School Choice under Partial Fairness*

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Abstract

Some school districts have been considering recently to allow violations of priorities at certain schools to improve students' welfare. Inspired by this, we generalize the school choice problem by allowing such violations. We characterize the set of constrained efficient outcomes for a school choice problem in this setting. We introduce a class of algorithms, denoted *Student Exchange under Partial Fairness (SEPF)*, which guarantees to find a constrained efficient matching for any problem. Moreover, any constrained efficient matching which Pareto dominates the Student Optimal Stable Matching can be obtained via an algorithm within the SEPF class. A similar approach to improve students' welfare is to ask students' consent for violation of their priorities (Kesten, 2010). The idea is that each student weakly benefits from this weakening of stability. Clearly, this welfare gain depends on students' having incentives to consent. We identify the unique rule, the *Top Priority rule*, within the SEPF class, which gives each student incentive to consent. This uniqueness result implies that it is equivalent to the generalized version of Kesten's EADAM (Efficiency Adjusted Deferred Acceptance Mechanism) algorithm (Kesten, 2010), thus justifying the seemingly ad hoc construction of EADAM.

Keywords : School choice, stability, efficiency

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1 Introduction

School choice has become an important policy tool for school districts in providing the parents with the opportunity to choose their child's school. School districts adopting public school choice programs allow parents to select schools in other residence areas as well. However, school capacities are limited and it is clearly not possible to enroll each student in his or her first choice school, indicating that achieving efficiency is nontrivial. Moreover, the schools' priorities over the students should be respected, which indicates that achieving fairness is nontrivial as well. Therefore, the central issue in school choice is the design of mechanisms for assigning the schools to students so that efficiency and fairness criteria are met. But, unfortunately, it is usually not possible to meet the requirements of efficiency and fairness at the same time. The current work addresses the school choice problem by focusing on the mechanisms compromising between efficiency and fairness.

In a school choice problem, students submit their preferences over a list of schools to a central placement authority and the authority decides on the assignment based on schools' priorities over the students. A school choice mechanism is a systematic way of matching students with schools for each school choice problem. However, there are several concerns and most of the time, it is impossible to design a mechanism to achieve all of these goals. A major concern is fairness in the sense that students' priorities at schools should not be violated: at the matching chosen by the central authority, there shouldn't be a student who prefers a school, say s, to her assigned school and another student with lower priority at s who is assigned to s. There are mechanisms which always select fair matchings. The well-known student-proposing deferred acceptance (DA) mechanism is such an example. The student-proposing DA gives the student-optimal stable matching (SOSM) (Gale and Shapley, 1962). Actually, the SOSM not only prevents priority violations, but also is the best matching in terms of students' welfare among all the matchings without any priority violation; that is, each student prefers the SOSM to any matching without any priority violation (Gale and Shapley, 1962; Balinski and Sönmez, 1999). Furthermore, the student-proposing DA

mechanism is not open to strategic manipulations by the students: revealing preferences truthfully is a dominant strategy for each student (Dubins and Freedman, 1981; Roth, 1982). However, there is a serious drawback of the SOSM: there might be another matching which is preferred by each student to the SOSM, i.e., it is not *Pareto efficient* (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Although the extent of this inefficiency depends on schools' priorities and students' preferences, from a theoretical perspective, the welfare loss can be quite large (Kesten, 2010). Moreover, it is observed that, based on the student preference data from the NYC school district, potential welfare gains over the SOSM are significant (Abdulkadiroğlu, Pathak, and Roth, 2009). The welfare loss under the SOSM is actually a deeper issue. Fairness and efficiency are incompatible: a fair and efficient matching does not exist in general (Gale and Shapley, 1962; Roth, 1982; Balinski and Sönmez, 1999). This incompatibility naturally raises the question of how to compromise either one of these properties to avoid high extents of priority violations or welfare losses. Our approach is to improve students' welfare by considering certain priority violations, which are either allowed by school districts or consented by students themselves.

Some school districts have recently been considering to allow some priority violations to improve students' welfare (Abdulkadiroğlu, 2011). One such example is a setting where the centralization of assignments to public and private (or exam and regular (Abdulkadiroğlu, 2011)) schools is possible. Whereas the priorities to public (exam) schools are legal constraints which cannot be violated, the private (regular) schools are more flexible in terms of their priorities (and efficiency is more of a first-order concern for these schools). In this case, school choice problem becomes the problem of designing compromise mechanisms to improve students' welfare while still respecting the priority violation constraints where such violations are not allowed (*partial fairness*) (Section 3).

Another approach towards this goal is to ask students' consent for priority violations and design a mechanism based on the consent of students (Kesten, 2010). The idea here is to design mechanisms such that students have incentives to consent for their priorities to be violated and each student's welfare can be weakly improved thanks to the relaxation of

fairness due to students' own consent for it. This approach has additional strategic component (compared to the case where the acceptable priority violations are set by school districts) in terms of each student's decision on whether to consent or not (Section 4).

An important point in these two approaches is not to ignore priorities completely for which violations are allowed, but rather to ignore them only if they lead to welfare losses.[1] Our first contribution is to propose a general class of mechanisms, the *Student Exchange under Partial Fairness (SEPF)*, for the school choice problem with priority violations. Each rule in our class gives a *partially fair* matching, and is not *Pareto dominated* by another *partially fair* matching (that is, *constrained efficient* in the class of *partially fair* matchings). Moreover, each such matching can be obtained by the class we propose (Theorem 1).

Our proposal is not the first focusing on *partially fair* matchings. Another such mechanism, the *Efficiency Adjusted Deferred Acceptance Mechanism (EADAM)* (Kesten, 2010), is based on students' consent for priority violations rather than taking the acceptable priority violations as given. The EADAM also finds a *partially fair* and *constrained efficient* matching.[2] By our characterization result (Theorem 1), the EADAM is in the SEPF class (Proposition 1). Second, we point out a particular rule, the *Top Priority Rule*, in the SEPF class. For the interpretation of priority violations with consent, this rule satisfies an important property: a consenting student is never better off by instead not consenting. This is an indispensable incentive-compatibility property, which assures that the idea of consent is operational. It's useful to point out that the EADAM also satisfies this property. Our second contribution is to show that the Top Priority Rule is the unique *partially fair* and *constrained efficient* rule which gives students incentives to consent (Theorem 2). An immediate corollary to our theorem is the equivalence of the EADAM and Top Priority Rule.

Even though the Top Priority Rule is immune to violations through consenting, in general, no cycle selection rule within the SEPF is immune to violations through misrepresentation

[1] We discuss this issue further in Sections 6.1 and 5.

[2] In our model, students can give consent for any set of schools, whereas in Kesten (2010) whenever a student consents, she consents for all schools. We introduce a trivial generalization of the EADAM to capture this extension and this generalized version selects *partially fair* and *constrained efficient* outcome (Appendix C).

of preferences (Proposition 4). This incompatibility is indeed more general: a *constrained efficient* mechanism can never be *strategy-proof* (Theorem 3).[3]

Related literature

The school choice problem is introduced by Abdulkadiroğlu and Sönmez (2003). Since then, one of the main questions has been the inefficiency of the DA. It has been demonstrated that theoretically, the level of inefficiency can be quite high (Kesten, 2010) and there is empirical support for this insight: in NYC high school match, possible welfare gains over the SOSM is significant (Abdulkadiroğlu, Pathak, and Roth, 2009). Since efficiency and fairness are incompatible in the school choice context, the only remedy for this problem is to relax fairness. One alternative in this direction is to focus on efficiency via Top Trading Cycles (TTC) mechanisms (Abdulkadiroğlu and Sönmez (2003), Hakimov and Kesten (2014), Morrill (2015a), Morrill (2015b)).

The second alternative is to weaken the fairness notion. Alcade and Romero-Medina (2015) introduce such a weakening: a matching with a priority violation is not deemed as unfair if student's objection to that priority violation is counter-objected by another student. The authors propose the Student Optimal Compensating Exchange mechanism to give a fair (in this weak sense) and efficient matching. A different approach to weaken fairness is to consider certain priority violations as acceptable. Such an example is NYC school match: motivated by the observation that the efficiency of the DA is significant, school districts have been considering to allow such violations anywhere but exam schools (Abdulkadiroğlu, 2011).

A different interpretation of acceptable priority violations is the proposal by (Kesten, 2010): ask students to consent for violations of their priorities. The author develops a mechanism (the EADAM) which guarantees each student that she will not be worse off by consenting. It is also shown that students assigned to certain schools (*underdemanded* schools[4]) are not Pareto improvable at the SOSM and the EADAM can be redefined by

[3]Kesten (2010) shows any mechanism which selects a *Pareto efficient* matching *Pareto dominating* SOSM cannot be *strategy-proof*. Since any such mechanism is *constrained efficient*, our imposibility result is stronger than Kesten's.

[4]See Section 3.3 for a discussion of underdemanded schools.

taking these schools into account (Tang and Yu, 2014). Moreover, the EADAM outcome is supported as the strong Nash Equilibrium of the preference revelation game under the DA (Bando, 2014). In the affirmative action context, a variant of the EADAM is recently proposed as a *minimally responsive* rule (that is, a rule such that changing the affirmative action parameter in favor of the minorities never results in a matching which makes each minority student weakly worse off) (Doğan, 2015).

The idea that possible welfare gains can be captured by improvement cycles is first discovered by Erdil and Ergin (2008) in the context of coarse priorities of schools. This idea inspired the mechanisms proposed in some other works (Ehlers, Hafalir, Yenmez, and Yildirim (2014), Abdulkadiroğlu (2011)) and our work as well.

The rest of this paper is structured as follows: Section 2 introduces the notion of school choice problem with priority violation. Section 3 introduces SEPF and characterizes its properties. Section 4 introduces TP Rule and demonstrates the associated no-consent-proofness properties. Section 5 discusses manipulation through preference misrepresentations, and Section 6 discusses extensions related to relaxation of improvement of SOSM, and priority structures that allow for indifferences. Section 7 concludes.

2 The Model

We first present the standard school choice problem and a simple example demonstrating that efficiency and fairness are incompatible. We then introduce the extended model with priority violations.

2.1 School Choice Problem

A **school choice problem** (introduced by Abdulkadiroğlu and Sönmez (2003)) consists of the following elements:

- a finite set of students $I = \{i_1, i_2, ..., i_n\}$,
- a finite set of schools $S = \{s_1, s_2, ..., s_m\}$,

- a strict priority structure of schools $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the complete priority order of school s over I,
- a capacity vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats at school s,
- a strict preference profile of students $P = (P_i)_{i \in I}$ such that P_i is student i's preferences over $S \cup \{\emptyset\}$, where $\emptyset$ stands for the option of being unassigned to a school in S.[5]

Let R_i denote the at-least-as-good-as preference relation associated with P_i:

$$s \; R_i \; s' \Leftrightarrow s \; P_i \; s' \text{ or } s = s'.$$

A **matching** $\mu : I \to S \cup \{\emptyset\}$ is a function such that for each $s \in S$, $|\mu^{-1}(s)| \leq q_s$. A **rule** is a systematic procedure which selects a matching for each problem.

A matching μ **violates the priority** of student $i \in I$ at school $s \in S$ if there exists another $j \in I$ such that: (i) $\mu(j) = s$, (ii) $s \; P_i \; \mu(i)$, and (iii) $i \succ_s j$. A matching μ is **fair** if for each $i \in I$ and $s \in S$, it doesn't violate the priority of student i at school s. A matching μ is **individually rational** if for each $i \in I$, $\mu(i) \; R_i \; \emptyset$. A matching μ is **non-wasteful** if there does not exist a student $i \in I$ and a school $s \in S$ such that $s \; P_i \; \mu(i)$ and $|\mu^{-1}(s)| < q_s$. A matching μ is **stable** if it is (i) *fair*, (ii) *individually rational* and (iii) *non-wasteful*.

A matching μ **weakly Pareto dominates** matching μ' if for each $i \in I$, $\mu(i) \;\; R_i \;\; \mu'(i)$. A matching μ **Pareto dominates** μ' if μ *weakly Pareto dominates* μ' and for some $j \in I$, $\mu(j) \; P_j \; \mu'(j)$. A matching μ is **Pareto efficient** if there does not exist another matching $\mu' \in \mathcal{M}$ which Pareto dominates μ.

The properties of *Pareto efficiency* and *fairness* are clearly desirable from a normative point of view. Unfortunately, for some problems, a *Pareto efficient* and *fair* matching may not exist (Balinski and Sönmez, 1999). We illustrate this situation in the following example .

Example 1 *Let $S = \{s_1, s_2, s_3\}$, $I = \{i_1, i_2, i_3\}$, and let $q_s = 1$ for each $s \in S$. The preference profile and priority structure are as follows:*

[5] Alternatively one can think of $\emptyset$ as a school with $q_\emptyset = \infty$

P_{i_1}	P_{i_2}	P_{i_3}	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
s_1	s_2	s_1	i_2	i_3	i_1
s_3	s_1	s_2	i_1	i_2	i_2
s_2	s_3	s_3	i_3	i_1	i_3

There are three Pareto efficient matchings in this example:

$$\mu = \{(i_1, s_1),\ (i_2, s_2),\ (i_3, s_3)\},$$
$$\mu' = \{(i_1, s_1),\ (i_2, s_3),\ (i_3, s_2)\},$$
$$\mu'' = \{(i_1, s_3),\ (i_2, s_2),\ (i_3, s_1)\}.$$

But, each Pareto efficient matching violates the priority of a student: μ violates the priority of i_3 at s_2, μ' violates the priority of i_2 at s_1, and μ'' violates the priority of i_1 at s_1. Therefore, none of the Pareto efficient matchings are stable.

Example 1 demonstrates that in some problems each *fair* matching can be *Pareto dominated* by another matching. But, for each problem there always exists a *fair* matching which *Pareto dominates* all *fair* matchings (Gale and Shapley, 1962; Abdulkadiroğlu and Sönmez, 2003). This matching is called the *Student Optimal Stable Matching (SOSM)* and it is is determined through the following **Student-Proposing Deferred Acceptance (DA)** algorithm (Gale and Shapley, 1962).

Student-Proposing DA Algorithm:

Step 1: Each student applies to her most preferred school. Each school s tentatively accepts the best students according to its priority list, up to q_s, and rejects the rest.

Step $k > 1$: Each student rejected in Step $k-1$ applies to her next best school. Each school s tentatively accepts the best students among the new applicants and the ones tentatively accepted in step $k-1$ according to its priority list, up to q_s, and rejects the rest.[6]

[6]The SOSM for the problem in Example 1 is the matching $\tilde{\mu} = \{(i_1, s_3),\ (i_2, s_1),\ (i_3, s_2)\}$ and it is *Pareto dominated* by the matching $\mu'' = \{(i_1, s_3),\ (i_2, s_2),\ (i_3, s_1)\}$.

Suppose the SOSM is not *Pareto efficient*. Since the SOSM *Pareto dominates* each *fair* matching, students' welfare can be improved only if priorities are violated. Throughout the rest of the paper, we focus on the incompatibility between *Pareto efficiency* and *fairness* and we follow the idea of compromising *fairness* by allowing violations of some priorities. The set of acceptable priority violations is obviously not part of the standard school choice problem. We next present the necessary extension to capture this added structure.

2.2 School Choice Problem with Priority Violation

A **school choice problem with priority violation** (or simply a **problem**) is a school choice problem where acceptable priority violations are given by a correspondence $C : S \rightrightarrows I \cup \{\emptyset\}$, where $C(s)$ denotes the set of students for whom a priority violation at school s is acceptable.

Throughout the paper, we will fix $I, S, \succ$ and q for expositional simplicity. Thus, a problem is defined by a preference profile R and a correspondence C. For problem (R, C), we denote the matching selected by rule ψ with $\psi_{(R,C)}$ and the match of student i in $\psi_{(R,C)}$ with $\psi_{(R,C)}(i)$.

There are two interpretations of acceptable priority violations given by the correspondence C. The first interpretation is that school districts have been recently considering to allow certain priority violations since there are substantial efficiency losses due to those priorities (see (Abdulkadiroğlu, 2011) for such cases). Thus, the correspondence C is determined by the school districts and the only revelation made by the students is their preferences. The second interpretation is to ask students for consent for violation of their priorities (see (Kesten, 2010)). Thus, the acceptable violation of student i's priority at school s is interpreted as student i has consented for this violation. Clearly, consenting (or not consenting) for priority violations is a strategic decision and it depends on the particular school choice mechanism: if consenting for a priority violation causes a student to be assigned to a worse school than she would have without consenting, one wouldn't expect her to consent for that priority violation. The extent of such reasoning by students is indeed questionable; nevertheless, it's clear

that guaranteeing that a student would never be worse off by consenting is a reasonable and important property. In Section 4, we formalize this property and analyze mechanisms which provide incentives for consent.[7]

The priorities for which violations are acceptable (according to C) do not have to be taken into account when fairness is considered. By ignoring these priorities, weaker notions of fairness and stability can be defined as follows: A matching μ **violates the priority** of student $i \in I$ at school $s \in S$ if there exists another $j \in I$ such that: (i) $\mu(j) = s$, (ii) $s \; P_i \; \mu(i)$, (iii) $i \succ_s j$ and (iv) $i \notin C(s)$.[8] A matching μ is **partially fair** if for each $i \in I$ and $s \in S$, it doesn't *violate the priority* of student i at school s. A matching μ is **partially stable** if it is (i) *partially fair*, (ii) *individually rational* and (iii) *non-wasteful*.

A matching μ is **constrained efficient** if (i) it is *partially stable*, and (ii) it is not *Pareto dominated* by any other *partially stable* matching. We are interested in rules and mechanisms generating a *constrained efficient* matching for each instance of a school choice problem.

The introduction of acceptable priority violations and the notion of *partial stability* extends the standard school choice model as to compromise between *stability* and *Pareto efficiency* – the two notions which are incompatible with each other. Indeed, one can easily see how the interpolation between the two ends works by investigating the extremes. In one extreme, when no priority violation is acceptable (i.e. when $C(s) = \emptyset$ for all $s \in S$), the notion of *partial stability* collapses to that of *stability*. In this case, the only *constrained efficient* matching is the SOSM. In the other extreme, when each priority violation is acceptable, (i.e. when $C(s) = I$ for each $s \in S$), *partial fairness* is vacuously satisfied by each matching. In this case, *constrained efficiency* is equivalent to *Pareto efficiency*, and one can implement a *constrained efficient* matching through the Top Trading Cycles (TTC) mechanism while preserving *strategy-proofness* (Shapley and Scarf, 1974; Abdulkadiroğlu and Sönmez, 2003). Thus, one can easily see that the standard school choice problem can be embedded within

[7]The students actually reveal two pieces of information simultaneously: their preferences and the set of schools for which they consent for priority violation. This is a much complicated game and we assume away this complication in Section 4. We consider the preference revelation game in Section 5.

