of agents in the previous step. Also, whenever endowment updates take place in the mechanism, possible gains from trade are to be realized. Thus, when a coalition *blocks* an allocation produced by the TTC class, it will make some agent, who is removed earlier or has an updated endowment, worse off. This worse off agent will be able to construct a counter-blocking coalition.

Theorem 2 *For each problem* (e, R), $TTC(e, R) \subseteq B(e, R)$. *Moreover, there exists an assignment problem such that the inclusion is strict.*

Proof. See Appendices C and D. ■

5 Conclusion

In the context of allocation problems with private endowments, when weak preferences are allowed, we study stability concepts and their relationship to the well-known *Top Trading Cycles class*. In such an environment, *core*, a central stability notion, can be empty. On the other hand, *weak core*, another stability notion that is a weakening of *core*, does not possess desirable properties. We show that *weak core* does no even satisfy *top-trade property*, a fundamental property for stability. Thus, we argue that *core* is too strong (it can be empty) and *weak core* is too weak (it does not satisfy *top-trade property*).

As an alternative stability notion, we consider the *bargaining set* à la Aumann and Maschler (1964): a blocking by a coalition is justified only if there is no counter-objection to it and an allocation is in the *bargaining set* if there does not exist a justified blocking. We prove that any allocation obtained by the *Top Trading Cycles class* is in the *bargaining set*, but not all allocations in the *bargaining set* can be obtained by this class.

Appendix A Proof of Proposition 1

Let $N = \{i_1, i_2, i_3, i_4\}$ and $O = \{o_1, o_2, o_3, o_4\}$ where $e(i_k) = o_k$. The preferences are given below with each set in the table being an indifference set:

R_{i_1}	R_{i_2}	R_{i_3}	R_{i_4}
$\{o_3, o_4\}$	$\{o_1, o_4\}$	$\{o_2\}$	$\{o_2\}$
$\{o_2\}$	$\{o_3\}$	$\{o_1, o_4\}$	$\{o_1, o_3\}$
$\{o_1\}$	$\{o_2\}$	$\{o_3\}$	$\{o_4\}$

The assignment $\mu = (o_2, o_1, o_4, o_3)$ is not strictly blocked by any coalition. Note that since i_2 receives a best object under μ, any strictly blocking coalition cannot contain i_2. Since $e(i_2)$ is the unique best object for agents i_3 and i_4, this implies that these agents cannot be in a strictly blocking coalition. We claim that any blocking of μ is blocked. Let S block μ via η. Coalitions $\{i_1, i_2\}$ and $\{i_2, i_3\}$ cannot block μ. We check the remaining coalitions.

Case 1: $S = \{i_1, i_2, i_4\}$

The assignment η is (o_4, o_1, o_3, o_2) and it is blocked by $S' = \{i_1, i_2, i_3\}$.

Case 2: $S = N$

There are two candidates for η: (o_3, o_4, o_2, o_1) and (o_3, o_4, o_1, o_2). The former is blocked by $\{i_1, i_2, i_4\}$, the latter by $\{i_1, i_2, i_3\}$.

Case 3: $S \neq \{i_1, i_2, i_4\}$ and $S \neq N$

First note that, at μ, agents i_1 and i_2 are assigned each other's endowments (the same for agents i_3 and i_4). Thus, by definition of blocking, at η, at least one of the agents in S is assigned her endowment. Since the agents in the coalition $\{i_1, i_2, i_4\}$ can reallocate their endowments such that each receives a best object, $\{i_1, i_2, i_4\}$ blocks η.

Appendix B Proof of Proposition 2

Lemma 1 *Let (e, R) be an assignment problem and $\mu \in B(e, R)$. Let G be a graph where each object points to its owner and each agent points to her best objects. If A is a covered minimal self-mapped set in G, then μ allocates the objects in A to the agents in A such that each agent receives one of her best objects.*

Proof. Suppose not. Then, since A is *covered*, agents in A, say S, block μ, say via η, under which each agent in S receives a best object. Since $\mu \in B(e, R)$, there is a coalition $C(S)$ which *counter-blocks* η via μ'. By definition, $C(S) \cap S \neq \emptyset$ and for each $i \in C(S)$, $\mu'(i)\ I_i\ \mu(i)$. Thus, each agent in $C(S) \cap S$ receives a best object under both μ and η. Note that $C(S) \nsubseteq S$, since otherwise $C(S)$ cannot block η. Thus, there is an agent in $C(S) \cap S$ who receives, under η, an object owned by some agent in $C(S) \setminus S$. But since S is a *minimal self-mapped* set, this object cannot be a best object for this agent, contradicting with the fact that each agent in $C(S) \cap S$ receives a best object under η. ■

Since any cycle $S = \{a_1, a_2, ..., a_K\}$, where $e(a_k)$ is agent a_{k-1}'s single best object for $k = 2, ..., K$ and $e(a_1)$ is agent a_K's single best object, is a *minimal self-mapped* set that is *covered*, by Lemma 1, μ assigns each agent in S a best object from the endowment set of the agents in S. Since each agent has a single best object, the *top-trade property* is satisfied.