[8]Throughout the rest of the paper, whenever we say a matching *violates the priority* of a student at a school, we refer to this definition.

this framework.

3 The Algorithm: Student Exchange under Partial Fairness

We present a class of algorithms to characterize the set of *constrained efficient* matchings, which improve the students' welfare upon the SOSM. First, we introduce notions that we use in the definition of this class.

Given a matching μ, for each $s \in S$, let

- $D_\mu(s) = \{i \in I : s \; P_i \; \mu(i)\}$ (the set of students who prefer school s to the school to which they are assigned under μ)
- $X_\mu(s) = \{i \in D_\mu(s) : \forall j \in D_\mu(s) \setminus (C(s) \cup \{i\}),\, i \succ_s j\}$ (the set of students who are eligible for a *partially fair* exchange involving school s).[9]

Let $G = (V, E)$ be a directed graph with the set of vertices V, and the set of directed edges E, which is a set of ordered pairs of V. A **trail** is a set of edges $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ in E. A trail $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is

- a **path** if the vertices $i_1, i_2 \ldots, i_{n+1}$ are distinct,
- a **cycle** if the vertices $i_1, i_2 \ldots, i_n$ are distinct and $i_1 = i_{n+1}$,

A path $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is a **chain** if for each $j \in V$, $j i_1, i_{n+1} j \notin E$. Given a chain $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\} \subseteq E$, vertex i_1 is called the **tail**.

For each matching μ, let $G(\mu) = (I, E(\mu))$ be the **(directed) application graph** associated with μ where the set of directed edges $E(\mu) \subseteq I \times I$ is defined as follows: $ij \in E(\mu)$ (that is, i *points to* j) if and only if $s = \mu(j)$ and $i \in X_\mu(s)$.

Remark 1 *In the graph $G(\mu)$, if i points to j, then i points to each student who is assigned to the school $\mu(j)$ at the matching μ.*[10]

[9] Note that this set is always well-defined. In particular, when $D_\mu(s) \setminus C(s) = \emptyset$, we have $X_\mu(s) = D_\mu(s)$.

[10] This follows from the following: i points to j if and only if $i \in X_\mu(\mu(j))$ and thus, i also points to student i' if i' is assigned to school $\mu(j)$.

We say that **cycle** $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_1\} \subseteq E(\mu)$ **is solved** when for each $ij \in \phi$, student i is assigned to $\mu(j)$ towards a new matching. Formally, we denote the *solution of a cycle* by the operation $\circ$; that is, $\eta = \phi \circ \mu$ if and only if for each $ij \in \phi$, $\eta(i) = \mu(j)$, and for each $i' \notin \{i_1, i_2, \ldots, i_k\}$, $\eta(i') = \mu(i')$. The following class of algorithms are defined by *solving* cycles inductively in the appropriately defined graph.

SEPF (Student Exchange under Partial Fairness) Algorithm:

Step 0 Let μ_0 be *student optimal stable* matching.

Step k Given a matching μ_{k-1},

(k.1) if there is no cycle in $G(\mu_{k-1})$, stop: μ_{k-1} is the matching obtained;

(k.2) otherwise, *solve* one of the cycles in $G(\mu_{k-1})$, say ϕ_k and let $\mu_k = \phi_k \circ \mu_{k-1}$.

We provide an example of school choice problem with priority violation to see how the SEPF works (see Appendix A). The SEPF is a class of algorithms and each particular cycle selection in this class generates a matching (the multiplicity is due to Step k.2 of the algorithm, which requires "one of the cycles" to be solved without specifying which one)[11].

3.1 A characterization result

We present our main characterization result: each matching obtained by the SEPF is *constrained efficient* and *weakly Pareto dominates* the SOSM; moreover, each *constrained efficient* matching which *weakly Pareto dominates* the SOSM is attainable through some cycle selection rule within the SEPF.

For a school choice problem (R, C), let $\Psi(R, C)$ denote the set of all *constrained efficient* matchings which *weakly Pareto dominate* the SOSM. Let $\Pi(R, C)$ denote the set of all matchings than can be obtained via SEPF algorithm for a school choice problem (R, C).

[11] In the example in Appendix A, there are two matchings obtained by the SEPF.

Theorem 1 *For each school choice problem, a matching is constrained efficient and weakly Pareto dominates the SOSM if and only if it is obtained by the SEPF. Thus,* $\Psi(R, C) = \Pi(R, C)$.

The proof of this theorem is provided in Appendix B. This result states that SEPF indeed characterizes the set of matchings which satisfy certain normative properties.

An important remark that is worth discussing at this point is that the characterization result concerns the set of *constrained efficient* matchings which *weakly Pareto dominate* the SOSM, rather than any *constrained efficient* matching. In general, there exist *constrained efficient* matchings which do not *weakly Pareto dominate* the SOSM.[12] These matchings are excluded in Theorem 1. The reason for this restriction that since the DA is the mostly used mechanism in school choice in the US and in other countries, it is natural to interpret the SOSM as the *outside option* of each student, which renders the SOSM a reasonable starting point. We remove this restriction in Section 6.1 and analyze *constrained efficient* mechanisms by focusing incentive compatibility. The impossibility result we obtain suggest that the restriction of *weak Pareto dominance* over the SOSM is not costly at all as far as the main properties of fairness, efficiency and incentive compatibility are concerned.

3.2 SEPF and EADAM

The SEPF class gives the set of all *constrained efficient* matchings which *weakly Pareto dominate* the SOSM. Another mechanism which attains the same properties is the **Efficiency Adjusted Deferred Acceptance (EADAM)** algorithm (Kesten, 2010). The motivation behind the EADAM is to explore the source of inefficiency of the DA due to fairness constraints. To explain this idea, let us reconsider the problem in Example 1. The DA algorithm selects a *Pareto inefficient* matching in order not to violate the priority of i_1 at s_1. Here, a crucial observation is that the priority of student i_1 at s_1 does not help i_1 to get a better

[12] In the example in Appendix A, since for each $s \in S$, $C(s) = I$, *constrained efficiency* is equivalent to *Pareto efficiency*, and there are *Pareto efficient matchings* other than μ and μ': such a matchings is $\mu'' = \{(i_1, s_2), (i_2, s_5), (i_3, s_3), (i_4, s_4), (i_5, s_1), (i_6, s_5)\}$ (note that since i_2 prefers her allocation under μ_0 over s_5, the matching μ'' is *Pareto incomparable* to μ_0).

school at all. If i_1 had the lowest priority at s_1 instead of her current priority, she would be assigned to the same school, s_3, and the DA would select μ'' which is a *Pareto efficient* matching. Motivated by this observation, Kesten (2010) introduces EADAM in a setting that allows students to *consent* for the violation of their own priorities that do not affect their assignment. In Example 1, this would correspond to i_1 consenting for a priority violation at s_1. This discussion demonstrates that there is a clear connection between our setup and the one considered in Kesten (2010).

Proposition 1 *The EADAM belongs to the SEPF class; that is, for each problem, the matching obtained by EADAM can also be obtained by a particular selection of cycles in the SEPF class.*

Proof. This is a direct consequence of Theorem 1 and Proposition 7 in Appendix C. ■

It needs to be pointed out that the setup considered in Kesten (2010) is slightly different from ours: Kesten (2010) assumes that a student can either consent for priority violation at every school, or consent for it at none of the schools (in other words, either $i \in C(s)$ for each $s \in S$ or $i \notin C(s)$ for each $s \in S$). We introduce the ***generalized EADAM***, a trivial generalization of the EADAM, to accommodate our setup (see Appendix C). Since the EADAM is based on students making their consenting decisions in a specific way and the generalized EADAM belongs to the SEPF class (Proposition 7), the EADAM belongs to the SEPF class as well.

3.3 The concept of underdemanded schools

The SEPF class is defined by an algorithm based on iterative selection of cycles and at each iteration, only the welfare of each student in the selected cycle improves. Clearly, a necessary condition for a student to be in a cycle is that her current school is demanded by other students. If this does not hold, then a welfare improvement for this student is not possible under the SEPF class. To formalize this idea, we introduce the concept of *underdemanded*

schools.[13]

A school, say s, has **no demand** at μ if there does not exist a student i who prefers s to $\mu(i)$. A school is **underdemanded** at μ if it has *no demand* at μ or each path to a student assigned to that school starts with a student assigned to a school with *no demand* at μ.

If student i is not pointed by another student in the graph $G(\mu_k)$, then school $\mu_k(i)$ has *no demand* at $\mu_{k'}$ for each $k' \geq k$.[14] Consequently, a student assigned to an *underdemanded* school at step k is not part of any cycle at any step $k' \geq k$. This implies that the students assigned to *underdemanded* schools at the SOSM (μ_0) are not part of any cycle throughout the SEPF algorithm. Thus, by Theorem 1, a student assigned to an *underdemanded* school at μ_0 is assigned to the same school at each *constrained efficient* matching which *weakly Pareto dominates* the SOSM.

We say that a student is **permanently matched at** μ if she is assigned to an *underdemanded* school at μ, and **temporarily matched at** μ if she is not *permanently matched*.

4 No-Consent-Proofness

We now focus on the second interpretation of priority violation discussed in Section 2.2: to ask students for consent for violation of their priorities (see (Kesten, 2010)) and to interpret the acceptable violation of student i's priority at school s as student i has consented for this violation. As discussed before, consenting (or not consenting) for priority violations is a strategic decision and the main issue for a school choice mechanism based on the idea of consent is whether students have incentives for *consenting*. If consenting for a priority violation causes a student to be assigned to a worse school than she would have without consenting, then school districts can only proceed without her consent and this hinders possible welfare gains, which is the whole point behind the idea of consent. Thus, in order the idea of consent to become operational, the mechanism should provide the students incentives to consent.

[13] See also Kesten and Kurino (2013) and Tang and Yu (2014) for a discussion of the same concept. The terminology in Tang and Yu (2014) is different: the authors call a school with *no demand* as a **tier-0 underdemanded** school and an *underdemanded* school as a **tier-k underdemanded** school.

[14] This easily follows from Remark 2 in Appendix B.

The following notion formalizes this.

Definition 1 *A rule f is* **no-consent-proof** *if for any problem, for each $i \in I$ and $s \in S$, student i who consents for s does not get a better assignment by not consenting for s.*

Our goal is to search for *no-consent-proof* rules which satisfy *constrained efficiency* and *weak Pareto dominance* over the SOSM. Our characterization result (Theorem 1) reduces this search to the specific cycle selection rules within the SEPF class. But, a cycle selection rule within the SEPF class may not satisfy *no-consent-proofness*. Such a cycle selection rule is provided in Appendix D. Next, we introduce a *no-consent-proof* rule in the SEPF class.

4.1 The Top Priority Rule

For each matching μ, let $G^T(\mu) = (I, E^T(\mu))$ be the **Top Priority (TP) graph** associated with μ, a subgraph of $G(\mu) = (I, E(\mu))$ where the set of directed edges $E^T(\mu) \subseteq E(\mu)$ is defined as follows: $ij \in E^T(\mu)$ if and only if, among the students who are *temporarily matched* at μ and point to j in $G(\mu)$, student i has the highest priority for school $\mu(j)$. The **Top Priority (TP) Algorithm** is based on iterative selection of specific cycles in the TP-graph.

The TP Algorithm:

Step 0 Let μ_0 be the *student optimal stable* matching.

Step k Given a matching μ_{k-1},

- **(k.1)** if there is no cycle in $G(\mu_{k-1})$, stop: μ_{k-1} is the matching obtained;
- **(k.2)** otherwise, *solve* one of the cycles in $G^T(\mu_{k-1})$, say ϕ_k and let $\mu_k = \phi_k \circ \mu_{k-1}$.

Appendix E.1 illustrates an example for how the TP Algorithm works. Since the TP Algorithm is based on randomly selecting one of the (in general) multiple cycles in Step k.2, it is not clear whether this algorithm defines a rule. We argue that the outcome of the TP Algorithm does not depend on the order of cycles solved. Thus, the TP Algorithm defines a rule.

Proposition 2 *The TP Algorithm defines a rule in the SEPF class.*

The proof of this result is provided in Appendix F. We call the rule defined by the *Top Priority Algorithm* as the **Top Priority (TP) Rule**. The TP Rule satisfies *no-consent-proofness*. Moreover, it is the *unique non-consent-proof* rule within the SEPF class.

Theorem 2 *A rule is constrained efficient, no-consent-proof and improves the SOSM if and only if it is the TP Rule.*

The proof of this result is provided in Appendix G.

By Theorem 1, a matching is *constrained efficient* and *weakly Pareto dominates* the SOSM if and only if it is an outcome of the SEPF. By definition, the TP rule is in the SEPF class. Another such rule is EADAM: it is *constrained efficient* and *weakly Pareto dominates* the SOSM (Proposition 1). Morever, EADAM also satisfies the following property: for any student, *consenting* for all schools weakly dominates *not consenting* for any of the schools (Proposition 3 by Kesten (2010)). The following trivial result states that the same property holds for the generalized EADAM as well.

Proposition 3 *The generalized EADAM is no-consent-proof.*

Proof. The proof is identical to the proof of Proposition 3 in Kesten (2010). The proof in (Kesten, 2010) relies on the consent of a student at a particular school only when that consent is relevant. Consequently, the same argument in the proof applies here as well: under the generalized EADAM, the placement of a student does not change whether she consents or not. This implies that generalized EADAM is *no-consent-proof.* ■

This result implies an important equivalence.

Corollary 1 *The generalized EADAM is equivalent to the TP rule.*

Proof. The generalized EADAM is in the SEPF class (Proposition 7) and it is *no-consent-proof.* Thus, by Theorem 2, it is equivalent to the TP-rule; that is, for each problem (R, C), the TP-rule and the generalized EADAM give the same matching. ■

The following important implication of this result is immediate: one cannot do better than EADAM without sacrificing *no-consent-proofness*. This provides a strong case for using EADAM in a school choice problem with consent.

5 Strategy-proofness

For the model based on the idea of students' consent, *no-consent-proofness* is a notion regarding incentives for consenting. This does not fully capture the strategic component in school choice with priority violation based on students' consent. Students reveal not only consent decision but also preferences. This is a complicated game and we argue that it is not possible to prevent manipulation in this game. Actually, we obtain stronger (negative) results by considering the first interpretation where the acceptable priority violations are determined by the school districts and students reveal only preferences. We say that a mechanism is **strategy-proof** if, given a profile of acceptable priority violations C, for each preference profile R, truthful revelation of preferences R_i is a dominant strategy for each $i \in I$. Clearly, *strategy-proofness* is a desirable property, but it's not always satisfied. We have the following negative result.

Proposition 4 *There is no cycle selection rule within the SEPF class, which satisfies strategy-proofness.*

The proof of this proposition is provided in Appendix H. This result, combined with Theorem 1, demonstrates that *constrained efficiency*, *weakly Pareto dominating* the SOSM and *strategy-proofness* are incompatible. This result is hardly surprising, since know from earlier literature (Theorem 1 of Abdulkadiroğlu, Pathak, and Roth (2009), Proposition 4 of Kesten (2010), and Theorem 1 of Kesten and Kurino (2013)) that there is no mechanism that is *strategy-proof* and *Pareto dominates* the SOSM. By Theorem 1, each SEPF outcome *weakly Pareto dominates* the SOSM. Thus, there cannot be a *strategy-proof* cycle selection rule within the SEPF class.

Given this incompatibility result, since *constrained efficiency* is an indispensable property in our model, the only possible way to gain *strategy-proofness* is to consider all the *constrained efficient* matchings instead of only the ones which *weakly Pareto dominate* the SOSM. We show that the impossibility extends.

Theorem 3 *In the school choice problem with priority violation, there is no strategy-proof mechanism which always yields a constrained efficient matching.*

The proof of this result is provided in Appendix J. An alternative is to relax the dominant-strategy incentive compatibility requirement, and consider the Nash Equilibrium. In this case, one can adopt the information setting offered in Section V.B. of Kesten (2010), which is an intermediate between the "complete information" and "symmetric incomplete information" setting. In this setup, the set of schools are partitioned into quality classes. Each student unambiguously prefers a school in a higher quality class over a school in a lower class; yet, the comparison of schools within the same class are not common knowledge, and each student has symmetric information about these schools. Here, *symmetric* has a particular meaning: it means that for any two schools s and s' in the same quality class, encountering a student who prefers s over s' is equally likely as encountering a student who prefers s' over s. Moreover, students' information about the set of acceptable priority violations are also uniform across the schools in the same quality class. Note that the extreme case where each quality class consists of only one school corresponds to the the complete information-common preferences setting, whereas the other extreme with only one quality class corresponds to the symmetric incomplete information setting.

One can now analyze the preference revelation game in this setup. Given the preferences P_i of a student, we say that a strategy P_i' stochastically dominates another strategy P_i'' if the probability distribution over the outcomes induced by P_i stochastically dominates the probability distribution induced by P_i''. The following is an adaptation of Theorem 2 of Kesten (2010), which demonstrates that EADAM has truthful revelation in an ordinal Nash Equilibrium under such a setting. By Corollary 1, we know that generalized EADAM is

equivalent to TP Rule, so the following result is not surprising.

Proposition 5 *Suppose that the following is common knowledge among students: The set of schools is partitioned into quality classes as follows: Let* $\{S_1, S_2, \ldots, S_m\}$ *be a partition of* S*. Given any* $k, l \in 1, \ldots, m$ *such that* $k < l$*, each student prefers any school in* S_k *to any school in* S_l*. Moreover, each student's information is symmetric for any two schools* s *and* s' *such that* $s, s' \in S_r$ *for some* $r \in 1, \ldots, m$*. Then for any student the strategy of truth telling stochastically dominates any other strategy when other students behave truthfully under TP Rule. Thus, truth telling is an ordinal Bayesian Nash equilibrium of the preference revelation game under TP Rule.*

The proof of this proposition is provided in Appendix I.

6 Extensions

6.1 Relaxing Weak Dominance over SOSM

Strategy-proofness and *constrained efficiency* are incompatible in school choice with priority violation (Theorem 3). Thus, since each cycle selection rule within the SEPF class is *constrained efficient* and the TP rule is the (unique) *no-consent-proof* rule within this class, weak dominance over the SOSM is not restrictive for other properties considered in the context of priority violation. Moreover, although the SEPF class is defined by taking the SOSM as the initial matching, there is nothing special about the SOSM as far as the SEPF-type exchange mechanisms are considered. Thus, the SEPF can be defined in a more general way where the initial matching is any *stable* matching. The weak dominance over the SOSM, on the other hand, is appealing, particularly for school districts using the DA; it gives them to argue for (weakly) improving each student's welfare compared to the school assignment given by the existing mechanism, the DA.

6.2 Extension to Weak Priority Orders

We now consider the case in which schools have weak priority order over the students. School districts usually rank students using some predetermined criteria such as proximity and sibling status. For instance, Boston Public School system group students into five priority classes:

1. Guaranteed students are the ones continuing on at their current schools,
2. Sibling-walk zone students are the ones with sibling currently attending a school and living in the walk zone,
3. Sibling students are the ones with sibling currently attending a school and not living in the walk zone,
4. Walk zone students are the ones without sibling currently attending a school and living in the walk zone,
5. Other students are ones not belonging any of the first four priority classes.

Since the number of applicants is more than the number of the priority classes, many students end up being grouped under the same priority classes. However, the student assignment mechanisms used by the school districts, such as Boston mechanism, Deferred Acceptance mechanism and Top Trading Cycles mechanism, are defined under the strict priority orders. Therefore, school districts use random lottery numbers to order students within priority classes. Erdil and Ergin (2008), Abdulkadiroğlu, Pathak, and Roth (2009), and Kesten (2010) point out that the random tie breaking between the students in the same priority classes causes efficiency loss, and that the particular tie-breaking rule may have dramatic effects on the outcome. In particular, Abdulkadiroğlu, Pathak, and Roth (2009) shows that in general a single tie-breaking rule is favored over a multiple-tie breaking rule. Yet, even the outcome of the DA mechanism with single tie breaking rule might be *Pareto dominated* by another matching in which there does not exist any student preferring the assignment of another student from lower priority class to her own assignment. In order to overcome the

efficiency loss caused by single random tie breaking, Erdil and Ergin (2008) and Kesten (2010) propose two solutions which are built on the DA mechanism. In particular, Erdil and Ergin (2008) propose a class of mechanisms called Stable Improvement Cycles (SIC) algorithm. The SIC algorithm takes the SOSM for a given tie breaking rule and then improves the assignment by utilizing trade cycles between students, where solving these cycles do not cause any priority violation. In any step of the SIC algorithm, there may exist more than one trade cycles and there is no certain rule for the selection of the cycle that will be solved in that step. On the other hand, Kesten (2010) modifies the EADAM mechanism to deal with the efficiency losses caused by single tie breaking rule. Different from Erdil and Ergin (2008), Kesten's algorithm for weak priorities selects a unique outcome. In this section, we show that both algorithms introduced by Erdil and Ergin (2008) and Kesten (2010) are equivalent to SEPF algorithm and TP Rule, respectively (with a restriction on the correspondence C).

We extend our model by allowing each school s to have coarse priority order over students denoted by $\succsim_s$. We denote the strict priority order of school s on set of students by $\succ_s$ and the associated indifference relation by $\sim_s$. Following this extension, we also revisit the standard notion of violation of priorities, defined in Section 2.1. In particular, we say a matching μ **violates the priority** of $i \in I$ for $s \in S$ if there exists another $j \in I$ such that: (i) $\mu(j) = s$, (ii) $sP_i\mu(i)$, and (iii) $i \succ_s j$. Note that this is the regular definition of priority violation in school choice literature (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). In Appendix K, we provide an example which illustrates the welfare loss caused by the DA mechanism with single tie-breaking rule.

By slightly changing the SEPF algorithm introduced in Section 3, we are able to propose an alternative way to improve the DA mechanism with single tie breaking.

SEPF for Weak Priorities:

Given a weak priority order $\succsim$ and a random draw π over students, denote the strict priority profile attained from $\succsim$ by using π with $\succ'$. Let $I(i, s, \succsim_s)$ be the set of students who have same priority with i for school s, i.e., $I(i, s, \succsim_s) = \{j \in I : j \sim_s i\}$. Note that

$i \in I(i, s, \succsim_s)$ for all $i \in I$. Given a matching μ, for each $s \in S$, let

- $D_\mu(s) = \{i \in I : sP_i\mu(i)\}$
- $X_\mu(s) = \{k \in (I(i, s, \succsim_s) \cap D_\mu(s)) : i \text{ is s.t. for each } j \in (D_\mu(s) \setminus C(s)) \setminus \{i\},\ i \succ'_s j\}$.

For each matching μ, let $G(\mu) = (I, E(\mu))$ be the **(directed) application graph** associated with μ where the set of directed edges $E(\mu) \subseteq I \times I$ is defined as follows: $ij \in E(\mu)$ (that is, *i points to j*) if and only if $s = \mu(j)$ and $i \in X_\mu(s)$.

Note the basic difference: Under SEPF with weak priorities, in step k if student $i \in D_{\mu_{k-1}}(s)$ has the highest priority among the students in $D_{\mu_{k-1}}(s)$ according to $\succ'$, then all students in $I(i, s, \succsim_s) \cap D_{\mu_{k-1}}(s)$ point to s in $G(\mu_{k-1})$ regardless of whether i consents for s or not. On the other hand, students in $D_{\mu_{k-1}}(s) \setminus I(i, s, \succsim_s)$ cannot point s in $G(\mu_{k-1})$ if any student in $I(i, s, \succsim_s) \cap D_{\mu_{k-1}}(s)$ does not consent for s.

The **SEPF** for weak priorities is class of algorithms defined by inductively with the following steps:

Step 0 Let μ_0 be the *student optimal stable* matching for $\succ'$.

Step k Given a matching μ_{k-1},

(k.1) if there is no cycle in $G(\mu_{k-1})$, stop: μ_{k-1} is the matching obtained;

(k.2) otherwise, *solve* one of the cycles in $G(\mu_{k-1})$, say ϕ_k and let $\mu_k = \phi_k \circ \mu_{k-1}$.

The SEPF for weak priorities aims to overcome the inefficiencies caused by the random tie breaking and rejection cycles caused due to priorities which do not have any role on the assignment of the students. Student i points to the assignees of school s in directed graph $G(\mu_{k-1})$ if only if her assignment to s does not violate *partial fairness*. Moreover, the algorithm terminates whenever there does not exist any swap of the assignments between students which does not violate *partial fairness*. Thus, it inherits *constrained efficiency* of the SEPF algorithm defined in Section 3.1. When our particular focus is to recover the efficiency losses due to the single tie breaking rule, we do not need the consents of the students.

Thus, in this case, we can exclude $C(s)$ from the calculation of $X_\mu(s)$. Or alternatively, we can set $C(s) = \emptyset$ for each $s \in S$ and in that case the SEPF for weak priorities and SIC algorithm of Erdil and Ergin (2008) are equivalent, i.e. for the same tie breaking rule and chain selection rule they select the same matching. We formally state these results in the following proposition.

Proposition 6 *For each school choice problem with random tie breaking rule π,*

(1) the SEPF for weak priorities selects a constrained efficient matching which Pareto dominates SOSM obtained under tie breaking rule π, and

(2) it is equivalent to the SIC algorithm when $C(s) = \emptyset$ for all $s \in S$.

Proof. For the proof of the part (1), we refer to the proof of the "if part" of Theorem 1. For the second part, when $C(s) = \emptyset$ for all $s \in S$ in each step k of the SEPF for weak priorities and SIC we have the same directed graph as long as the same cycle is selected in step $k-1$. Thus, for the same cycle selection order, both algorithms select the same outcome. ■

7 Conclusion

This study introduces the *school choice problem with priority violation.* The two main results are (i) characterization of a class of algorithms, each of which always yields a *constrained efficient* matching *weakly Pareto dominating* the SOSM, and (ii) characterization of the unique *no-consent-proof* rule within this class. The mechanism is easily applicable to settings where priority violations are deemed feasible. One such example is a setting where the centralization of assignments to public and private (or exam and regular) schools is possible. Whereas the priorities to public (exam) schools are legal constraints which cannot be violated, the private (regular) schools are more flexible in terms of their priorities (and efficiency is more of a first-order concern for these schools). One can then simply adopt the framework offered in Section 2.2 and specify that priority violation in private schools are allowed. Each cycle selection rule within SEPF, including the "uniform cycle selection rule" which solves

each cycle at each step with equal probabilities,[15] is guaranteed to produce a *constrained efficient* matching in this case.

Another case in which the SEPF is applicable is the setup where the mechanism designer asks for the consents of students (Kesten, 2010). Clearly, this setup generates the need to incentivize (or at least avoid punishing) students for consenting and *no-consent-proofness* is indispensable in this setting. Our proposal, the TP Rule is the unique rule satisfying within the SEPF class. Indeed, the mechanism designer can also attempt to provide more incentives by designing (perhaps stochastic) cycle selection rules within SEPF. One such rule may be the one which solves the cycles which include the consenting students with higher probability. That is, the mechanism designer may favor consenting students in the cycle selection process, which in turn would provide incentives to consent. The characterization of such rules is left for future work.

One deficiency of the SEPF is that no rule in this class satisfies *strategy-proofness*, which is indeed the deficiency of any constrained efficient mechanism (Theorem 3). Consequently, the school choice problem with priority violations is in general prone to manipulation via misrepresentation of preferences. One could perhaps follow some "large market" results (as in Kojima and Pathak (2009)) and characterize the extent to which the students can gain by such manipulations. The incentive-compatibility properties in large markets of school choice problem with priority violation remains an open question.

[15] One may also expect better incentive-compatibility properties from the uniform cycle selection rule, at least under the symmetric incomplete information setup in Proposition 5, which is left for future work.

Appendix A The SEPF algorithm: An Example

Example 2 *(This is based on Example 3 in Kesten (2010), pp. 1310) Let* $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$*,* $S = \{s_1, s_2, s_3, s_4, s_5\}$*,* $q_{s_i} = 1$ *for* $i = 1, \ldots, 4$ *and* $q_{s_5} = 2$*. Assume that, for each* $s \in S$*,* $C(s) = I$*. The students' preferences and schools' priorities are as follows:*

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_3	s_3	s_1	s_1	s_4
s_1	$\boxed{s_1}$	s_4	s_2	$\boxed{s_5}$	s_1
$\boxed{s_3}$	s_5	$\boxed{s_2}$	$\boxed{s_4}$	$\vdots$	s_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\boxed{s_5}$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_2	i_3	i_1	i_4	$\vdots$
i_1	i_6	i_6	i_3	$\vdots$
i_5	i_4	i_2	i_6	$\vdots$
i_6	i_1	i_3	$\vdots$	$\vdots$
i_4	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i_3	$\vdots$	$\vdots$	$\vdots$	$\vdots$

The student proposing DA algorithm gives the following SOSM (marked with boxes above) for this problem:

$$\mu_0 = \{(i_1, s_3), (i_2, s_1), (i_3, s_2), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$$

The sets X_{μ_0} are as follows:

$$X_{\mu_0}(s_1) = \{i_1, i_4, i_5, i_6\}$$

$$X_{\mu_0}(s_2) = \{i_1, i_4, i_6\}$$

$$X_{\mu_0}(s_3) = \{i_2, i_3, i_6\}$$

$$X_{\mu_0}(s_4) = \{i_3, i_6\}$$

$$X_{\mu_0}(s_5) = \emptyset,$$

and these sets yield the following graph $G(\mu_0)$[16]:

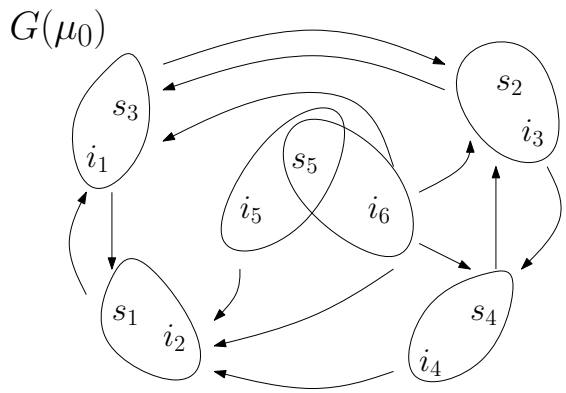

Figure 1: Graph $G(\mu_0)$

There are four cycles in this graph:

$$\phi_1 = (i_3 i_4, i_4 i_3)$$

$$\phi_2 = (i_1 i_3, i_3 i_1)$$

$$\phi_3 = (i_1 i_2, i_2 i_1)$$

$$\phi_4 = (i_1 i_3, i_3 i_4, i_4 i_2, i_2 i_1)$$

[16] As defined in Section 3, each graph in the algorithm is on the set of students. For convenience and tractability, we include the school which is assigned to the student in the current matching as well.

First, we demonstrate how the algorithm proceeds when cycle ϕ_3 is selected in the graph $G(\mu_0)$. Let $\mu_1 = \{(i_1, s_3), (i_2, s_1), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$ be the resulting matching. The sets X_{μ_1} are as follows:

$$X_{\mu_1}(s_1) = \{i_1, i_4, i_5, i_6\}$$
$$X_{\mu_1}(s_2) = \{i_1, i_6\}$$
$$X_{\mu_1}(s_3) = \{i_2, i_3, i_6\}$$
$$X_{\mu_1}(s_4) = \{i_6\}$$
$$X_{\mu_1}(s_5) = \emptyset,$$

and these sets yield the following graph $G(\mu_1)$:

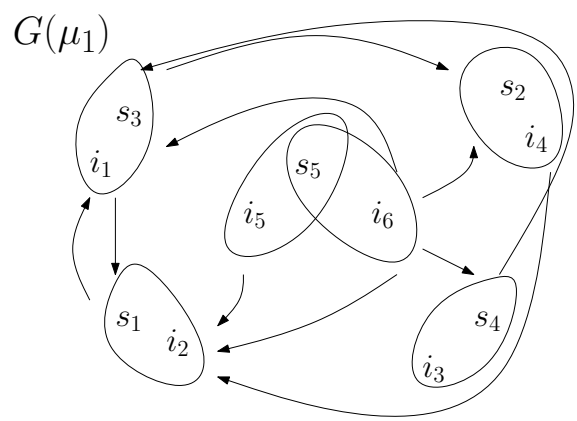

Figure 2: Graph $G(\mu_1)$

In the graph $G(\mu_1)$, there are two cycles: $\phi_1' = (i_1 i_4, i_4 i_2, i_2 i_1)$ and $\phi_1'' = (i_1 i_2, i_2 i_1)$. Let cycle ϕ_1' be selected and $\mu_2 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$ be the resulting matching. Then, the following graph is obtained:

Since there exists no cycle in the graph $G(\mu_2)$, the algorithm stops.

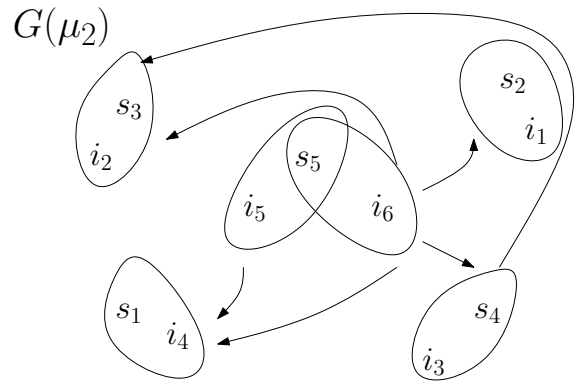

Figure 3: Graph $G(\mu_2)$

Next, by considering all possible cycle selections, we list all matchings obtained by the SEPF algorithm.

1. If cycle ϕ_1 is selected, then the following matching is obtained:

   $\mu_1 = \{(i_1, s_3), (i_2, s_1), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$.

   In the graph $G(\mu_1)$, there are two cycles: $\phi_1' = (i_1 i_4, i_4 i_2, i_2 i_1)$ and $\phi_1'' = (i_1 i_2, i_2 i_1)$.

   1.1 If cycle ϕ_1' is selected, then the following matching is obtained:

   $\mu_2 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$.

   Since there is no cycle in the graph $G(\mu_2)$, the algorithm stops.

   1.2 If cycle ϕ_1'' is selected, then the following matching is obtained:

   $\mu_3 = \{(i_1, s_1), (i_2, s_3), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$.

   In the graph $G(\mu_3)$, there is only one cycle: $(i_1 i_4, i_4 i_1)$.

   1.2.1 By selecting this cycle, the following matching is obtained:

   $\mu_4 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$.

   Since there is no cycle in the graph $G(\mu_4)$, the algorithm stops.[17]

2. If cycle ϕ_2 is selected, then the following matching is obtained:

   $\mu_5 = \{(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$.

[17] This is the path that EADAM follows.

Since there is no cycle in the graph $G(\mu_5)$, the algorithm stops.

3. If cycle ϕ_3 is selected, then the following matching is obtained:

   $\mu_6 = \{(i_1, s_1), (i_2, s_3), (i_3, s_2), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$.

   In the graph $G(\mu_6)$, there are two cycles: $\phi_3' = (i_3 i_4, i_4 i_3)$ and $\phi_3'' = (i_1 i_3, i_3 i_4, i_4 i_1)$.

   3.1 If cycle ϕ_3' is selected, then the following matching is obtained:

   $\mu_7 = \{(i_1, s_1), (i_2, s_3), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$.

   In the graph $G(\mu_7)$, there is only one cycle: $(i_1 i_4, i_4 i_1)$.

   3.1.1 By selecting this cycle, the following matching is obtained:

   $\mu_8 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$

   Since there exists no cycle in the graph $G(\mu_8)$, the algorithm stops.

   3.2 If cycle ϕ_3'' is selected, then the following matching is obtained:

   $\mu_9 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$

   Since there is no cycle in the graph $G(\mu_9)$, the algorithm stops.

4. If cycle ϕ_4 is selected, then the following matching is obtained:

   $\mu_{10} = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$.

   Since there is no cycle in the graph $G(\mu_{10})$, the algorithm stops.

There are two different matchings generated by the SEPF algorithm and these matchings are depicted in the preference table (μ is marked with boxes and μ' is marked with underlines):

$$\mu = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$$

$$\mu' = \{(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$$

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
$\boxed{s_2}$	$\boxed{s_3}$	s_3	$\boxed{s_1}$	s_1	s_4
s_1	s_1	$\boxed{s_4}$	s_2	$\boxed{s_5}$	s_1
s_3	s_5	s_2	s_4	$\vdots$	s_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\boxed{s_5}$

Appendix B Proof of Theorem 1

We begin by introducing a few remarks which will be useful in the following argument.

A cycle is *solved* at each step of the SEPF algorithm, which implies that the students in the cycle are better off and no student is worse off at the new matching obtained by solving a cycle. Thus, the matching achieved at each step *Pareto dominates* the matching in the previous step. This implies that for a student i, if a school s is better than $\mu_k(i)$, then it is also better than $\mu_{k-1}(i)$.