Appendix C Each outcome of the TTC is in the bargaining set.

Let (e, R) be an assignment problem and let $\mu \in TTC(e, R)$. We need to show that μ satisfies the conditions in Definition 2: (i) μ is not *strictly blocked* by any coalition, and (ii) if S *blocks* μ via η, then there exists a coalition $C(S)$ which *counter-blocks* η.

In the TTC algorithm, at each step k, a *minimal self-mapped* set T_k is chosen. This set may be *covered* or not. If it is *covered*, then it is removed. If not, an endowment update takes place among agents in a selected subset (cycle) of T_k, and there is no removal in this step.[7] At the next step, there may be a removal of a new *covered minimal self-mapped* set or there may be a further endowment update with no removal. Thus, potentially, at some steps of the algorithm, there is no removal of agents, while in others there is. We consider the endowment updates and removal of agents in the following way: Let L_0 be the set of agents who are removed before any endowment update takes place. It is the union of a series of *covered minimal self-mapped* sets. If there is no removal before any endowment update, L_0 is empty. After each agent in L_0 and their original endowments are removed, there is a consecutive series of endowment updates, which is followed by a consecutive series of removal of agents, which is followed by another consecutive series of endowment updates, and so on.[8]

[7] When a set of agents is removed, the objects they are assigned to are also removed. In the rest of the proof, when we say a set of agents is removed, we mean that the objects they are assigned to are also removed.

[8] These updates and removals may be a single update or a single removal.

For $t = 1, 2, ..., M$, let U_t be the set of agents whose endowments are updated in a series of updates with no removal in between, and let L_t be the set of agents who are removed in a series of removals with no updates in between. More precisely, L_t is the set of all agents who are removed after the endowment updates of the agents in U_t and before the endowment updates of the agents in U_{t+1} take place. U_t is the set of all agents whose endowments are updated after each agent in L_{t-1} is removed and before each agent in L_t is removed. We set $U_0 = \emptyset$.[9]

Also, define V_t to be the set of all non-removed agents whose endowments are updated sometime before the agents in L_t are removed. Thus, for $t = 1, 2, ..., M$,

$$V_t = (\cup_{\tau=1}^{t} U_\tau) \setminus (\cup_{\tau=0}^{t-1} L_\tau)$$

By definition, $U_1 = V_1$. L_M is the last set of agents who are removed after the last series of endowment updates, U_M. Note that

$$N = L_0 \cup \bigcup_{t=1}^{M} (V_t \cup L_t)$$

By definition, for each $t = 1, 2, ..., M$, $U_t \neq \emptyset$ and $L_t \neq \emptyset$. Otherwise, if $U_t = \emptyset$, then L_{t-1} and L_{t+1} are essentially equal to L_{t-1}, contradicting the definition of L_t. Likewise, $L_t = \emptyset$ contradicts the definition of U_t. Thus, for each $t = 1, 2, ..., M$, $V_t \neq \emptyset$. Also, for each $t = 1, ..., M$, $U_t \cap L_t \neq \emptyset$. Otherwise, by definition, no agent in L_t has an endowment update after L_{t-1} is removed. Thus, L_t is essentially removed before the endowment update(s) of U_t, contradicting the definitions of L_{t-1}, U_t and L_t. Thus, for each $t = 1, ..., M$, we have $V_t \cap L_t \neq \emptyset$.

Lemma 2 *Let for some t, $i \in U_t$ and $i \notin V_{t-1} \cup L_t$. Then, i is assigned to a best object among the remaining objects after the agents in L_{t-1} are removed.*

Proof. Suppose an agent i's endowment is updated for the first time before L_t and after L_{t-1} are removed, and agent i is not removed with L_t. Then, agent i is tentatively endowed (at the end of updates in U_t) with a best object among the remaining objects after the removal of L_{t-1}. After step t, at each step before the agent is assigned an object and removed, the agent always points to her best

[9] An agent's endowment may be updated multiple times and one update may be before some L_t is removed and another update may be after L_t is removed, provided that the agent is not removed with L_t or before.

objects among the remaining objects. Thus, she is never tentatively endowed with an object that is worse than her tentative endowment after the removal of L_{t-1}. Thus, she is assigned to a best object among those remaining ones after L_{t-1} is removed. ■

Lemma 3 *Let for some t, $i \in U_t$. At the end of updates at t, let o_j be the updated endowment of agent i. Let agent j be the original owner of o_j. If j is not removed at or before t, then she is assigned to an object that is at least as good as a best object among the remaining objects after the agents in L_{t-1} are removed.*

Proof. Suppose $i \in U_t$ for some t. At the end of updates at t, let agent i's updated endowment be o_j. Let j be the original owner of o_j. Suppose j is not removed at or before t. Thus, agent j has an updated endowment during or before the updates at t. Suppose her first endowment update was at some $t' \leq t$. Thus, $j \in U_{t'}$ and $j \notin V_{t'-1} \cup L_{t'}$. By Lemma 2, she is assigned to a best object among those remaining ones after $L_{t'-1}$ is removed. Since $t' \leq t$, the object she is assigned to is at least as good as a best object among the remaining objects after the agents in L_{t-1} are removed. ■