Remark 2 *For each $k \geq 1$ and each $s \in S$, $D_{\mu_k}(s) \subseteq D_{\mu_{k-1}}(s)$.*

A consequence of this remark is that, if student i points to student j in the graph $G(\mu_{k-1})$ and i is not better off at step k, then in the graph $G(\mu_k)$, i points to the students who are assigned at μ_k to school $\mu_{k-1}(j)$. In particular, if i points to j and both are not better off at a step, then i points to j in the next step as well.

Remark 3 *If i points to j in $G(\mu_{k-1})$ and both students' assignment do not change at step k, then i points to j in $G(\mu_k)$.*

To see this, let cycle $\phi_k = \{i_1 i_2, i_2 i_3 \ldots, i_n i_1\}$ be *solved* in the graph $G(\mu_{k-1})$ such that $\mu_k = \phi_k \circ \mu_{k-1}$. Suppose i points to j in $G(\mu_{k-1})$ and $i, j \notin \{i_1, i_2, \ldots, i_n\}$. By definition of the graph $G(\mu_{k-1})$, $i \in X_{\mu_{k-1}}(s)$ where $s = \mu_{k-1}(j)$. Since $\mu_k(i) = \mu_{k-1}(i)$, $i \in D_{\mu_k}(s)$. Let $i' \in D_{\mu_k}(s)$ be such that $i' \succ_s i$. By Remark 2, $i' \in D_{\mu_{k-1}}(s)$. Thus, since $i \in X_{\mu_{k-1}}(s)$ and $i' \succ_s i$, we have $i' \in C(s)$. Thus, each student in $D_{\mu_k}(s)$ with a higher priority than student i

at school s is in the set $C(s)$. Thus, $i \in X_{\mu_k}(s)$. Since $s = \mu_k(j) = \mu_{k-1}(j)$, i points to j in the graph $G(\mu_k)$.

Now we can start with the proof.

(Proof of the "if" part)

Lemma 1 *Each matching obtained by the SEPF algorithm is partially stable.*

Proof. (i) *Partial fairness.* Let $\mu_0, \mu_1, \ldots, \mu_k, \ldots, \mu_K$ be the number of matchings obtained by SEPF at each step of the algorithm. We prove this statement by induction on k. The SOSM (μ_0) is a *stable* matching. Thus, for each $i \in I$ and $s \in S$, it doesn't violate the priority of student i at school s. Thus, μ_0 is *partially fair*.

As an inductive hypothesis, suppose μ_{k-1} is *partially fair*. Suppose there is a student i and school s such that $s \; P_i \; \mu_k(i)$ and $i \notin C(s)$. At each step of the algorithm, each student is either better off (she is in the selected cycle) or she is assigned to the same school as in the previous step. Thus, for each $\ell \in I$, $\mu_k(\ell) \; R_\ell \; \mu_{k-1}(\ell)$. Since $s \; P_i \; \mu_k(i)$, this implies that $s \; P_i \; \mu_{k-1}(i)$ and $i \in D_{\mu_{k-1}}(s)$. Take any $j \in \mu_k^{-1}(s)$. If $j \in \mu_{k-1}^{-1}(s)$, then by *partial fairness* of μ_{k-1}, $j \succ_s i$. Alternatively, suppose $j \notin \mu_{k-1}^{-1}(s)$. Since $j \in \mu_k^{-1}(s)$, student j is in the cycle selected in step k of the algorithm. Thus, $j \in X_{\mu_{k-1}}(s)$. By assumption, $i \notin C(s)$, thus, $i \in D_{\mu_{k-1}}(s) \setminus C(s)$. Since $j \in X_{\mu_{k-1}}(s)$, by definition, $j \succsim_s i$. Since the priorities are strict and $j \neq i$, we obtain that $j \succ_s i$. Thus, μ_k does not violate the priority of student i at school s and it is *partially fair*. The induction follows.

(ii) *Individual rationality.* Since μ_0 is *individually rational* and each student is weakly better off at each step of the SEPF algorithm, its outcome is *individually rational*.

(iii) *Non-wastefulness.* By the definition of the SEPF algorithm, for each school s, the number of students assigned to s remains at each step the same as it is under the SOSM; that is, for each step k of the algorithm, $|\mu_k^{-1}(s)| = |\mu_0^{-1}(s)|$. Thus, if $|\mu_0^{-1}(s)| = q_s$, then each matching obtained by the SEPF assigns q_s students to school s and it does not violate *non-wastefulness* for school s. Suppose $|\mu_0^{-1}(s)| < q_s$. Since μ_0 is *non-wasteful*, the set $D_{\mu_0}(s)$ is empty. By Remark 2, at each step k, $D_{\mu_k}(s)$ is empty. Thus, each matching obtained by the SEPF

satisfies *non-wastefulness.* ■

Lemma 2 *For a matching μ and $s \in S$, $X_\mu(s) = \emptyset$ if and only if $D_\mu(s) = \emptyset$*

Proof. *(Only if)* Let $X_\mu(s) = \emptyset$. Suppose $D_\mu(s) \neq \emptyset$. Then, for each $i_m \in D_\mu(s)$, there exists $i_{m+1} \in D_\mu(s) \setminus C(s)$ such that $i_{m+1} \succ_s i_m$. Note that $i_{m+1} \in D_\mu(s)$ as well. Let $i_1 \in D_\mu(s)$. Then, there is a sequence of students $(i_1, i_2, \ldots, i_n, \ldots)$ such that each student is in $D_\mu(s)$. Since the problem is finite, the sequence repeats at least one student, without loss of generality, say $i_1 = i_n$. This contradicts with the binary relation $\succ_s$ being a *strict linear order. (If)* It follows directly from the definition of the set $X_\mu(s)$. ■

Lemma 3 *Each matching obtained by the SEPF algorithm is constrained efficient.*

Proof. Let μ_k be a matching obtained by the SEPF algorithm. We will show that there does not exist a partially stable matching which Pareto dominates μ_k. Suppose there exists a *partially stable* matching $\widetilde{\mu}$ and it *Pareto dominates* μ_k.

We know that, by definition of the SEPF, there is no cycle in the graph $G(\mu_k)$. There are two possible cases.

Case 1: *There is no chain in $G(\mu_k)$.* Then, for each $s \in S$, $X_{\mu_k}(s) = \emptyset$. By Lemma 2, this implies that $D_{\mu_k}(s) = \emptyset$. Thus, at μ_k, each student is assigned to her best school. Thus, μ_k is *Pareto efficient* and $\widetilde{\mu}$ cannot Pareto dominate μ_k.

Case 2: *There is a chain in $G(\mu_k)$.* Let I_1 be the set of students who is the tail of some chain in $G(\mu_k)$. Let ϕ be a chain in $G(\mu_k)$ with the tail $i_1 \in I_1$ such that $\mu_k(i_1) = s_1$. Since i_1 is not pointed by any student, by definition of the graph $G(\mu_k)$, $X_{\mu_k}(s_1) = \emptyset$. By Lemma 2, this implies that $D_{\mu_k}(s_1) = \emptyset$. Then, since $\widetilde{\mu}$ *Pareto dominates* μ_k, the following must hold: there does not exist $i \in I$ such that $\mu_k(i) \neq s_1$ but $\widetilde{\mu}(i) = s_1$. Thus, $\widetilde{\mu}^{-1}(s_1) \subseteq \mu_k^{-1}(s_1)$. Suppose first that $\mu_k^{-1}(s_1) \setminus \widetilde{\mu}^{-1}(s_1) \neq \emptyset$. Then, there exists a school s such that $(q_s \geq)|\widetilde{\mu}^{-1}(s)| > |\mu_k^{-1}(s)|$. Since $\widetilde{\mu}$ *weakly Pareto dominates* μ_k, the second inequality implies that there exists $j \in \widetilde{\mu}^{-1}(s)$ such that $s \; P_j \; \mu_k(j)$. Since $q_s > |\mu_k^{-1}(s)|$, this violates *non-wastefulness* of μ_k. Therefore, we must have $\mu_k^{-1}(s_1) = \widetilde{\mu}^{-1}(s_1)$. Since i_1 is

chosen arbitrarily, this holds for each $s \in S$ such that $\mu_k^{-1}(s) \subseteq I_1$. Let S_1 denote the set of these schools. That is, for each $s \in S_1$, $\mu_k^{-1}(s) = \widetilde{\mu}^{-1}(s)$.

There exists at least one student in $I \setminus I_1$ such that she is pointed only by students in I_1. (Otherwise there is a cycle in $G(\mu_k)$, a contradiction.) Let I_2 be the set of such students and take some $i_2 \in I_2$, $s_2 = \mu_k(i_2)$. We first show the following: there does not exist $j \in I$ such that $\mu_k(j) \neq s_2$ but $\widetilde{\mu}(j) = s_2$. To see why, suppose there is such a j. Since $\widetilde{\mu}$ *Pareto dominates* μ_k, this implies that $s_2 \; P_j \; \mu_k(j)$ and thus $j \in D_{\mu_k}(s_2)$. Nevertheless, we must have $j \notin X_{\mu_k}(s_2)$. This is because otherwise $j \in I_1$ (recall that i_2 is pointed only by students in I_1), thus, by the above paragraph, we must have $\mu_k(j) = \widetilde{\mu}(j)$, a contradiction. We conclude that $j \notin X_{\mu_k}(s_2)$ and it implies that for some $j' \in D_{\mu_k}(s_2) \setminus C(s_2)$, $j' \succ_{s_2} j$. Note that i_2 is pointed by student $i \in D_{\mu_k}(s_2) \setminus C(s_2)$, who, among the students in $D_{\mu_k}(s_2) \setminus C(s_2)$, has the top priority at s_2. Moreover, since i_2 is pointed only by students in I_1, $i \in I_1$. Since student i is assigned to the same school both under μ_k and $\widetilde{\mu}$, matching $\widetilde{\mu}$ violates the priority of student i at school s_2, which contradicts with *partial fairness* of $\widetilde{\mu}$. Thus, there does not exist a student j such that $\mu_k(j) \neq \; s_2$ but $\widetilde{\mu}(j) = s$. By *non-wastefulness* of μ_k (repeating the same argument in the previous paragraph), $\mu_k^{-1}(s_2) = \widetilde{\mu}^{-1}(s_2)$. Let S_2 denote the set of the schools such that for each $s \in S_2$, $\mu_k^{-1}(s) \subseteq I_2$.

Now we can continue in the same manner. If there is a student in $I \setminus (I_1 \cup I_2)$, who is pointed by a student in $G(\mu_k)$, then at least one of them, say i_3, is pointed only by a student in $I_1 \cup I_2$. By same argument above, the same students are assigned to school $\mu_k(i_3)$ both under μ_k and $\widetilde{\mu}$. Once again, each student in a chain is assigned to the same school both under μ_k and $\widetilde{\mu}$. Repeating the same argument, we take care of all the students in a chain in $G(\mu_k)$.

Now, consider the students who are not in a chain in $G(\mu_k)$. If such a student assigned to school s, then $X_{\mu_k}(s) = \emptyset$, and by Lemma 2, $D_{\mu_k}(s) = \emptyset$. Thus, under μ_k, each student in $I \setminus \mu_k(s)$ prefers her assignment to s. Since μ_k and $\widetilde{\mu}$ coincide on the students who are in a chain, if a student (not in a chain) is assigned to a different school under $\widetilde{\mu}$, then she is assigned to a school s' such that $X_{\mu_k}(s') = \emptyset$, and by Lemma 2, $D_{\mu_k}(s') = \emptyset$. Since she prefers

s to s' and $\widetilde{\mu}$ *Pareto dominates* μ_k, this is a contradiction. Since s is chosen arbitrarily, this holds for each such school which is assigned to a student who is not in a chain. Therefore, the matchings μ_k and $\widetilde{\mu}$ coincide also on the students who are not in a chain. Thus, $\mu_k = \widetilde{\mu}$ and $\widetilde{\mu}$ cannot Pareto dominate μ_k. ■

Since the SEPF is such that the matching achieved at each step *improves* the matching in the previous step, with the initial step being the SOSM, clearly it *improves* the SOSM. This completes the proof of the "if" part of the theorem.

(Proof of the "only if" part)

Definition 2 *An* ***improvement cycle*** ϕ ***over a matching*** μ *is a set of ordered pairs of students* $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_1\}$ *such that for each* $ij \in \phi$, $\mu(j)\ P_i\ \mu(i)$.

Lemma 4 *Let* μ *and* η *be partially stable matchings such that* η Pareto dominates μ. *Then, there exists a set of distinct improvement cycles* $\Phi = \{\phi_1, \ldots, \phi_m\}$ *such that* $\eta = \phi_m \circ \ldots \circ \phi_1 \circ \mu$.

Proof. We first claim that the number of students who are assigned to each school is the same under μ and η. That is, for each $s \in S$, $|\eta^{-1}(s)| = |\mu^{-1}(s)|$. Take a school $s \in S$. To show that $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$, assume the contrary: $(q_s \geq)|\eta^{-1}(s)| > |\mu^{-1}(s)|$ and i be a student who is assigned to this school under η but not under μ. Since η *Pareto dominates* μ and preferences are strict, this implies that $s\ P_i\ \mu(i)$. But since $|\mu^{-1}(s)| < q_s$, this violates *non-wastefulness* of μ. This implies we must have $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$ for each $s \in S$. To show that $|\eta^{-1}(s)| \geq |\mu^{-1}(s)|$, again assume the contrary, i.e. assume that $|\eta^{-1}(s)| < |\mu^{-1}(s)|$. Adding over all schools and using the previous finding that $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$, we have: $\sum_{s \in S} |\eta^{-1}(s)| < \sum_{s \in S} |\mu^{-1}(s)|$. However, since η *Pareto dominates* μ and since both matchings are partially stable, if a student is assigned to a school under μ, then she is also assigned to a school under η. This means we have $\sum_{s \in S} |\eta^{-1}(s)| \geq \sum_{s \in S} |\mu^{-1}(s)|$, a contradiction.

Let N be the set of students who are better off under η. Let $G(\mu, \eta)$ be the graph with the set of vertices N and the set of edges, where student $i \in N$ points to a unique student in $N \cap \mu^{-1}(\eta(i))$ such that each student in N is pointed by a unique student. We claim that the

graph $G(\mu, \eta)$ is well-defined. Since for each school s, $|\mu^{-1}(s)| = |\eta^{-1}(s)|$, if $\mu^{-1}(s) \neq \eta^{-1}(s)$, then clearly, $|\mu^{-1}(s) \setminus \eta^{-1}(s)| = |\eta^{-1}(s) \setminus \mu^{-1}(s)|$. Moreover, each $i \in \mu^{-1}(s) \setminus \eta^{-1}(s)$ is pointed by one of the students in $\eta^{-1}(s) \setminus \mu^{-1}(s)$. Thus, it is possible to construct the graph $G(\mu, \eta)$ as defined. Since each student in N is pointed by a unique student and points to a unique student, each student is in a cycle and no two cycles intersect. Each of these distinct cycles is an *improvement cycle* over μ, and the matching η is obtained by *solving* these cycles in any order, so that the numbering of these cycles is not important. ■

We next prove that each *constrained efficient* matching can be obtained by the SEPF algorithm. For each k, a cycle in the graph $G(\mu_k)$ of the SEPF algorithm is called a **SEPF-cycle**. The previous lemma states that each *constrained efficient* matching which *improves* the SOSM can be obtained by *solving* a sequence of *improvement cycles*. To complete our proof, we prove a similar result using the *SEPF-cycles*.

Lemma 5 *Let μ and η be partially stable matchings such that η* Pareto dominates *μ. Then, there exists a sequence of* SEPF-cycles *$(\gamma_1, \ldots, \gamma_n)$ such that:*

- *γ_1 appears in $G(\mu)$;*
- *for each $i \in \{2, \ldots, n\}$, γ_i appears in $G(\gamma_{i-1} \circ \ldots \circ \gamma_1 \circ \mu)$;*
- *$\gamma_n \circ \ldots \circ \gamma_1 \circ \mu = \eta$.*

Proof. By Lemma 4, there is a set of distinct *improvement cycles* $\Phi = \{\phi_1, \ldots, \phi_m\}$ such that $\eta = \phi_m \circ \ldots \circ \phi_1 \circ \mu$. The proof is trivial for the case where the matching η is achieved by *solving* a *SEPF-cycle* at each step. To prove the other case, we assume that none of the cycles in $\Phi = \{\phi_1, \ldots, \phi_m\}$ is a *SEPF-cycle*. This assumption is without loss of generality because of the following: If some of these cycles are *SEPF-cycles* at μ, then first a *SEPF-cycle* is *solved*. At the matching obtained, if some of the remaining cycles are *SEPF-cycles*, then first a *SEPF-cycle* is *solved*. This continues until none of the remaining *improvement cycles* is a *SEPF-cycle* at the matching obtained.

Let $\phi \in \Phi$. Since ϕ is not a *SEPF-cycle*, there exists a student i with $ij \in \phi$ such that $i \notin X_\mu(\eta(i))$. We call student i a **prevented** student. We claim that there exists $i_p \in X_\mu(\eta(i))$ such that i_p is in an *improvement cycle* in Φ. Since $i \notin X_\mu(\eta(i))$, there exists a student i' such that $i' \in D_\mu(\eta(i)) \setminus C(\eta(i))$ and $i' \succ_{\eta(i)} i$. Let i_p be the student with the highest priority for the school $\eta(i)$ among such students. Clearly, $i_p \in X_\mu(\eta(i))$. If $\eta(i_p) = \mu(i_p)$, then, since $\eta(i)\ P_{i_p}\ \eta(i_p)$, $i_p \notin C(\eta(i))$ and $i_p \succ_{\eta(i)} i$, there is a priority violation at η, contradicting *partial stability* of η. Thus, $\eta(i_p)\ P_{i_p}\ \mu(i_p)$, which implies that i_p is in an *improvement cycle* in Φ. We call student i_p as the **preventer of** i. By definition of the *preventer*, for each *prevented* student i, there exists a unique *preventer*, denoted by i_p.

Let $ij \in \phi_k$ and i be a *prevented* student. We consider the sequence, which starts with student j and ends with the next *prevented* student in the cycle ϕ_k. If there is no other *prevented* student in this cycle, then this sequence ends at student i. Similarly, if j is a *prevented* student, then the sequence consists only of student j.

Let $G(\mu, \eta)$ be the directed graph defined in Lemma 4 and note that it consists of *improvement cycles* in Φ.[18] We next construct the directed graph $G^{SEPF}(\mu, \eta)$ by using $G(\mu, \eta)$. First, we break each cycle in the graph $G(\mu, \eta)$ into its sequences,[19] each of which starts with a student who is pointed by a *prevented* student and ends with a *prevented* student. Clearly, each student in an *improvement cycle* is in a sequence. Second, for each *prevented* student i with $ij \in \phi_k \in \Phi$, the directed edge $i_p j$ is added to the graph $G(\mu, \eta)$. Thus, since student i is *prevented*, student j is the first member of a sequence and it is pointed by the *preventer* i_p of student i. Since for each *prevented* student there exists a unique *preventer*, the graph $G^{SEPF}(\mu, \eta)$ is such that the first student in each sequence is pointed by a (unique) student, who is also in a sequence. Thus, there exists a cycle γ_1 in this graph.

We claim that γ_1 is a *SEPF-cycle*, that is each edge in γ_1 is in the set of edges $E(\mu)$ of the application graph $G(\mu)$. First note that, the edges in γ_1 are such that either

[18] By the definition of the graph $G(\mu, \eta)$ and by Lemma 4, each cycle in the graph corresponds to an *improvement cycle* in Φ. Thus, we refer to cycles in the graph $G(\mu, \eta)$ and *improvement cycles* in Φ interchangeably.

[19] Note that this is achieved simply by removing each ij from the set of edges of $G(\mu, \eta)$ such that i is a *prevented* student.

(i) a student i' points to the next student in the sequence; that is, to the student who is assigned to school $\eta(i')$ under μ, or

(ii) a *preventer* i_p in a sequence points to the first student of a (possibly different) sequence; that is, to the student who is assigned to school $\eta(i)$ under μ.

By definition of a sequence, since only the last student is *prevented*, (i) implies that $i' \in X_\mu(\eta(i'))$. Moreover, by definition of a *preventer*, (ii) implies that $i_p \in X_\mu(\eta(i))$. Thus, each edge in the cycle γ_1 is also an edge in the *directed application graph* $G(\mu)$. Thus, γ_1 is a *SEPF-cycle*.

We next show that the matching $\gamma_1 \circ \mu$ *Pareto dominates* μ and is *(weakly) Pareto dominated* by η. First note that under the matching $\gamma_1 \circ \mu$, each student i' who (in γ_1) points to the next student in the sequence is assigned to school $\eta(i')$. Also, in the cycle γ_1, a *preventer* i_p points to a student, who is assigned to school $\eta(i)$ under μ. We claim $\eta(i_p)\ R_{i_p}\ \eta(i)$. Suppose $\eta(i)\ P_{i_p}\ \eta(i_p)$, that is $i_p \in D_\eta(\eta(i))$. By definition of a *preventer*, $i_p \notin C(\eta(i))$ and $i_p \succ_{\eta(i)} i$. Thus, matching η *violates the priority of student* i_p *at school* $\eta(i)$, a contradiction. Thus, under the matching $\gamma_1 \circ \mu$, each student in γ_1 is better off than the matching μ and weakly worse off than the matching η; each remaining student is assigned to the same school to which she is assigned under μ, which implies that the matching $\gamma_1 \circ \mu$ *Pareto dominates* μ and is *weakly Pareto dominated* by η. Moreover, by the same argument in Lemma 1, $\gamma_1 \circ \mu$ is *partially stable*. If the matching $\gamma_1 \circ \mu$ is equivalent to η, the proof is complete. If not, we use the same argument inductively: By Lemma 4, there is a set of distinct *improvement cycles* such that the matching η is obtained by *solving* these cycles over $\gamma_1 \circ \mu$ and one can construct a *SEPF-cycle*. ■

The idea behind Lemma 5 can perhaps be best illustrated via an example. The following is such an illustration which demonstrates how to construct the SEPF-cycles out of the improvement cycles.

Example 3 *Let* $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$, $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$, $q_s = 1$ *for each* $s \in S$. *Assume that* $i_3 \notin C(s_1)$, $i_3 \notin C(s_6)$ *and* $i_5 \notin C(s_2)$.

The students' preferences and schools' priorities are as follows:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
s_2	s_3	s_4	s_1	s_6	s_5
s_1	s_2	s_1	s_4	s_2	s_6
$\vdots$	$\vdots$	s_6	$\vdots$	s_5	$\vdots$
$\vdots$	$\vdots$	s_3	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$	$\succ_{s_6}$
i_1	i_2	i_3	i_4	i_5	i_6
i_3	i_5	i_2	i_3	i_6	i_3
i_4	i_1	$\vdots$	$\vdots$	$\vdots$	i_5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Consider the matchings:

$$\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4), (i_5, s_5), (i_6, s_6)\}$$

$$\eta = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_6), (i_6, s_5)\}$$

It's easy to see that both matchings are *partially stable* and η *Pareto dominates* μ. Thus, by Lemma 4, there is a set of distinct improvement cycles such that solving those under μ yields η. Indeed, two improvement cycles which can be solved to get η from ν are: $\{i_1i_2, i_2i_3, i_3i_4, i_4i_1\}$ and $\{i_5i_6, i_6i_1\}$. Note that every student is a part of an improvement cycle (this is not necessarily true in general). The graph $G(\mu, \eta)$ is as follows:

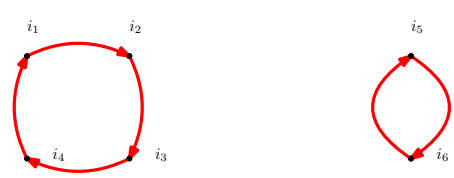

Figure 4: Graph $G(\mu, \eta)$

Even though these improvement cycles can always be found, as discussed in Lemma 5, these cycles need not appear in $G(\mu)$, because some agents may *prevent* some others. In the example, i_3 prevents i_4 because $i_3 \notin C(s_1)$ and $i_3 \succ_{s_1} i_4$. Similarly, i_3 prevents i_5 since $i_3 \notin C(s_6)$, and i_5 prevents i_1 since $i_5 \notin C(s_2)$. The prevented students are i_1, i_4, i_5 and their preventers are i_5, i_3, i_3, respectively. Note that all the preventers are in an improvement cycle. (This is a direct consequence of everyone being a part of an improvement cycle, but indeed Lemma 5 demonstrates that this is always the case.) Then, as in Lemma 5, we can construct the graph $G^{SEPF}(\mu, \eta)$. As defined in the lemma, we break the cycle $\{i_1i_2, i_2i_3, i_3i_4, i_4i_1\}$ into its sequences. Take the prevented student i_1 and the student she points to in the improvement cycle, i_2. The sequence therefore begins with i_2 and ends with the next prevented student, who is i_4. Thus, (i_2i_3, i_3i_4) is a sequence we add. Similarly, (i_1) is another sequence, and (i_6i_5) is another sequence we add to $G^{SEPF}(\mu, \eta)$. Finally, we add the sequences that include preventers, which are (i_5i_2), (i_3i_1) and (i_3i_6). To sum up, the graph $G^{SEPF}(\mu, \eta)$ includes vertices I and edges: $\{i_2i_3, i_3i_4, i_6i_5, i_5i_2, i_3i_1, i_3i_6\}$. It is demonstrated in the following figure:

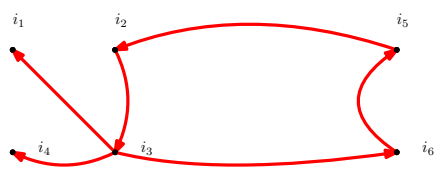

Figure 5: Graph $G^{SEPF}(\mu, \eta)$

Clearly, the cycle $\gamma_1 := (i_2i_3, i_3i_6, i_6i_5, i_5i_2)$ appears in the graph. The main contribution of Lemma 5 is proving that a cycle constructed in such a manner will also appear in $G(\mu)$. For the example, one can see this by directly drawing $G(\mu)$. The following figure demonstrates $G(\mu)$ and $G(\mu, \eta)$. The edges in red are those in $G(\mu)$, and the edges in green appear in

$G(\mu, \eta)$. This figure makes it clear that even if the improvement cycles do not appear in $G(\mu)$, there is a cycle which appears in $G(\mu)$.

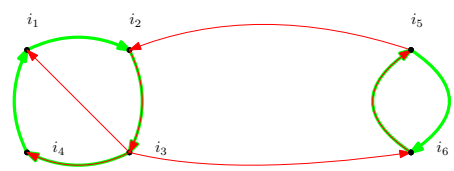

Figure 6: Graph $G(\mu)$ (in red) and $G(\mu, \eta)$ (in green).

The rest of the exercise is straightforward. One can solve the cycle $\{i_2 i_3, i_3 i_6, i_6 i_5, i_5 i_2\}$ to obtain the matching:

$$\gamma_1 \circ \mu = \{(i_1, s_1), (i_2, s_3), (i_3, s_6), (i_4, s_4), (i_5, s_2), (i_6, s_5)\}$$

It's trivial to check that $\gamma_1 \circ \mu$ *Pareto dominates* μ and is *Pareto dominated* by η. One can then repeat the same argument: There is an improvement cycle $\{i_1 i_5, i_5 i_3, i_3 i_4, i_4 i_1\}$, the solution of which under $\gamma_1 \circ \mu$ gives η. The graph $G(\gamma_1 \circ \mu, \eta)$ is demonstrated in the following figure.

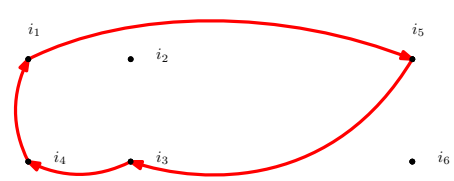

Figure 7: Graph $G(\gamma_1 \circ \mu, \eta)$

The cycle $\{i_1 i_5, i_5 i_3, i_3 i_4, i_4 i_1\}$ does not appear under $G(\gamma_1 \circ \mu)$ because i_4 is prevented by i_3. One can again follow the same procedure to derive the SEPF cycle $\gamma_2 := \{i_1 i_5, i_5 i_3, i_3 i_1\}$

which appears under $G(\gamma_1 \circ \mu)$. Graphs $G(\gamma_1 \circ \mu)$ and $G(\gamma_1 \circ \mu, \eta)$ are demonstrated in the following figure.

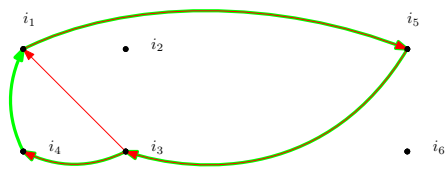

Figure 8: Graph $G(\gamma_1 \circ \mu)$ (in red) and $G(\gamma_1 \circ \mu, \eta)$ (in green).

Solution of γ_2 yields the following matching:

$$\gamma_2 \circ \gamma_1 \circ \mu = \{(i_1, s_2), (i_2, s_3), (i_3, s_1), (i_4, s_4), (i_5, s_6), (i_6, s_5)\}$$

Again, $\gamma_2 \circ \gamma_1 \circ \mu$ *Pareto dominates* $\gamma_1 \circ \mu$ and is *Pareto dominated* by η. Indeed, solving the cycle $\gamma_3 := \{i_3 i_4, i_4 i_3\}$ under $\gamma_2 \circ \gamma_1 \circ \mu$ yields η. One can check that γ_3 appears under $G(\gamma_2 \circ \gamma_1 \circ \mu)$ (i.e. there are no preventers.) Thus, it is a SEPF-cycle, and $\gamma_3 \circ \gamma_2 \circ \gamma_1 \circ \mu = \eta$. Let η be a *constrained efficient* matching which *weakly Pareto dominates* μ_0. By Lemma 5, η can be obtained from μ_0 by *solving SEPF-cycles* and thus, the proof of the "only if" part follows.

Appendix C The Efficiency Adjusted Deferred Acceptance Algorithm (EADAM)

An important observation on the DA is that a student's (say i) priority at a school (say s) could prevent another student from enrolling to s although i is not enrolled to s at the matching given by the DA. To formalize this point, Kesten (2010) introduces the following definition: If student i is tentatively accepted by school s at some step t and is rejected by s in a later step t' of DA and there exists another student j who is rejected by s in

step $t'' \in \{t, t+1, ..., t'-1\}$, then student i is called **interrupter** for school s and (i, s) is called an **interrupting pair** of step t'. For a given problem and a set of consenting students, EADAM selects its outcome through the following algorithm:

Efficiency-Adjusted Deferred Acceptance Algorithm:

Round 0**:** Run the DA algorithm.

Round $k > 0$: Find the last step of the DA run in Round $k-1$ in which a consenting interrupter is rejected from the school for which she is an interrupter. Identify all the interrupting pairs of that step with consenting interrupters. For each identified interrupting pair (i, s), remove s from the preferences of i without changing the relative order of the other schools. Rerun DA algorithm with the updated preference profile. If there are no more consenting interrupters, then stop.

We argue that the matching given by the EADAM is *constrained efficient* and thus, it belongs to the SEPF class. Actually, a more general result can be obtained by providing a slight generalization of the EADAM. To this end, we generalize the idea of *consent* adopted in Kesten (2010) by allowing each student to *consent* for violation of her priorities at *selected* schools, instead of restricting students to *consent* for all schools or not to *consent* for any school. That is, differently from Kesten (2010), we assume that when a student *consents* for a school s she does not have to *consent* for violation of her priorities at all schools. This difference is important also for strategic *consenting* which is analyzed in Section 4.

Generalized Efficiency-Adjusted Deferred Acceptance Algorithm:

Round 0**:** Run the DA algorithm.

Round $k > 0$: Find the last step of the DA run in Round $k-1$ in which an interrupter is rejected from the school for which she is an interrupter and she consents. Identify all the interrupting pairs of that step with interrupters who consent for the schools in those pairs. For each identified interrupting pair (i, s), remove s from the preferences of i without

changing the relative order of the other schools. Rerun DA algorithm with the updated preference profile. If there are no more interrupters who consent for the schools they are an interrupter to, then stop.

Proposition 7 *For each problem (R, C), the matching obtained by the generalized EADAM is constrained efficient and weakly Pareto dominates the SOSM.*

Proof. We first show that the outcome of the generalized EADAM *weakly Pareto dominates* the SOSM, which is the tentative outcome of generalized EADAM in Round 0. When the preference profile is updated in Round $k > 0$ at most one school is removed from each student's preferences while keeping the relative order of the other schools fixed and the removed school has rejected her in Round $k - 1$. That is, each student's assignment in Round $k - 1$ is not removed. The priority order does not change between rounds. Thus, the matching selected in Round $k - 1$ is *stable* under the updated preference profile in Round k. Since the DA selects the SOSM under the updated preference profile, in Round k, no student is assigned to a school worse than her assignment in Round k. This shows that matching in Round k weakly Pareto dominates matching in Round $k - 1$, and the result follows by transitivity of Pareto dominance relation.

Since the generalized EADAM outcome *Pareto dominates* the DA outcome, it is *individually rational* and *non-wasteful.* At each step, the DA is run and the generalized EADAM allows a student's priorities to be violated only if she *consents* for the violation of those priorities. Thus, the final outcome of generalized EADAM is *partially stable.*

What remains to be shown is that the outcome is *constrained efficient,* that is, it cannot be Pareto dominated by another partially stable matching. Suppose the generalized EADAM terminates in Round K. Let $(\tilde{R}, C)$, and $\tilde{\mu}$ be the problem considered in Round K and the matching selected by the DA in Round K, respectively. We need that $\tilde{\mu}$ cannot be *Pareto dominated* by another *partially stable* matching in problem (R, C).

We first show that $\tilde{\mu}$ is constrained efficient in problem $(\tilde{R}, C)$. Since $\tilde{\mu}$ is the SOSM for problem $(\tilde{R}, C)$, it is *stable* for problem $(\tilde{R}, C)$, which implies that it is *partially stable* for

$(\tilde{R}, C)$. We now show that there does not exist another *partially stable* in problem $(\tilde{R}, C)$ which Pareto dominates $\tilde{\mu}$. Take another *partially stable* matching, $\bar{\mu}$, for problem $(\tilde{R}, C)$. For any school $s \in S$, let $r_n(s)$ and $t_n(s)$ denote the sets of rejected and tentatively accepted students by s, respectively, in the nth step of the DA run in the last round (Round K) of the generalized EADAM. By induction we show that for each $n > 0$, students in $r_n(s)$ cannot be assigned to s in $\bar{\mu}$ and thus, $\bar{\mu}$ cannot *Pareto dominate* $\tilde{\mu}$.

Consider Step 1 of the DA run in Round K of the generalized EADAM. If for each $s \in S$, $r_1(s) = \emptyset$, then each student is assigned to her top-ranked school in problem $\tilde{R}$ and $\tilde{\mu}$ is *Pareto efficient*, that is, it cannot be *Pareto dominated* by another matching with respect to preference profile $\tilde{R}$. Otherwise, let $s \in S$ be a school such that $r_1(s) \neq \emptyset$. Since the generalized EADAM terminates at the end of this round (Round K), there does not exist an *interrupting pair* (i, s) such that $i \in C(s)$. Thus, each student in $t_1(s)$ is either (i) permanently accepted by s at $\tilde{\mu}$, or (ii) they are in $t_1(s) \setminus \tilde{\mu}(s)$. But no one in $t_1(s) \setminus \tilde{\mu}(s)$ *consents* for s (Otherwise, the consenting student in $j \in t_1(s) \setminus \tilde{\mu}(s)$ is an *interrupter* for s this contradicts generalized EADAM terminating in Round K.) In either case, students in $r_1(s)$ cannot be assigned to s at any partially stable matching, in particular $\bar{\mu}$. Now suppose that students in $r_{\bar{n}}(s)$ are not assigned to s in $\bar{\mu}$ for each $n \leq \ell$ and $s \in S$. Consider step $\ell + 1$. Students who are tentatively accepted by school s in step ℓ and students applying to school s in step ℓ have already been rejected by their better options, and by inductive hypothesis, they cannot be assigned to a better school than s in $\bar{\mu}$. If for each $s \in S$, $r_{\ell+1}(s) = \emptyset$, then for each $s \in S$, $t_{\ell+1}(s) = \tilde{\mu}^{-1}(s)$ and any student i cannot be assigned to a better school than $\tilde{\mu}(i)$ at $\bar{\mu}$. Suppose for some $s \in S$, $r_{\ell+1}(s) \neq \emptyset$. Then, each student in $t_{\ell+1}(s)$ is either (i) permanently accepted by s at $\tilde{\mu}$ or (ii) they are in $t_{\ell+1}(s) \setminus \tilde{\mu}(s)$. Again, no one in $t_{\ell+1}(s) \setminus \tilde{\mu}(s)$ consents for s, otherwise K cannot be the last step. In either case students in $r_{\ell+1}(s)$ cannot be assigned to s at $\bar{\mu}$. The induction follows, and we have the result that any $\bar{\mu}$ which is *partially stable* in $(\tilde{R}, C)$ cannot *Pareto dominate* $\tilde{\mu}$; hence, $\tilde{\mu}$ is *constrained efficient* in $(\tilde{R}, C)$.