Lemma 4 *If S* blocks *μ, then $S \cap L_0 = \emptyset$.*

Proof. Suppose S *blocks* μ via some η. First, we show $S \setminus L_0 \neq \emptyset$. Suppose $S \subseteq L_0$. Thus, each strictly better off (under η than under μ) agent is also in L_0. Since L_0 consists of a series of *covered minimal self-mapped* sets, each of which is removed in some step k in the TTC algorithm: $L_0 = \cup_{k=1}^{K} T_k$, where each T_k is a *covered minimal self-mapped* set. Now, pick one of those strictly better off (under η than under μ) agents in L_0. She is removed at some step $k' > 1$, and she is assigned, under η, to an object that is removed in some earlier step $k'' < k'$. Since there is no endowment update within these K stages, some other agent, say agent j, who is removed at step k'' is assigned, under η, to an object that is outside the endowments of agents in $\cup_{k=1}^{k''} T_k$. Agent j is in S and worse off under η than under μ. This contradicts our supposition that S *blocks* μ via η. Thus, $S \setminus L_0 \neq \emptyset$. Now, suppose $S \cap L_0 \neq \emptyset$. Since $S \setminus L_0 \neq \emptyset$, an agent in $S \cap L_0$ is assigned, under η, to an object that is outside the original endowments of all agents in L_0. But, since L_0 is a union of consecutive *covered minimal self-mapped* sets, this agent is worse off under η than under μ. This contradicts our supposition that S *blocks* μ via η. Thus, $S \cap L_0 = \emptyset$. ■

Proof of part (i) of Definition 2: We show that there is no S that *strictly blocks* μ. Suppose μ is *strictly blocked* by some coalition S via η. Thus, each agent in S is assigned, under η, to some object that is strictly better than the object she is assigned to under μ. When S *strictly blocks* μ via η, it also *blocks* μ via η. By Lemma 4, $S \cap L_0 = \emptyset$. There exits a $k \geq 1$ such that $S \cap \bigcup_{t=0}^{k-1} L_t = \emptyset$ and $S \cap L_k \neq \emptyset$. Consider the following two exhaustive cases:

Case 1. No agent in $\bigcup_{t=0}^{k-1} L_t$ has an updated endowment (before L_k is removed) which is originally endowed by some agent in S. Thus, the original endowments of agents in $S \cap L_k$ are not removed before L_k is removed. Thus, each agent in $S \cap L_k$ is assigned, under μ, to a best object among the remaining objects after $\bigcup_{t=0}^{k-1} L_t$ is removed, since $S \cap L_k \subseteq L_k$. Also, each agent in $S \cap L_k$ is assigned, under η, to an object which is an original endowment of some agent in S, since S is a coalition and $S \cap L_k \subseteq S$. By the supposition of this case, for an agent in $S \cap L_k$ original endowment of some agent in S can be at most a best object among the remaining objects after $\bigcup_{t=0}^{k-1} L_t$ is removed. Thus, for an agent $i \in S \cap L_k$, it is not possible to have $\eta(i)\ P_i\ \mu(i)$.

Case 2. There is an agent in $\bigcup_{t=0}^{k-1} L_t$ who has an updated endowment (before L_k is removed) which is originally endowed by some agent in S. Denote the set of such agents in S by $\hat{S}$. Since the coalition S creates a cycle via η, there is an agent $i \in \hat{S}$, who is assigned, under η, to an object which is the original endowment of some agent $j \in S \setminus \hat{S}$. By the definition of $\hat{S}$, agent j's original endowment is among the remaining objects after $\bigcup_{t=0}^{k-1} L_t$ is removed. Thus, agent i is assigned, under η, to an object that cannot be strictly better than a best object among the remaining objects after $\bigcup_{t=0}^{k-1} L_t$ is removed. By Lemma 3, each agent in $\hat{S}$, thus agent i as well, is assigned, under μ, to an object that is at least as good as a best object among the remaining objects after $\bigcup_{t=0}^{k-1} L_t$ is removed. Thus, for agent $i \in \hat{S} \subset S$, it is not possible to have $\eta(i)\ P_i\ \mu(i)$. ■

Proof of part (ii) **of Definition 2:** Suppose μ does not satisfy part (ii) of Definition 2. Denote this supposition with $SUPP$. Under $SUPP$, there exists an S which *blocks* μ via some η, for which there is no $C(S)$ that *counter-blocks* η.[10] Thus, there is no $C(S)$ such that $C(S)$ *blocks* η via some μ' such that (i) $S \cap C(S) \neq \emptyset$, and (ii) for each $i \in C(S)$, $\mu'(i)\ I_i\ \mu(i)$. Under this supposition $SUPP$, we show

$$S \cap \bigcup_{t=1}^{M} V_t \cup L_t = \emptyset \tag{1}$$

[10]If there is no S that *blocks* μ, then (ii) is already satisfied. Thus, we assume that there is some S that *blocks* μ.

through proof by induction. Lemma 4 ($S \cap L_0 = \emptyset$) and Equation 1 together imply

$$S \cap [L_0 \cup \bigcup_{t=1}^{M} V_t \cup L_t] = \emptyset$$

But since $L_0 \cup \bigcup_{t=1}^{M} V_t \cup L_t = N$, this implies $S \cap N = \emptyset$, which is a contradiction. Thus, this contradiction implies that our supposition $SUPP$ cannot be true, proving that μ satisfies part (ii) of Definition 2.