Now, our final claim is that *constrained efficiency* in problem $(\tilde{R}, C)$ implies *constrained*

efficiency in (R, C). We prove this by backward induction. Let μ_k be the matching selected in Round k, with $k \in \{1, \ldots, K\}$. By Tang and Yu (2014) (Section 3.2), the students whose preferences are updated in Round k are assigned to *underdemanded* schools in matching μ_{k-1}. That is, only difference between problems considered in Round k and $k-1$ is in the preference profiles of some students who are assigned to *underdemanded* schools in Round $k-1$, and these are the students whose assignments can't be made better off in any partially stable matching. Thus, if a matching is *constrained efficient* in problem considered in Round k, then it is also *constrained efficient* in problem considered in Round $k-1$. The induction follows, and we obtain the result. ■

Appendix D The SEPF Class and No-Consent-Proofness

Example 4 ***(A student may gain by not consenting for some cycle selection under the SEPF class.)*** *Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, and $q_s = 1$ for all $s \in S$. Assume that $C(s_1) = \{i_3, i_4\}$ and $C(s_j) = \emptyset$ for all $j \in \{2, 3, 4\}$.*

The preferences are as given below:

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
$\underline{s_2}$	$\underline{s_1}$	$\boxed{s_1}$	s_1
$\boxed{s_3}$	$\boxed{s_2}$	$\underline{s_3}$	$\boxed{s_4}$
s_1	s_4	s_4	$\vdots$
s_4	s_3	s_2	$\vdots$

and the priority structure is:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
i_1	i_2	i_3	$\vdots$
i_4	i_1	i_1	$\vdots$
i_3	$\vdots$	$\vdots$	$\vdots$
i_2	$\vdots$	$\vdots$	$\vdots$

The SOSM for this problem is $\mu_0 = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4)\}$. *Given the SOSM* μ_0, $X_{\mu_0}(s_1) = \{i_2, i_3, i_4\}$, $X_{\mu_0}(s_2) = \{i_1\}$, $X_{\mu_0}(s_3) = \{i_1\}$ *and the graph* $G(\mu_0)$ *has two cycles:* $\phi_1 = (i_1 i_2, i_2 i_1)$ *and* $\phi_2 = (i_1 i_3, i_3 i_1)$. *By solving* ϕ_1 *(*ϕ_2*), underlined matching* μ_1 *(boxed matching* μ_2*) is obtained. Both matchings are constrained efficient.*

Note that matching μ_2 *is the one preferred by* i_3. *Now, suppose the cycle selection rule solves* ϕ_1 *in this example. If* i_3 *refuses to consent for* s_1, *she can guarantee that matching* μ_2 *is obtained. This is because, by not consenting,* i_3 *guarantees that* $X_{\mu_0}(s_1) = \{i_3, i_4\}$ *and there is only one cycle (*ϕ_2*) to be* solved, *and any rule within SEPF must solve it. But then, any rule that solve* ϕ_1 *fails to satisfy no-consent-proofness.*

Appendix E The Top Priority Algorithm

E.1 An example

Let us consider the problem given in Example 2 in Appendix A. The SOSM for this problem is $\mu_0 = \{(i_1, s_3), (i_2, s_1), (i_3, s_2), (i_4, s_4), (i_5, s_5), (i_6, s_5)\}$ and the application graph associated with μ_0, $G(\mu_0)$, is given in Figure 1 in Appendix A. The TP-graph $G^T(\mu_0)$ is obtained from $G(\mu_0)$ in the following way: (i) the students who are *permanently matched* at μ_0 are removed, and (ii) if, in the remaining graph, more than one student point to j, then only the one with the highest priority for $\mu_0(j)$ points to j. The crucial point here is the order in which (i) and (ii) are conducted. Suppose that step (ii) is conducted first such that among the students pointing to a particular student, say i, in $G(\mu_0)$, the top priority student is selected, and only this student points to i . This gives the graph in Figure 5.

This graph has no cycles but the application graph $G(\mu_0)$ has. Thus, when step (i) is skipped, in general, we end up with a matching which is not *constrained efficient*. The TP-algorithm, on the other hand, ignores the *permanently matched* students when selecting the student with the highest priority for a given school: students i_5 and i_6 are *permanently matched*

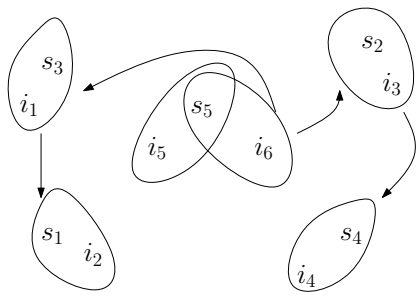

Figure 9: The highest priority students are chosen before *permanently matched* students are removed from the application graph.

at μ_0 (note that $\mu_0(i_5) = \mu_0(i_6) = s_5$ has *no demand* at μ_0) and the edges that originate from these students are removed in step (i) resulting in the subgraph of $G(\mu_0)$ in Figure 6.

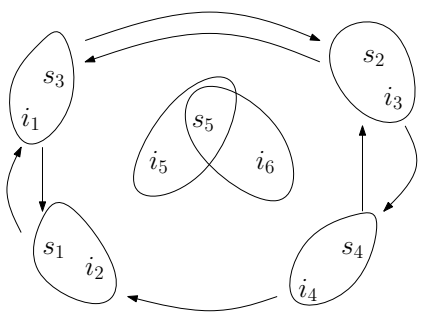

Figure 10: The subgraph of $G(\mu_0)$ after *permanently matched* students' demands are ignored.

Among the students pointing to a student i in this graph, the student with the highest priority at school $\mu_0(i)$ is selected and in the graph $G^T(\mu_0)$, only that student points to i (Figure 7).

There are two cycles in this graph: $\phi_1 = (i_3 i_4, i_4 i_3)$ and $\phi_3 = (i_1 i_2, i_2 i_1)$ (for ease of comparison, we denote the cycles by the same letters as in Example A.) The TP-algorithm proceeds by *solving* both of these cycles simultaneously and the matching

$$\mu_1 = \{(i_1, s_1), (i_2, s_3), (i_3, s_4), (i_4, s_2), (i_5, s_5), (i_6, s_5)\}$$

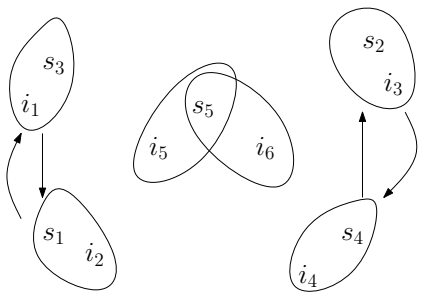

Figure 11: $G^T(\mu_0)$

is obtained.[20] The graph $G(\mu_1)$ is given in Figure 8.

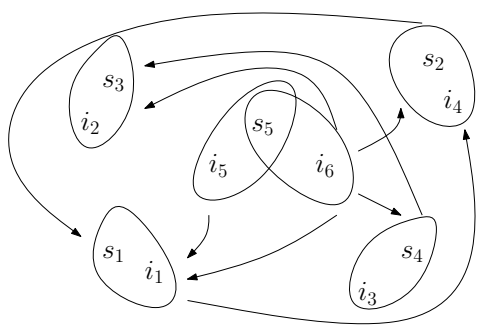

Figure 12: Graph $G(\mu_1)$

Once again, the TP-algorithm proceeds by first ignoring the demands of *permanently matched* students, who are i_2, i_3, i_5 and i_6 (Figure 9).

Since no student is pointed by more than one student, the TP-graph $G^T(\mu_1)$ is the same as in Figure 9. By *solving* the only cycle (i_1i_4, i_4i_1) in the graph $G^T(\mu_1)$, the matching

$$\mu_2 = \{(i_1, s_2), (i_2, s_3), (i_3, s_4), (i_4, s_1), (i_5, s_5), (i_6, s_5)\}$$

[20] Actually, in the TP-algorithm, only one cycle is *solved* at each step. But, as we argue in the proof of Proposition 2 (see Appendix F), the order of cycles *solved* is not consequential. Thus equivalently, each cycle in the TP-graph can be *solved* simultaneously.

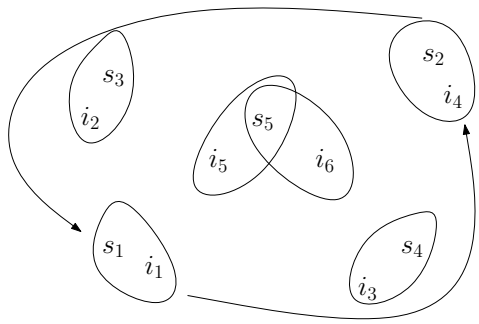

Figure 13: The subgraph of $G(\mu_1)$ after *permanently matched* students' demands are ignored.

is obtained. In the graph $G(\mu_2)$, there is no cycle (see Figure 10). Thus, the TP-algorithm stops and the matching obtained by the TP-algorithm is μ_2. This matching is also the one obtained by the (generalized) EADAM (see footnotes in Appendix A).

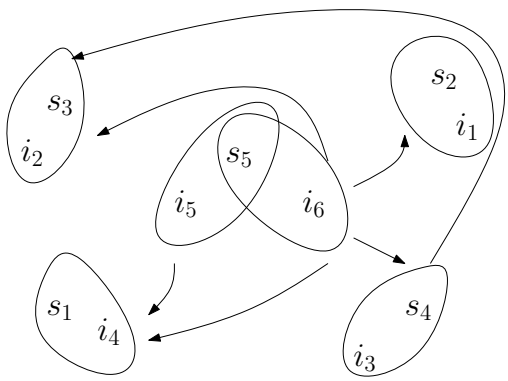

Figure 14: $G(\mu_2)$

E.2 An insight for how the TP-rule works

A cycle may form at a later step throughout the TP-algorithm. The reason is simple: a *temporarily matched* student, say i, might prevent other students to form a cycle at a step since she has the highest priority at a particular school, say s, among the students who prefer s to their current school. It could be however that student i is part of a cycle at a

later step, where she points to students assigned to a better school than s (with respect to i's preferences), and a new cycle forms because student i no longer prevents another student from pointing to the students assigned to school s. We argue that if a cycle forms at a step, it forms for each order of cycles solved throughout the algorithm. We begin by introducing a simple remark.

We say that a trail $\{i_1i_2, i_2i_3, \ldots, i_ni_{n+1}\}$ is a **cycle-trail** if for some $m < n+1$, the vertices $i_1, i_2 \ldots, i_{m-1}, i_{m+1} \ldots i_{n+1}$ are distinct and $i_1 = i_m$ (for the cycle $\{i_1i_2, i_2i_3, \ldots, i_{m-1}i_1\}$ denoted by ϕ, we call the associated cycle-trail also as **ϕ-trail**).

A *cycle-trail* is depicted in the following figure:

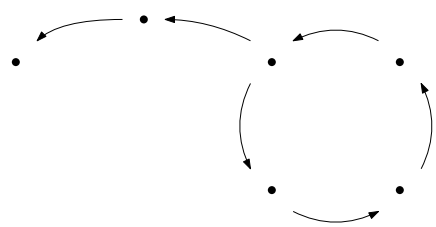

Figure 15: A cycle-trail.

Remark 4 *Student i is temporarily matched at μ_k and not part of a cycle in the graph $G^T(\mu_k)$ if and only if i is the endpoint of a cycle-trail at μ_k.*

Proof. (If) By definition, the school of i is not *underdemanded* at μ_k. Since each student is pointed by at most one student, the only cycle with a student on the ϕ-trail is ϕ and student i is not part of it. (Only if) This follows directly from the definition of an *underdemanded* school and the fact that the TP-graph at each step is such that each student is *temporarily matched* if and only if she is pointed by a unique student. ■

Suppose cycle $\omega = \{i_1i_2, i_2i_3 \ldots, i_ni_1\}$ exists in graph $G^T(\mu_k)$ but not in graph $G^T(\mu_{k-1})$; that is, cycle ω *forms* at step k. We claim that for each order of cycles *solved*, cycle ω forms at some step k' (before the TP-algorithm terminates) such that for each $i \in \{i_1, i_2 \ldots, i_n\}$, $\mu_{k'}(i) =$

$\mu_k(i)$.[21]

Take a particular order of cycles *solved*. let $\omega_1 = \{i_1 i_2, i_2 i_3 \ldots, i_n i_1\}$ be the first cycle such that for some k, ω_1 existed in graph $G^T(\mu_k)$ but not in graph $G^T(\mu_{k-1})$. We will show that that ω_1 forms for each order of cycles *solved*.[22]

By Lemma 7 (Appendix F), each $i \in \{i_1, i_2 \ldots, i_n\}$ is *temporarily matched* at μ_0. By Remark 4, this implies that if i is not part of a cycle in the graph $G^T(\mu_0)$, then i is the endpoint of a cycle-trail. Suppose i is part of a cycle ϕ' in the graph $G^T(\mu_0)$. First, note that the cycles in any TP-graph do not intersect. Second, by Lemma 9 (Appendix F), each cycle which is not *solved* at some step exists in the TP-graph at the next step. Thus, ϕ' is *solved* before ω_1 forms. When ϕ' is *solved*, there are two cases: i is part of a new cycle (that is, a new cycle forms including i and since ω_1 is the first cycle that forms, this new cycle can only be ω_1) or not (by Lemma 7 and Remark 4, this implies that i is the endpoint of a cycle-trail). The graph $G^T(\mu_0)$ has the structure depicted in Figure 12.

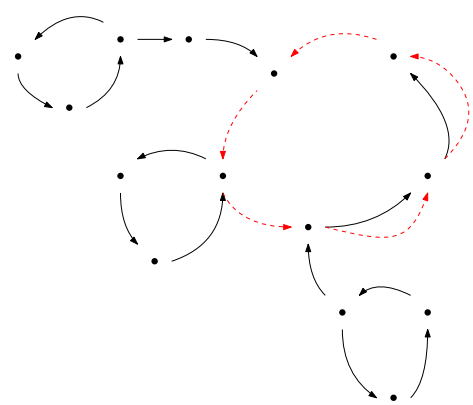

Figure 16: Graph $G^T(\mu_0)$: the dotted cycle is ω_1, which forms at step k of the algorithm.

[21]This intuition suggests that if a cycle forms for a particular order of cycles *solved*, then the same cycle forms for each order of cycles *solved* as well. The 'same' cycle means that not only the edges are the same, but also each student in the cycle is assigned to the same school under the corresponding matchings as well. (Note that in general, two cycles might have the same set of edges although it is possible that a student is assigned to different schools in these two different cycles under the corresponding matchings.)

[22]A more general result is proved in Appendix F: each cycle that forms at some step, forms for each order of cycles formed. Here, we aim to provide an intuition for how this result holds.

By Remark 4 and the fact that ω_1 is the first cycle that forms, if student $i \in \{i_1, i_2 \ldots, i_n\}$ is the endpoint of a cycle-trail in the TP-graph at a step before ω_1 forms, then i is the endpoint of a cycle-trail (possibly a different cycle-trail) until ω_1 forms.

Let $ij \in \omega_1$ such that $ij \notin G^T(\mu_0)$. Suppose that under a different order of cycles *solved*, i is part of a cycle before edge ij forms. Suppose that i is the endpoint of a cycle-trail before it is part of a cycle. By Remark 4, at the step when i is part of a cycle, the cycle in the cycle-trail with the endpoint i is *solved*; otherwise, i cannot be in a cycle. Suppose that student i points to a student i' on the *solved* cycle so that a cycle different than ω_1 forms.[23] Note that the *solved* cycle is in the graph $G^T(\mu_0)$. Thus, once it is *solved* under the original order of cycles *solved*, student i (if not, another student) points to i'. If another student points to student i', then, since i is not part of a cycle, then by Remark 4, she is the endpoint of a *cycle-trail*. But, student i' cannot be in an *underdemanded* school since student i prefers the school of student i' to her current school. Thus, at some step, i points to i' and a cycle including i (other than ω_1) forms also under the original order of cycles *solved*. Since i is an end-point of a cycle-trail until that step by Remark 4, the cycle ω_1 does not form before that. Thus, the cycle including i and i' forms before cycle ω_1 forms under the original order of cycles selected. This contradicts with ω_1 being the first cycle formed. Thus, student i cannot be in a cycle other than ω_1 before the edge ij forms. Since the edge ij is arbitrarily chosen, this holds for any edge in the cycle ω_1.

Lemma 6 *In the graph $G^T(\mu_{k-1})$, let cycle $\phi_k = \{i_1 i_2, i_2 i_3 \ldots, i_n i_1\}$ be* solved *such that $\mu_k = \phi_k \circ \mu_{k-1}$. Then, i points to j in $G^T(\mu_k)$ but not in $G^T(\mu_{k-1})$ implies that there exists $i' \in I$ where i' points to j in $G^T(\mu_{k-1})$ such that either (i) $i' \in \{i_1, i_2, \ldots, i_n\}$, or (ii) $\mu_{k-1}(i')$ is underdemanded at μ_k.*

Proof. Let i point to j in $G^T(\mu_k)$ but not in $G^T(\mu_{k-1})$. First note that, by definition of the graph $G^T(\mu_k)$, both students i and j are *temporarily matched* at μ_k. By Lemma

[23] It could be that in the new formed cycle there is a path from i to i', instead of an edge between i and i'. But, the same argument that follows continue to hold. Thus, without loss of generality, we assume that i points i'.

7, schools $\mu_k(i)$ and $\mu_k(j)$ are not *underdemanded* at μ_{k-1}. If $\mu_{k-1}(j) \neq \mu_k(j)$, then $j \in \{i_1, i_2, \ldots, i_n\}$, so she was in cycle and pointed by another agent, and the result follows. Suppose $\mu_{k-1}(j) = \mu_k(j) = s$. This implies that $j \notin \{i_1, i_2, \ldots, i_n\}$. Since i points to j in $G^T(\mu_k)$, $i \in D_{\mu_k}(s)$. By Remark 2, $i \in D_{\mu_{k-1}}(s)$ as well. Since i does not point to j in the graph $G^T(\mu_{k-1})$ and neither of the students i and j is *permanently matched* at μ_{k-1}, there is a student in the set $D_{\mu_{k-1}}(s) \setminus C(s)$ with a higher priority than student i at school s. Let i' be the student with the highest priority among these students who are *temporarily matched* at μ_{k-1}. By definition, $i' \in X_{\mu_{k-1}}(s)$ and i' points to j in $G^T(\mu_{k-1})$. Since i points to j in the graph $G^T(\mu_k)$, student i' is assigned under μ_k either (i) to a better school than s, which is possible only if $i' \in \{i_1, i_2, \ldots, i_n\}$, or (ii) to an *underdemanded* school. If $i' \notin \{i_1, i_2, \ldots, i_n\}$, then $\mu_k(i') = \mu_{k-1}(i')$ and $\mu_{k-1}(i')$ is *underdemanded* at μ_k. ■

By Lemma 6, the edge $ij \in \omega_1$ does not form unless the cycle in a cycle-trail with the endpoint j is *solved*. Note that depending on the order of the cycles *solved*, the cycle-trails with the endpoint j may form in a different order. But, by Lemma 9 (Appendix F), the cycle in each of these cycle-trails is *solved*, thus the edge ij forms independently from the order of these cycles *solved*. Thus, cycle ω_1 forms for each order of cycles *solved*.