In what follows, we assume that S *blocks* μ via some assignment η. We now prove Equation 1 in a series of lemmas, Lemma 5 through Lemma 9.

Lemma 5 *Suppose* $S \cap \bigcup_{\tau=0}^{t} V_\tau \cup L_\tau = \emptyset$ *and* $S \cap (V_{t+1} \cup L_{t+1}) \neq \emptyset$. *If* $i \in S$ *with* $\eta(i)\ P_i\ \mu(i)$, *then* $i \in S \setminus (V_{t+1} \cup L_{t+1})$.[11]

Proof. Pick an agent $i \in S$ with $\eta(i)\ P_i\ \mu(i)$. First note that, each agent in V_{t+1} is assigned, under μ, to one of her best objects among the remaining objects after L_t is removed, by Lemma 2. Thus, $i \notin V_{t+1}$. Now, suppose $i \in L_{t+1}$. Note that each agent in L_{t+1} is removed as part of a *covered minimal self-mapped* set given the endowment profile after the updates in V_{t+1}. Since $i \in L_{t+1}$ and i is strictly better off under η than under μ, she is assigned, under η, to an object which is removed as part of a *covered self-mapped* set that is removed before she is removed. Thus, there is an agent, say j, in a previously removed *covered self-mapped set* T, who is assigned, under η, to an object outside this *covered self-mapped set* T. Agent j is in S and is worse off under η than under μ. This contradicts with S *blocks* μ via η. Thus, $i \notin L_{t+1}$. Thus, $i \in S \setminus (V_{t+1} \cup L_{t+1})$. Thus, each strictly better off (under η than under μ) agent in S is in $S \setminus (V_{t+1} \cup L_{t+1})$. ■

Given an initial endowment profile e, an allocation μ essentially reallocates the original endowments among the agents. This can be described as a collection of disjoint cycles, where each cycle has some agents and only the original endowments of those agents in that cycle. Thus, a cycle is a swap of original endowments of those agents in that cycle among those agents only.

For a given allocation μ, let $\{C_1^\mu, C_2^\mu, ..., C_Y^\mu\}$ be the set of cycles that **describe** μ, where for each $y = 1, 2, ..., Y$, $\mu(C_y^\mu) = e(C_y^\mu)$ and μ is obtained by carrying out all of these cycles. Note that these cycles are disjoint. Also, each agent $i \in N$ leaves the TTC algorithm as part of some cycle C_y^μ

[11] By Lemma 4, $S \cap L_0 = \emptyset$. Thus, if $i \in S$ and $\eta(i)\ P_i\ \mu(i)$, then $i \in S \setminus (V_1 \cup L_1)$.

where $y \in \{1, 2, ...Y\}$. This cycle C_y^μ may have some agent whose endowment has been updated, or may have no agent with an updated endowment. We emphasize this distinction: If for some t, $C_y^\mu \cap V_t \neq \emptyset$, then, we call cycle C_y^μ a C^u**-type cycle**; otherwise, we call it a C^n**-type cycle**.[12]

When an assignment μ is *blocked* by a coalition S via some η, we assume that under η,
(i) each agent in the coalition S is assigned to the original endowment of another agent in S,
(ii) each agent in a cycle C_y^μ, with $C_y^\mu \cap S = \emptyset$, is assigned to the object she is assigned to under μ,
(iii) each agent in $C_y^\mu \setminus S$, with $C_y^\mu \cap S \neq \emptyset$, is assigned to her original endowment. We say that a cycle C_y^μ, with $C_y^\mu \cap S \neq \emptyset$ is **broken** by S via η, equivalently, S **breaks** C_y^μ via η.

Lemma 6 *Suppose $S \cap (V_1 \cup L_1) \neq \emptyset$. Then, each agent in $L_1 \setminus S$ who belongs to some C^u-type cycle and each agent in $V_1 \setminus S$ is assigned, under η, to her original endowment. Each agent in $L_1 \setminus S$ who is in some C^n-type cycle is assigned, under η, to either her original endowment (if this cycle intersects with S), or her assignment under μ (if this cycle does not intersect with S).*

Proof. It cannot be the case that $S \cap (V_1 \cup L_1) \subseteq C_y^\mu$ where $C_y^\mu \subset L_1$ is some C^n*-type* cycle. Suppose otherwise. By Lemma 5, $S \setminus (V_1 \cup L_1) \neq \emptyset$. Thus, an agent in $S \cap C_y^\mu$ is assigned, under η, to an object which is the original endowment of some agent outside of C_y^μ. But since cycle C_y^μ is a *minimal self-mapped* set, this agent is worse off under η than under μ. Thus, S cannot *block* μ. Thus, $S \cap (V_1 \cup L_1)$ has a nonempty intersection with some C^u*-type* cycle. Thus, S *breaks* a cycle which includes entire V_1 via η. S also *breaks* each C^u*-type* cycle in L_1, via η. Thus, each agent in $L_1 \setminus S$ who belongs to some C^u*-type* cycle and each agent in $V_1 \setminus S$ is assigned, under η, to her original endowment.[13] ■

In terms of the intersection of the cycles, C_y^μ, with L_t and V_t, there is a crucial difference between $t = 1$ and $t > 1$: For $t = 1$, a C^u-type cycle that is *broken* by S via η cannot have a nonempty intersection with L_0. But for $t > 1$ and $0 < \tau < t$, a C^u-type cycle that is *broken* by S via η can have a nonempty intersection with $V_\tau \cup L_\tau$.

For Lemma 7 below, let $\{C_1^\mu, C_2^\mu, ..., C_Y^\mu\}$ be the set of cycles that *describe* μ. Note that for any agent $i \in N$, there is some C_y^μ such that $i \in C_y^\mu$: any agent i belongs to one of the cycles that *describe* μ.

Lemma 7 *Suppose $S \cap \bigcup_{\tau=0}^{t} V_\tau \cup L_\tau = \emptyset$ and $S \cap (V_{t+1} \cup L_{t+1}) \neq \emptyset$. Suppose for each $z = 1, ..., Z$, $C_{y_z}^\mu$ has a nonempty intersection with $S \cap (V_{t+1} \cup L_{t+1})$. Each agent in $(\cup_{z=1}^{Z} C_{y_z}^\mu) \setminus S$ is assigned, under η,*

[12]Note that a cycle of either type is not necessarily a subset of some L_t.

[13]The latter part of the argument in the lemma holds by the assumption on how η assigns objects to $N \setminus S$.

to her original endowment.[14]

Proof. Since S *breaks* the cycles $\{C^{\mu}_{y_1}, C^{\mu}_{y_2}, ..., C^{\mu}_{y_Z}\}$ via η, each agent in $C^{\mu}_{y_z} \setminus S$, with $C^{\mu}_{y_z} \cap S \neq \emptyset$, is assigned to her original endowment under η. Thus, the result follows. ■

We are now ready to prove Equation 1, $S \cap \bigcup_{t=1}^{M} V_t \cup L_t = \emptyset$, through an induction argument: We first show that Equation 1 holds for $t = 1$ (Lemma 8 below). Then, we assume that Equation 1 holds for t, and we show that it also holds for $t + 1$ (Lemma 9 below).

Lemma 8 $S \cap (V_1 \cup L_1) = \emptyset$.

Proof. Suppose $S \cap (V_1 \cup L_1) \neq \emptyset$. We show that there is a coalition $C(S)$, which *counter-blocks* η, that is, $C(S)$ *blocks* η via $\mu' \in M_{C(S)}(\mu)$ with $S \cap C(S) \neq \emptyset$.

Let $C(S) = V_1 \cup L_1$. Define μ' as follows: for each $i \in L_1$, let $\mu'(i) = \mu(i)$, and for each $i \in V_1 \setminus L_1$, let $\mu'(i) = u^{V_1}(i)$, where $u^{V_1}(i)$ is the updated endowment of agent i at the end of all updates in V_1. Note that $e(V_1 \cup L_1) = \mu'(V_1 \cup L_1)$. Thus, $C(S) = V_1 \cup L_1$ is a coalition that assigns all of its original endowments among its members. By our supposition, $S \cap C(S) \neq \emptyset$. By definition of μ', for each agent $i \in L_1$, $\mu'(i)\ I_i\ \mu(i)$. For each agent $i \in V_1 \setminus L_1$, the updated endowment $u^{V_1}(i) = \mu'(i)$, is one of her best objects among the remaining objects after L_0 is removed. By Lemma 2, each agent in V_1 is assigned, under μ, to one of her best objects among the remaining objects after L_0 is removed. Thus, for each agent $i \in V_1 \setminus L_1$, we have $\mu'(i)\ I_i\ \mu(i)$. Thus, $\mu' \in M_{V_1 \cup L_1}(\mu)$.