Appendix F Proof of Proposition 2

Lemma 7 *Suppose student i is assigned to an underdemanded school at step k of the SEPF algorithm. Then, at each step $t \geq k$, she is assigned to an underdemanded school, thus she is not part of any cycle.*

Proof. Let $\mu_k(i) = s$. Remember that if school s is underdemanded at step k, either s has no demand at μ_k or each path to i starts with a student assigned to a school with no demand. That is, i is a part of a path in $G(\mu_k)$, and she may be located at any part of this path (including the tail).[24] We prove the result by induction on the location of i within the path.

[24] One can interpret a student who is not pointed by anyone and who doesn't point to anyne as a path with 1 vertex, so this is a comprehensive statement.

Initial step: Student i is at the tail: she assigned to a school with no demand. Since i is not pointed in $G(\mu_k)$, $X_{\mu_k}(s) = \emptyset$. By Lemma 2, $D_{\mu_k}(s) = \emptyset$. Moreover, by Remark 2, $D_{\mu_k}(s) = \emptyset$ implies that, for each $t \geq k$, $D_{\mu_t}(s) = \emptyset$. Since by definition, $X_\mu(s) \subseteq D_\mu(s)$, for each $t \geq k$, $X_{\mu_t}(s) = \emptyset$, and student i is not part of any cycle.

Inductive hypothesis: Student i is pointed only by students who are not part of a cycle at each step $t \geq k$. Suppose $i' \in D_{\mu_k}(s)$ but i' does not point to i. Since $i' \in D_{\mu_k}(s) \setminus X_{\mu_k}(s)$, there exists $j \in D_{\mu_k}(s) \setminus C(s)$ such that $j \succ_s i'$ and j points to i. Since for each $t \geq k$, j is not part of a cycle, at each step $t \geq k$, i' does not point to i. Note that this is true for each student in $D_{\mu_k}(s)$ who does not point to student i at step k. Thus, at each step $t \geq k$, student i is pointed only by the students who point to her at step k, and thus she is not part of a cycle. By inductive hypothesis, if student i is assigned to an *underdemanded* school at step k, then at each step $t \geq k$, i is pointed only by students who are not part of a cycle. Thus, in each matching defined by the SEPF class, student i is matched to school $\mu_k(i)$. ■

Lemma 7 justifies the language we use for the students assigned to *underdemanded* schools at some matching μ_k: by definition, a *permanently matched* student, say i, at μ_k is assigned to an *underdemanded* school at μ_k and she is not part of any cycle through the end of the SEPF algorithm. Thus, at each *constrained efficient* matching which *weakly Pareto dominates* μ_k, student i is assigned to school $\mu_k(i)$.

Lemma 8 *In the graph $G^T(\mu_{k-1})$, let cycle $\phi_k = \{i_1 i_2, i_2 i_3 \ldots, i_n i_1\}$ be* solved *such that $\mu_k = \phi_k \circ \mu_{k-1}$. Then, if i points to j in $G^T(\mu_{k-1})$ where $i, j \notin \{i_1, i_2, \ldots, i_n\}$ and i is temporarily matched at μ_k, then i points to j in $G^T(\mu_k)$.*

Proof. Since the schools to which students i and j are assigned do not change at step k, by Remark 3, i points to j in $G(\mu_k)$. Suppose student $i' \neq i$ points to student j in $G^T(\mu_k)$. Thus, $i' \in D_{\mu_k}(\mu_k(j)) \setminus C(\mu_k(j))$ and i' is *temporarily matched* at μ_k. Since $\mu_k(j) = \mu_{k-1}(j)$, by Remark 2, $i' \in D_{\mu_{k-1}}(\mu_{k-1}(j)) \setminus C(\mu_{k-1}(j))$. Moreover since i is *temporarily matched* at μ_k, i' has a higher priority at the school $\mu_{k-1}(j)$ than i. Thus, since i (but not i') points to j in the graph $G^T(\mu_{k-1})$ implies that i' is *permanently matched* at μ_{k-1}. This contradicts

Lemma 7, which implies that for each $t \geq k-1$, i' must be *permanently matched* at the matching μ_t. ■

Lemma 9 *Suppose there are at least two cycles in the graph $G^T(\mu_{k-1})$. If a cycle ϕ in the graph $G^T(\mu_{k-1})$ is not* solved *at step $k-1$, then ϕ exists in the graph $G^T(\mu_k)$.*

Proof. Let ϕ be a cycle in the graph $G^T(\mu_{k-1})$ such that it is not *solved* at step $k-1$. Let $ij \in \phi$. By Remark 3, i points to j in $G(\mu_k)$. Since this holds for each edge in ϕ, ϕ is a cycle in $G(\mu_k)$. Thus, i is *temporarily matched* at μ_k. By Lemma 8, this implies that i points to j in $G^T(\mu_k)$. Since this holds for each $ij \in \phi$, the graph $G^T(\mu_k)$ has cycle ϕ. ■

Lemma 10 *Let μ_0 be the SOSM under problem (R, C). Consider a cycle selection order denoted by $\Phi = (\phi_1, \phi_2, ..., \phi_n)$ such that ϕ_1 occurs in $G^T(\mu_0)$, $\mu_k = \phi_k \circ \mu_{k-1}$ and ϕ_k occurs in $G^T(\mu_{k-1})$ for all $k \in \{1, 2, ..., n\}$. Denote the outcome of TP-algorithm under Φ with μ. If there exists $\tilde{k}$ such that $\phi_{\tilde{k}+1}$ occurs in $G^T(\mu_{\tilde{k}-1})$, then TP-algorithm selects μ in cycle selection order $\hat{\Phi} = (\phi_1, ...\phi_{\tilde{k}-1}, \phi_{\tilde{k}+1}, \phi_{\tilde{k}}, \phi_{\tilde{k}+2}, ..., \phi_n)$.*

Proof. Let ν_k be the matching selected in step k of TP-algorithm under $\hat{\Phi}$. Since in the first $\tilde{k}-1$ steps the same cycles are removed under both $\hat{\Phi}$ and Φ, we have $\mu_k = \nu_k$ for all $k \leq \tilde{k}-1$. Hence, $G^T(\mu_{\tilde{k}-1}) = G^T(\nu_{\tilde{k}-1})$. That is, $\phi_{\tilde{k}+1}$and $\phi_{\tilde{k}}$ exist in $G^T(\nu_{k-1})$. Moreover, $\phi_{\tilde{k}+1}$and $\phi_{\tilde{k}}$ are disjoint. When $\phi_{\tilde{k}+1}$ is solved in step k, by Lemma 9 $\phi_{\tilde{k}}$ exists in $G^T(\nu_k)$. We have $\mu_{\tilde{k}+1} = \nu_{\tilde{k}+1}$ and $G^T(\mu_{\tilde{k}+1}) = G^T(\nu_{\tilde{k}+1})$ since the cycles are disjoint and only the students in $\phi_{\tilde{k}+1}$and $\phi_{\tilde{k}}$ improved to the same schools. Then, ϕ_k occurs in $G^T(\nu_k)$ and $\mu_k = \nu_k$ for all $k \geq \tilde{k}+1$. ■

Lemma 11 *The outcome of TP-Algorithm is independent of the order of cycles solved in each step.*

Proof. We will prove this lemma by constructing a cycle selection order, Φ, which generates the same outcome as any other cycle selection order, $\hat{\Phi}$, under TP-Algorithm.

Take a given problem (R, C), and let μ_0 be the SOSM under this problem. Denote the set of cycles in $G^T(\mu_0)$ with A_0. The construction of the "universal cycle selection" order Φ

first requires a tie-breaker vector. Let $\pi = (\pi_i)_{i \in I}$ be such a tie-breaker vector where π_i is the number assigned to student $i \in I$. Given this, the order Φ is as follows:

"At round $k \geq 0$, given matching μ_k:

1. Let A_k be the set of cycles in $G^T(\mu_k)$.

2. Consider the cycles in $\cup_{\tilde{k} \leq k} A_{\tilde{k}}$ which are not yet solved. Among those, pick the cycle to solve according to the following (lexicographic) cycle selection rule:

   (a) for all m and m' such that $m < m' \leq k$, all cycles in A_m are solved before the cycles in $A_{m'}$;

   (b) for all $m \leq k$, the cycles in A_m are solved according to the highest tie breaker number of the student in the cycle.

3. Solve the cycle given according to this rule and obtain the new matching μ_{k+1}."

Suppose the cycle selection rule Φ given above ends at Round K, and yields the matching μ_K. Take any other cycle selection rule $\hat{\Phi}$. Now, we show that $\hat{\Phi}$ also produces the same matching. To see this, first realize that all cycles in A_0 necessarily appear under any cycle selection rule. By Lemma 9, they are solved under $\hat{\Phi}$. But then, by using Lemma 10, we can rearrange the order of cycles such that first $|A_0|$ rounds are the same as those of Φ, and the final outcome of $\hat{\Phi}$ is unchanged. This produces, say, $\tilde{\Phi}$, whose final outcome is the same as $\hat{\Phi}$ and whose first $|A_0|$ rounds are the same as Φ. But then, since first $|A_0|$ rounds are the same, the cycles in $\cup_{\tilde{k} \leq k} A_{\tilde{k}}$ all appear under $\tilde{\Phi}$. Once again, by Lemma 9, these cycles are solved under $\tilde{\Phi}$. One can then reapply Lemma 10 and get another cycle selection rule which yields the same outcome as $\hat{\Phi}$, and yields the same matchings in the first $|A_0| + |A_{|A_0|}|$ steps. One can then continue until the cycle selection rule whose final outcome is the same as $\hat{\Phi}$ and whose first K steps are the same as Φ. We conclude that $\hat{\Phi}$ and Φ must produce the same outcome. ■

Lemma 12 *For each $k \geq 1$, there is a cycle in the graph $G(\mu_k)$ if and only if there is a cycle in the graph $G^T(\mu_k)$.*

Proof. (Only if) Since there is a cycle in $G(\mu_k)$, the set of *temporarily matched* students, $I \setminus I^u_{\mu_k}$, is nonempty. By definition of the graph $G^T(\mu_{k-1})$, each student in $I \setminus I^u_{\mu_k}$ is pointed by a unique student in $I \setminus I^u_{\mu_k}$.[25] Thus, there exists a cycle in $G^T(\mu_k)$. In particular, each cycle in $G^T(\mu_k)$ is formed by the students in $I \setminus I^u_{\mu_k}$. (If) It follows directly from the fact that $G^T(\mu_k)$ is a subgraph of $G(\mu_k)$. ■

Proof of Proposition 2: By Lemma 12, the TP-algorithm is in the SEPF class. By Theorem 1, each matching produced by the TP-algorithm is *constrained efficient*. By Lemma 11, any cycle selection order gives the same matching under the TP-algorithm. Thus, the TP-algorithm produces a unique matching and it defines a rule.

Appendix G Proof of Theorem 2

(Proof of the "if" part)

Lemma 13 *Let i be a permanently matched student at μ for the problem (R, C). Then, student i is permanently matched at μ for each problem (R, C') where C and C' coincide except i's consent.*

Proof. Let (R, C) be a problem and μ be a matching. First, note that if i does not point to j in the graph $G(\mu)$, then i's *consent* for school $\mu(j)$ is irrelevant in terms of which students point to j.[26]

Suppose i is *permanently matched* at μ. Then, by the definition of an *underdemanded* school, either (i) $\mu(i)$ has no demand at μ or (ii) each path to i starts with a student assigned to a school with *no demand* at μ. For case (i), a school having *no demand* depends only on

[25] Note that by Remark 1, the students who are in $I \setminus I^u_{\mu_k}$ and are assigned to the same school at μ_k are pointed in $G^T(\mu_k)$ by the same student.

[26] The following argument clarifies this. Clearly, when $\mu(i) = \mu(j)$ the consent doesn't matter at all, so assume $\mu(i) \neq \mu(j)$. Now consider two cases. (i) If $\mu(i))\ P_i\ \mu(j)$, then $i \notin D_\mu(\mu(j))$, so i's consent is never used in the construction of $G(\mu)$. Therefore it doesn't determine who points to j. (ii) If $\mu(j)\ P_i\ \mu(i)$, there is another student i' pointing to j such that she has a higher priority than i at $\mu(j)$ and does not consent for $\mu(j)$. But then the consent of i is does not determine who points to j, because there is higher priority and non-consenting student.

the students' preferences, so there is no way to change it through the consenting behavior. Now assume case (ii). For this case, first realize that the only way to change the underdemanded status is through changing the arrows in the paths leading to student i. Nevertheless, consenting behavior for schools where no students in this group are assigned has no effect on these arrows. This means that we can restrict attention to changes in consents to schools where some students in the paths leading to i are assigned. Let j be such a student, i.e. a student on a path to i. Clearly, i does not point j (otherwise, i is not *permanently matched* at μ). Thus, since, by the argument in the previous paragraph, i's *consent* for school $\mu(j)$ is irrelevant in terms of which students point to j, each path to i remains the same regardless of the consent of i for the schools which the students on these paths are assigned. This means that i remains permanently matched at μ regardless of her consenting behavior for the schools of students in the paths leading to i. Therefore, $\mu(i)$remains underdemanded at μ for each problem (R, C') where C and C' coincide except i's consent. Thus, student i is *permanently matched* at μ for such a problem. ■

Proposition 8 *Under the TP-rule, the placement of a student does not change whether she consents or not. Consequently, the TP-rule is no-consent-proof.*

Proof. By the definition of the TP-rule, at each step k, the consent of only the *permanently matched* students at μ_{k-1} is relevant for the graph $G^T(\mu_{k-1})$.[27] Moreover, by Lemma 13, a student remains *permanently matched* at μ_{k-1} regardless of her consenting decisions. Also, by Lemma 7, each *permanently matched* student at μ_{k-1} is assigned to the same school under the matching given by the TP-rule. That is, whenever a student's *consent* matters at some step k of the TP-rule, then that student is already assigned to her school under the matching given by the TP-rule at an earlier step $k' < k$, and she can't affect this through her consenting decisions. ■

[27] A *temporarily matched* student can potentially affect the graph $G(\mu_{k-1})$ by her consenting decision, but not $G^T(\mu_{k-1})$. This is because the only way in which a temporarily matched student i affects $G^T(\mu_{k-1})$ by not consenting for s is by being the top priority agent among those who are temporarily matched and who point to $\mu_{k-1}^{-1}(s)$ in $G(\mu_{k-1})$. But in this case i points to $\mu_{k-1}^{-1}(s)$ under $G^T(\mu_{k-1})$ anyway, so her consenting decision is irrelevant.

(Proof of the "only if" part)

Fix a problem (R, C). Let TP denote the TP-rule and ψ denote a *constrained efficient* and *no-consent-proof* rule which gives a matching that *weakly Pareto dominates* the SOSM. The matchings given by the rules TP and ψ for problem (R, C) are denoted by $TP_{(R,C)}$ and $\psi_{(R,C)}$, respectively. Let μ_k be the matching selected at step k of the TP-rule. By the following lemma, we first show that for each k, $\psi_{(R,C)}$ *weakly Pareto dominates* the matching μ_k, which implies that $\psi_{(R,C)}$ *weakly Pareto dominates* $TP_{(R,C)}$. Since both matchings $\psi_{(R,C)}$ and $TP_{(R,C)}$ are *constrained efficient*, this implies that $\psi_{(R,C)} = TP_{(R,C)}$. This completes the proof.

Lemma 14 *For each step k of the TP-algorithm, $\psi_{(R,C)}$ weakly Pareto dominates μ_k.*

Proof. We prove this lemma by contradiction. In particular, we will start by assuming the contrary, and then we will generate a consent profile C^* for which $\psi_{(R,C^*)}$ does not produce a constrained efficient matching.

Let A_0 be an empty set. Let ϕ_k be the cycle *solved* in the graph $G^T(\mu_{k-1})$ and μ_k the matching obtained at step k of the TP-algorithm. Suppose TP-algorithm terminates at step K; that is, $\mu_K = TP_{(R,C)}$.

Suppose that, to get a contradiction, that there is a step $\tilde{k} \leq K$ where $\psi_{(R,C)}$ does not weakly Pareto dominate $\mu_{\tilde{k}}$. Let k be the first such step. That is, assume that $k \leq K$ is such that: for all all $k' < k$ and for all $i \in I$, $\psi_{(R,C)}(i)\ R_i\ \mu_{k'}(i)$ and $\mu_k(j)\ P_j\ \psi_{(R,C)}(j)$ for some $j \in I$. Let $\phi_k = \{i_1 i_2, i_2 i_3 \ldots, i_n i_1\}$. Since we chose k to be the *first* step which is not weakly Pareto dominated, there exists a student in $\{i_1, i_2 \ldots i_n\}$ who prefers her assignment under μ_k to $\psi_{(R,C)}$. Without loss of generality, suppose it is student i_1. That is, assume: $\mu_k(i_1) P_{i_1} \psi_{(R,C)}(i_1)$. Note that $\mu_k(i_1) = \mu_{k-1}(i_2)$, and denote $\mu_{k-1}(i_2)$ with s_1.