We show that $C(S) = V_1 \cup L_1$ *blocks* η via μ. By Lemma 5, if for an agent i, $\eta(i)\ P_i\ \mu(i)$, then $i \in S \setminus (V_1 \cup L_1)$. Thus, no agent in $C(S) = V_1 \cup L_1$ is strictly better off under η than under μ. Since $\mu' \in M_{V_1 \cup L_1}(\mu)$, no agent in $C(S) = V_1 \cup L_1$ is strictly better off under η than under μ'. Now, to show that $C(S) = V_1 \cup L_1$ *blocks* η via μ, we show that there is at least one agent in $C(S) = V_1 \cup L_1$ who is strictly better off under μ' than under η. Such an agent can be only in $(V_1 \cup L_1) \setminus S$. Thus, consider the agents in $(V_1 \cup L_1) \setminus S$. By Lemma 6, under η, each agent in $V_1 \setminus S$ is assigned to her original endowment. Also, each agent in $L_1 \setminus (S \cup V_1)$ who belongs to some C^u-type cycle is assigned, under η, to her original endowment. Each agent in $L_1 \setminus (S \cup V_1)$ who belongs to some C^n-type cycle is assigned, under η, to either her original endowment (if this cycle intersects with S), or her assignment

[14] Under η, each agent in L_{t+1} who belongs to some C^u*-type* cycle and each agent in V_{t+1} is assigned to her endowment. Each agent in L_{t+1} who belongs to some C^n*-type* cycle is assigned to her assignment under μ (if this cycle does not intersect with S) or to her endowment (if this cycle intersects with S). The reasoning is similar to that of Lemma 6.

under μ (if this cycle does not intersect with S).[15] Thus, an agent in $(V_1 \cup L_1) \setminus S$ is assigned, under η, either to her original endowment or to her assignment under μ. Suppose that each agent in $(V_1 \cup L_1) \setminus S$ who is assigned to her original endowment under η is indifferent between her original endowment and her assignment under μ.[16] Thus, each agent in $(V_1 \cup L_1) \setminus S$ is indifferent between μ and η. Also, each agent in S is weakly better off under η than under μ and one agent in $S \setminus (V_1 \cup L_1)$ is strictly better off under η than under μ. Also, no agent in $N \setminus (S \cup V_1 \cup L_1)$ is worse off under η than under μ. Otherwise, there is an agent $j \in N \setminus (S \cup V_1 \cup L_1)$ who is worse off under η than under μ. Agent j a member of some cycle C_y^μ (among the cycles that *describe* μ) that is *broken* by S.[17] Since S *breaks* C_y^μ, $C_y^\mu \cap S \neq \emptyset$. Also, C_y^μ allocates its own original endowments among its members via μ, by the definition of cycles that *describe* μ. Thus $\mu \in M_{C_y^\mu}(\mu)$. Agent j is strictly better off under μ than under η. Since, each agent in $C_y^\mu \setminus S$ is assigned, under η, to her original endowment, no agent in C_y^μ is worse off under μ than under η. Thus, C_y^μ *blocks* η via $\mu \in M_{C_y^\mu}(\mu)$ with $C_y^\mu \cap S \neq \emptyset$. Thus, C_y^μ *counter-blocks* η. This contradicts our supposition $SUPP$. Thus, no agent in $N \setminus (S \cup V_1 \cup L_1)$ is worse off under η than under μ. Thus η *Pareto dominates* μ, which contradicts with TTC algorithm producing a *Pareto efficient* allocation.[18]

Thus, there is an agent in $(V_1 \cup L_1) \setminus S$, who is assigned to her original endowment under η and her original endowment is worse than her assignment under μ. Thus, this agent is worse off under η than under μ. Since $\mu' \in M_{V_1 \cup L_1}(\mu)$, this agent is strictly better off under μ' than under η. Thus, $C(S) = V_1 \cup L_1$ *blocks* η via $\mu' \in M_{C(S)}(\mu)$ with $S \cap C(S) \neq \emptyset$. Thus, $C(S) = V_1 \cup L_1$ *counter-blocks* η. This contradicts with our supposition $SUPP$, proving $S \cap (V_1 \cup L_1) = \emptyset$. ■

Lemma 9 *If for each $\tau \leq t$, $S \cap (V_\tau \cup L_\tau) = \emptyset$, then $S \cap (V_{t+1} \cup L_{t+1}) = \emptyset$.*

Proof. Suppose for each $\tau \leq t$, $S \cap (V_\tau \cup L_\tau) = \emptyset$ and $S \cap (V_{t+1} \cup L_{t+1}) \neq \emptyset$. We show that there is a $C(S)$, which *counter-blocks* η. Suppose $\{C_1^\mu, C_2^\mu, ..., C_Y^\mu\}$ is the set of cycles that *describe* μ. Let $\{C_{y_1}^\mu, C_{y_2}^\mu, ..., C_{y_Z}^\mu\}$ be the set of all cycles that have a nonempty intersection with $S \cap (V_{t+1} \cup L_{t+1})$.[19]

[15] An agent who is assigned, under η, to her assignment under μ, can only be in $L_1 \setminus (S \cup V_1)$ since each agent in V_1 is removed as part of a C^u-*type* cycle.

[16] Since TTC algorithm produces individually rational assignments, μ is also individually rational. Thus, $e(i)\ P_i\ \mu(i)$ is not possible for any i.

[17] If C_y^μ is a cycle that is *not broken* by η, then each agent in C_y^μ is assigned to her assignment under μ. Thus, no such agent is worse under η than under μ, which is a contradiction.

[18] *Pareto efficiency* of TTC is shown in Saban and Sethuraman (2013).

[19] There exists at least one such cycle. This follows from the fact that each agent belongs to some cycle C_y^μ, which is among the cycles that *describe* μ, and that $S \cap (V_{t+1} \cup L_{t+1}) \neq \emptyset$.

Let $C(S) = \cup_{z=1}^{Z} C_{y_z}^{\mu}$ and $\mu'(i) = \mu(i)$ for all $i \in C(S)$. Note that $e(C(S)) = \mu'(C(S))$. Also, we have $\mu' \in M_{C(S)}(\mu)$ and $C(S) \cap S \neq \emptyset$. By Lemma 7, each agent in $C(S) \setminus S$ is assigned, under η, to her original endowment. Suppose each agent in $C(S) \setminus S$ is indifferent between her original endowment and her assignment under μ. Thus, no agent in $C(S)$ is worse off under η than under μ. Since S *blocks* μ via η, each agent in S is weakly better off under η than under μ and one agent in S is strictly better off under η than under μ. Also, no agent in $N \setminus (S \cup C(S))$ is worse off under η than under μ. Otherwise, there is an agent $j \in N \setminus (S \cup C(S))$ who is worse off under η than under μ. Agent j is a member of some cycle $C_{\hat{y}}^{\mu}$ that is *broken* by S.[20] With a similar argument as in the proof of Lemma 8, C_y^{μ} *counter-blocks* η. This contradicts our supposition $SUPP$. Thus, no agent in $N \setminus (S \cup C(S))$ is worse off under η than under μ. Thus, η *Pareto dominates* μ, which contradicts with TTC algorithm producing a *Pareto efficient* allocation. Thus, there is at least one agent in $C(S)$ who is worse off under η than under μ. Thus, $C(S) = \cup_{z=1}^{Z} C_{y_z}^{\mu}$ *blocks* η via $\mu' \in M_{C(S)}(\mu)$ with $S \cap C(S) \neq \emptyset$. Thus, $C(S) = \cup_{z=1}^{Z} C_{y_z}^{\mu}$ *counter-blocks* η. This contradicts with our supposition $SUPP$, proving $S \cap (V_{t+1} \cup L_{t+1}) = \emptyset$. ■

Induction arguments proven in Lemma 8 and Lemma 9 together imply

$$S \cap \bigcup_{t=1}^{M} V_t \cup L_t = \emptyset$$

By Lemma 4, $S \cap L_0 = \emptyset$. Thus,

$$S \cap [L_0 \cup \bigcup_{t=1}^{M} V_t \cup L_t] = \emptyset$$

Since $N = L_0 \cup \bigcup_{t=1}^{M} (V_t \cup L_t)$, we get $S \cap N = \emptyset$, which is a contradiction. Thus, our initial supposition $SUPP$ does not hold. Thus, μ satisfies part (ii) of Definition 2. Thus, $\mu \in B(e, R)$. This proves that for each assignment problem (e, R), $TTC(e, R) \subseteq B(e, R)$. ■

Appendix D $TTC(e, R) \subsetneq B(e, R)$ for some (e, R).

To show that the inclusion in Theorem 2 is strict for some assignment problem, we construct an assignment problem (e, R) such that $B(e, R) \subseteq TTC(e, R)$ does not hold.

Let $N = \{i, j, k, l, m, i_1, i_2\}$ and $O = \{x, y, z, a, b, e_1, e_2\}$.[21] The preferences are given below with

[20] $C_{\hat{y}}^{\mu}$ is among the cycles that *describe* μ, with $\hat{y} \notin \{y_1, ..., y_Z\}$.

[21] For expositional convenience, we deviate from the notational convention we have used in previous tables and use a mixed notation instead of $e(i_k) = o_k$ for each agent i_k.

each set in the table being an indifference set and a square box standing for the endowment of the corresponding agent:

$\underline{R_i}$	$\underline{R_j}$	$\underline{R_k}$	$\underline{R_l}$	$\underline{R_m}$	$\underline{R_{i_1}}$	$\underline{R_{i_2}}$
$\{y, z\}$	$\{e_1\}$	$\{e_2\}$	$\{b, x, z\}$	$\{x\}$	$\{x\}$	$\{x\}$
$\boxed{x}$	$\{a\}$	$\{b\}$	$\boxed{a}$	$\boxed{b}$	$\{e_2\}$	$\{e_1\}$
	$\boxed{y}$	$\boxed{z}$			$\boxed{e_1}$	$\boxed{e_2}$

We claim that the assignment μ (colored red in the table) is in the *bargaining set*. To prove this, we use the graph in Figure 1, where each agent points to her best objects, each object points to its owner, and solid edges represent best objects while dashed edges represent second-best objects except endowments. Dashed edges also represent a crucial feature of μ: an edge is a dashed line if and only if it starts with an agent such that she is assigned her second-best object and that object is not her endowment.

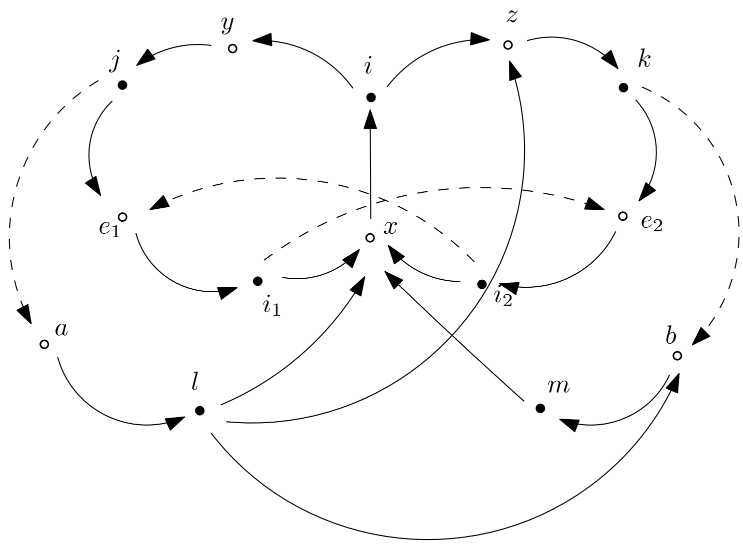

Figure 1: Initial graph

Let S block μ. We now find all possible blocking coalitions S. Note that each agent is assigned either her best or second-best object. Thus, any blocking coalition is a cycle in this figure including either solid or dashed edges. By definition of *blocking*, any such cycle has the following properties: (1) The cycle should contain at least one agent assigned to her second-best object at μ (in Figure 1, these are the vertices starting at a dashed edge, thus, agents j, k, i_1 and i_2) such that she is the starting vertex of a solid edge in the cycle. (2) Any agent who is not starting a dashed edge in Figure 1 should be starting a solid edge in the cycle. Next we find these cycles by checking four possible cases each representing one of the agents in (1) being better off at the new assignment.

Blocking coalitions with agent j better off: $\{j, i_1, i\}$ and $\{j, i_1, i_2, i\}$.

Blocking coalitions with agent k better off: $\{k, i_2, i\}$, $\{k, i_2, i_1, i\}$, $\{k, i_2, i, j, l\}$ and $\{k, i_2, i_1, i, j, l\}$.

Blocking coalitions with agent i_1 better off: $\{i_1, i, j\}$, $\{i_1, i, j, l, k, i_2\}$ and $\{i_1, i, k, i_2\}$.

Blocking coalitions with agent i_2 better off: $\{i_2, i, k\}$, $\{i_2, i, j, i_1\}$ and $\{i_2, i, j, l, k\}$.

Since some coalitions are such that multiple agents are better off after blocking, clearly, some of the above blocking coalitions above overlap. There are in total six blocking coalitions:

$S_1 = \{i, j, i_1\}$

$S_2 = \{i, k, i_2\}$

$S_3 = \{i, j, i_1, i_2\}$

$S_4 = \{i, k, i_1, i_2\}$

$S_5 = \{i, j, k, l, i_2\}$

$S_6 = \{i, j, k, l, i_1, i_2\}$

The assignment μ is given by two trading cycles: $\{i_1, i_2\}$ and $\{i, j, l, k, m\}$. By definition of the assignment after blocking, any agent who is not a member of a blocking coalition is assigned to her endowment after blocking. For $n = 1, \ldots, 6$, let S_n block μ via η_n. Moreover, by definition of *counter-blocking*, each agent in the counter-blocking coalition is indifferent between the object she is assigned after counter-blocking and the object she is assigned under μ. The assignments η_1 and η_3 are counter-blocked by $C_1 = \{i, k, m\}$. To see this, note that agents k and m are assigned their endowments at both η_1 and η_3 after blocking by S_1 and S_3, respectively. Also, agent i is assigned a best object both at μ and the assignment after agents $\{i, k, m\}$ *block* η_1 and η_3. By the same argument, assignments η_2 and η_4 are counter-blocked by $C_2 = \{i, j, l\}$. Again, note that each agent in C_2 is assigned after counter-blocking an object indifferent to the one she is assigned under μ. Finally, assignments η_5 and η_6 are *counter-blocked* by $C_3 = \{i, j, l, m\}$.

The assignment μ is such that whenever a coalition blocks it via an assignment, the latter is *counter-blocked* by a another coalition. Thus, μ is in the *bargaining set*. To complete the proof we need to show that μ is not an outcome of the TTC class for this problem. This can be seen from Figure 1.

By definition of the TTC, a *minimally self-mapped* set is chosen at the first step, when agents point to their best objects. Thus, in Figure 1, this is the subgraph of with only solid edges. The only *minimally self-mapped set* in this subgraph consists of agents i, j, k, i_1 and i_2. It is not *covered* since object x is the only best object for both agents i_1 and i_2. Thus, the TTC algorithm chooses a cycle in this *minimally self-mapped* set to assign them their best objects. There are two such cycles: $\{i, j, i_1\}$ and $\{i, k, i_2\}$. But, each of these cycles contains at least one agent, who is assigned a worse object than the object she points to. This implies that $\mu \notin TTC(e, R)$. This completes the proof.

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