We begin by adding the student-school pair (i_1, s_1) to A_0. Let $A_1 := A_0 \cup \{(i_1, s_1)\}$. Furthermore, consider consent profile C^1 such that $i_1 \notin C^1(s_1)$ and the *consent* profile for the remaining schools/students is the same as C. Now, we consider two possible cases:

Case 1: Suppose student i_1 does not *consent* for s_1; that is, $i_1 \notin C(s_1)$. (Note that in this case, $C = C^1$, so we don't change anything on the original consent profile. Moreover, we have: $s_1 P_{i_1} \psi_{(R,C^1)}(i_1)$ by assumption.) i_1 has the highest priority at s among the *temporarily matched* students who prefer s_1 to their assignment at μ_{k-1}. Thus, only i_1 points to students in $\mu_{k-1}^{-1}(s_1)$. Since $\psi_{(R,C^1)}$ is *constrained efficient* and *weakly Pareto dominates* μ_{k-1}, the matching $\psi_{(R,C^1)}$ is obtained by solving of a sequence of *SEPF-cycles* (Theorem 1). Thus, since i_1 is assigned a school worse than s_1 under $\psi_{(R,C^1)}$, she prevents each student not in $\mu_{k-1}^{-1}(s_1)$ from being assigned to school s_1 under $\psi_{(R,C^1)}$. Moreover, at each *partially stable* matching *weakly Pareto dominating* μ_{k-1}, the number of students assigned to school s_1 is $|\mu_0^{-1}(s_1)|$ (Lemmas 4 and 5). Thus, student i_1 prevents each student in $\mu_{k-1}^{-1}(s_1)$ from being better off under $\psi_{(R,C^1)}$ as well. Thus, each student in $\mu_{k-1}^{-1}(s_1)$ is assigned to s_1 under $\psi_{(R,C^1)}$. That is, *if $i_1 \notin C(s_1)$, then i_2 is assigned a school worse than $\mu_k(i_2) = \mu_{k-1}(i_3)$ in $\psi_{(R,C^1)}$.*

Case 2: Suppose student i_1 *consents* for s_1; that is, $i_1 \in C(s_1)$. By *no-consent-proofness* of ψ, we must have $\psi_{(R,C)}(i_1) R_{i_1} \psi_{(R,C^1)}(i_1)$. Since $s_1 P_{i_1} \psi_{(R,C)}(i_1)$, we have: $s_1 P_{i_1} \psi_{(R,C^1)}(i_1)$. There are two possibilities:

Case 2.1: $\psi_{(R,C^1)}^{-1}(s_1) = \mu_{k-1}^{-1}(s_1)$. In this case, all students in $\mu_{k-1}^{-1}(s_1)$ are assigned to s_1 in $\psi_{(R,C^1)}$. Then, *i_2 is assigned a school worse than $\mu_k(i_2) = \mu_{k-1}(i_3)$ in $\psi_{(R,C^1)}$.*

Case 2.2: $\psi_{(R,C^1)}^{-1}(s_1) \neq \mu_{k-1}^{-1}(s_1)$. In this case, there is another student j who is not in $\mu_{k-1}^{-1}(s_1)$ and who is assigned to s_1 in $\psi_{(R,C^1)}$. But by *partial fairness* of ψ, each student assigned to s_1 in $\psi_{(R,C^1)}$ must have higher priority than i_1. This means that $j \succ_{s_1} i_1$.

Consider the two possibilities: under (R, C), either $\mu_{k-1}(j) P_j s_1$ or $s_1 P_j \mu_{k-1}(j)$. In the former case, *j is assigned a school worse than $\mu_{k-1}(j)$ in $\psi_{(R,C^1)}$.* In the latter case, we must have: $j \in C(s_1)$ and j is *permanently matched* at μ_{k-1}. Then, the assignment of j to s_1 in $\psi_{(R,C^1)}$ implies that at least one student is assigned to an underdemanded school in $\psi_{(R,C^1)}$ which is worse than her assignment under μ_{k-1}.[28] Therefore, under both cases there exists

[28]The heuristics is as follows: Given that j is assigned to s_1, now, by constrained efficiency of ψ, someone must fill the seat that j left in $\mu_{k-1}(j)$ under $\psi_{(R,C^1)}$. Call this student j'. j' may prefer $\mu_{k-1}(j)$ to her assignment under $\mu_{k-1}(j')$, in which case we found the student. Alternatively, j' may prefer $\mu_{k-1}(j')$ to

$k' < k$ and a student $\tilde{j}$ such that $\mu_{k'}(\tilde{j})\ P_{\tilde{j}}\ \psi_{(R,C^1)}(\tilde{j})$.

Let's summarize everything we have done so far. We began with the first step k where ψ does not Pareto dominate μ_k. Then, we found a student-school pair (i_1, s_1), with the property that $s_1 P_{i_1} \psi_{(R,C)}(i_1)$. Then, we found a consent profile C^1 where $i_1 \notin C^1(s_1)$, and a step $k^1 \leq k$ with the following property: for some $\ell \in I$, $\mu_{k^1}(\ell)\ P_\ell\ \psi_{(R,C^1)}(\ell)$. Remember that at this point $A_1 = \{(i_1, s_1)\}$.

Now, we repeat the whole argument over again. Take step k^1 defined in the previous paragraph, and take the student-school pair $(i_2, s_2) := (\ell, \mu_{k^1}(\ell))$. Realize that by construction this pair satisfies the property that $s_2\ P_\ell\ \psi_{(R,C^1)}(i_2)$. Add this pair to A_1, and let $A_2 := A_1 \cup \{(i_2, s_2)\}$. Consider the C^2 where $i_2 \notin C^2(s_2)$ and the consent profile for the remaining schools/students is the same as C^1. Following the exact same argument, one can find a step $k^2 \leq k^1$ with the following property: for some $\ell \in I$, $\mu_{k^2}(\ell)\ P_\ell\ \psi_{(R,C^2)}(\ell)$.

In general, at each step m, given A_{m-1} and k^{m-1}, take this pair, and let $(i_m, s_m) := (\ell, \mu_{k^{m-1}}(\ell))$. Define $A_m = A_{m-1} \cup \{(i_m, s_m)\}$, find a consent profile C^m where $i_m \notin C^m(s_m)$, and a step $k^m \leq k^{m-1}$ with the following property: for some $\ell \in I$, $\mu_{k^m}(\ell)\ P_\ell\ \psi_{(R,C^m)}(\ell)$.

Realize that k^m is a weakly decreasing sequence, and A_m is expanding at each step. These two facts, combined with the finiteness of student and school sets, implies that eventually the next pair (i_{m+1}, s_{m+1}) will be a pair which is already in A_m. That is, the process will *cycle*. Fix the consent profile C^m and the step k^m at this moment, and denote them C^* and k^*, respectively. Now we have a consent profile C^*, a step k^*, and a cycle of agents $\phi = (i_1 i_2, i_2 i_3, \ldots, i_m i_1)$ which appears in $G^T(\mu_{k^*})$, with the following property: "for each $n \in \{1, \ldots, m\}$, $\mu_{k^*}(i_n)\ P_{i_n}\ \psi_{(R,C^*)}(\ell)$. Since the solution of this cycle ϕ does not violate *partial fairness* of $\psi_{(R,C^*)}$ and does not make any student worse off, $\psi_{(R,C^*)}$ cannot be *constrained efficient*. ■

$\mu_{k-1}(j)$. But then, since j is permanently matched under μ_{k-1}, either (i) j' is permanently assigned under μ_{k-1} too, (ii) j' is temporarily assigned under μ_{k-1}, but $j \notin X_{\mu_{k-1}}(\mu_{k-1}(j)$ because she is blocked by a higher-priority, non-consenting student (say, j''). In case (i), we continue with the seat that j' left. In case (ii), by partial stability, j'' must also be assigned to a better school, and we continue with the seat that j'' left. Because we always continue with seats in underdemanded schools, the process can't cycle and will eventually end up with such a student.

Appendix H Proof of Proposition 4

Proof. Consider the following problem, based on Example 7 of Kesten (2010) (p. 1319):

Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, and $q_s = 1$ for all $s \in S$. Assume that $C(s_1) = \{i_1\}$ and $C(s_2) = C(s_3) = \emptyset$.

The preferences are as given below:

P_{i_1}	P_{i_2}	P_{i_3}
s_1	s_1	s_3
s_2	s_2	s_1
s_3	s_3	s_2

and the priority structure is:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_2	i_2
i_1	i_3	i_1
i_2	i_1	i_3

The application of SOSM to this problem yields the initial matching:

$$\mu_0 = \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$$

where $G(\mu_0)$ has no cycles, hence the algorithm stops and μ_0 is identified as the end product of any cycle selection rule within SEPF.

Here, instead of truthful reporting, i_2 can manipulate the algorithm and misrepresent her preferences as: $s_1 P'_{i_2} s_3 P'_{i_2} s_2$ instead of $s_1 P_{i_2} s_2 P_{i_2} s_3$. Under this new preference profile, the SOSM yields:

$$\mu'_0 = \{(i_1, s_2), (i_2, s_3), (i_3, s_1)\}$$

where $X_{\mu'_0}(s_1) = \{i_1, i_2\}$, $X_{\mu'_0}(s_2) = \emptyset$ and $X_{\mu'_0}(s_3) = \{i_3\}$. The only cycle is $(i_3 i_2 i_2 i_3)$, and

and cycle selection rule within SEPF must solve it. Solving this cycle yields:

$$\mu = \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$$

which gives a better seat than μ_0 for s_2. Therefore, s_2 is able to gain by deviating under any cycle selection rule within SEPF. ■

Appendix I Proof of Proposition 5

Proposition 5 is almost the same as Theorem 2 of Kesten (2010), and unsurprisingly, its proof follows the proof of Proposition 2 in Kesten (2010) very closely, too. It most critically uses Theorem 3.1 of Ehlers (2008),[29] which demonstrates that a sense of strategy-proofness which is very reminiscent of the one hypothesized in the proposition is achieved by any mechanism which satisfies the two basic properties: anonymity and positive association. Anonymity is simply the requirement that the mechanism should treat the schools equally, up to the permutation of their names. Positive association is the requirement that the mechanism should be invariant to certain types of transformations in an agent's preferences. A substantial part of our discussion will contain showing that TP Rule satisfies these two properties, hence it is worth spending some time on digesting these definitions and the notation pertaining to Ehlers (2008).

Take a student $i \in I$, and fix the preferences of this student P_i. Let $\mathcal{P}_j$ be the set of all strict preferences of student $j \in I$, let $\mathcal{B}_s$ be the set of all strict priority orders for school s, and let $\mathcal{C}_s$ be the set of all consent profiles for school s. Let $\mathcal{X}_{-i} := (\mathcal{P}_j)_{j \in I \setminus \{i\}} \times (\mathcal{B}_s)_{s \in S} \times (\mathcal{C}_s)_{s \in S}$.[30] In this incomplete information setup, for each student $i \in I$, we interpret a *school choice problem with priority violation* as a probability distribution (e.g. information) $\tilde{P}_{-i}$ over $\mathcal{X}_{-i}$.[31] Let φ be a mechanism, and let $\varphi(P_i, \tilde{P}_{-i})(i)$ be the distribution of allocations that

[29] Which itself is a generalization of Roth and Rothblum (1999).

[30] Throughout this proof, for expositional simplicity, we will be assuming that each school has capacity one.

[31] The main text refers to a school choice problem with priority violation as (R, C), whereas here we add school priorities as well. The reason is that anonymity requires a setup where school names can be permuted, which implies that the priority orders of these schools need to be permuted as well.

$\varphi(P_i, \tilde{P}_{-i})$ induces over S (the set of i's possible placements). Given the preferences $P_i \in \mathcal{P}_i$ and information $\tilde{P}_{-i}$, we say that strategy $P_i' \in \mathcal{P}_i$ *stochastically dominates* strategy $P_i'' \in \mathcal{P}_i$ if for all $s \in S$, $Pr\{\varphi(P_i', \tilde{P}_{-i})(i) R_i s\} \geq Pr\{\varphi(P_i'', \tilde{P}_{-i})(i) R_i s\}$.

Given a student $i \in I$ and preferences $P_i \in \mathcal{P}_i$, for any two schools $s, s' \in S$, let $P_i^{s \leftrightarrow s'}$ be the preference profile where the positions of s and s' are exchanged and everything else remains the same. We define $P_{-i}^{s \leftrightarrow s'}$ analogously: it is the profile where each student exchanges the positions of s and s', and schools s and s' exchange priority orders and acceptable priority violations. Similarly, given a matching μ and two schools $s, s' \in S$, we let $\mu^{s \leftrightarrow s'}$ denote the matching where s and s' switch partners. Formally, $\mu^{s \leftrightarrow s'}$ is defined such that, for each $i \in I$: (i) if $\mu(i) \notin \{s, s'\}$, then $\mu^{s \leftrightarrow s'}(i) = \mu(i)$, (ii) if $\mu(i) = s$, then $\mu^{s \leftrightarrow s'}(i) = s'$, and (iii) if $\mu(i) = s'$, then $\mu^{s \leftrightarrow s'}(i) = s$.

We say that student i's information for s and s' are *symmetric* if P_{-i} and $P_{-i}^{s \leftrightarrow s'}$ are equally likely, i.e. $Pr\{\tilde{P}_{-i} = P_{-i}\} = Pr\{\tilde{P}_{-i} = P_{-i}^{s \leftrightarrow s'}\}$. The following are the formal definitions of the properties proposed by Ehlers (2008).

Definition 3 *A mechanism φ satisfies* ***anonymity*** *if, for each $i \in I$, for each $(P_i, P_{-i}) \in \mathcal{P}_i \times \mathcal{X}_{-i}$, and all $s, s' \in S$, if $\varphi(P_i, P_{-i}) = \mu$, then $\varphi(P_i^{s \leftrightarrow s'}, P_{-i}^{s \leftrightarrow s'}) = \mu^{s \leftrightarrow s'}$.*

Definition 4 *A mechanism φ satisfies* ***positive association*** *if, for each $i \in I$, for each $(P_i, P_{-i}) \in \mathcal{P}_i \times \mathcal{X}_{-i}$, and all $s, s' \in S$, if $\varphi(P_i, P_{-i})(i) = s$ and $s' P_i s$, then $\varphi(P_i^{s \leftrightarrow s'}, P_{-i}) = s$.*

The following is the critical theorem in Ehlers (2008). Note that Ehlers (2008) considers a setup where acceptable priority violations are non-existent, so we needs to allow for such an extension in this theorem. Accordingly, we re-state the proof but it remains mostly unchanged.

Theorem 4 ***(Ehlers (2008) Theorem 3.1, part (a).)*** *In a matching market that uses a mechanism φ which satisfies anonymity and positive association, if a student i's information for s and s' are symmetric, then strategy P_i stochastically dominates strategy $P_i^{s \leftrightarrow s'}$.*

Proof. Without loss of generality, assume that $s' P_i s$. Fix P_{-i} drawn from $\tilde{P}_{-i}$ where, by assumption, $\tilde{P}_{-i}$ is symmetric for s and s'. We first show that i is assigned to any $s'' \in S \setminus \{s, s'\}$ with equal probability under P_i and $P_i^{s \leftrightarrow s'}$. Second, we show that probability of being matched to s under P_i is lower than the corresponding probability under $P_i^{s \leftrightarrow s'}$.

To see the first point, first realize that anonymity requires: $\varphi(P_i, P_{-i})(i) = s''$ if and only if $\varphi(P_i^{s \leftrightarrow s'}, P_{-i}^{s \leftrightarrow s'}) = s''$. But since $\tilde{P}_{-i}$ is symmetric for s and s', P_{-i} and $P_{-i}^{s \leftrightarrow s'}$ are equally likely. Therefore, $Pr\{\varphi(P_i', \tilde{P}_{-i})(i) = s''\} = Pr\{\varphi(P_i^{s \leftrightarrow s'}, \tilde{P}_{-i})(i) = s''\}$ for any $s'' \in S \setminus \{s, s'\}$.

To see the second point: realize that by positive association, $s' P_i s$ and $\varphi(P_i, P_{-i})(i) = s$ implies: $\varphi(P_i^{s \leftrightarrow s'}, P_{-i})(i) = s$. This implies that for any realization P_{-i} where i is matched to s under P_i, she is also matched to s under $P_i^{s \leftrightarrow s'}$. This implies that $Pr\{\varphi(P_i', \tilde{P}_{-i})(i) = s\} \leq Pr\{\varphi(P_i^{s \leftrightarrow s'}, \tilde{P}_{-i})(i) = s\}$. ■

The following is obtained as a simple corollary of this theorem (which parallels Proposition A.1 of Kesten (2010)):

Corollary 2 *For a student i whose information $\tilde{P}_{-i}$ satisfies the conditions given in Proposition 5, the strategy P_i stochastically dominates any other strategy P_i' that ranks every school in S_r above every school in S_k for all $r < k$.*

Heuristically, this takes care of "simple" manipulations: the ones that exchange the places of schools within the same quality class. For this Corollary to be useful, we first need to make sure that the TP Rule satisfies monotonicity and positive association. This is what we demonstrate next.

Lemma 15 *Top Priority Rule satisfies anonymity and positive association.*

Proof. Anonymity is obvious, so we will just prove positive association.

The first thing to note that the Round 0 allocation, obtained by running the student-proposing DA, satisfies positive association. This is indeed a direct consequence of the strategy-proofness of DA algorithm (Suppose, to get a contradiction, that student i with

preferences P_i can receive a different school than s by submitting $P_i^{s\leftrightarrow s'}$. Let x be the school that she obtains by submitting $P_i^{s\leftrightarrow s'}$. If xP_is, then the student has a profitable deviation. If sP_ix, then the student with preferences $P_i^{s\leftrightarrow s'}$ has a profitable deviation, indicating the contradiction.)

Now, suppose that the student is placed to s under TP Rule, and assume $s'P_is$. We will show that the school that i obtains when she submits $P_i^{s\leftrightarrow s'}$ is also s. There are two possibilities: either student i receives s at the end of Round 0 when she submits P_i and never changes schools; or, she first receives a strictly worse school and then improves to s at the later rounds.

In the case of the first possibility, the student receives s in Round 0 when she submits P_i. Since, as we argued above, the student proposing DA satisfies positive association, the student also receives in Round 0 s when she submits $P_i^{s\leftrightarrow s'}$. Assume, to get a contradiction, that i receives another school when she submits $P_i^{s\leftrightarrow s'}$ under TP Rule. This is only possible when i takes part in a cycle in one of the following steps when she submits $P_i^{s\leftrightarrow s'}$. Because the other students are not changing strategies, schools they are pointing to i at the end of Round 0 are the same in both cases. The only difference is that, in the alternative case, because i ranks s higher when she submits $P_i^{s\leftrightarrow s'}$, she in pointing to one fewer school. In order for this to influence the outcome of TP Rule, at some step, a different cycle needs to be executed under P_i and under $P_i^{s\leftrightarrow s'}$. Let k be the earliest such step, i.e. assume that the matchings $\mu_0, \ldots, \mu_{k-1}$ are identical in both cases, but μ_k is different. This implies that there is a cycle ϕ_k which does not appear under $G^T(\mu_{k-1})$ when $i \in D_{\mu_{k-1}}(s')$, but appears under $G^T(\mu_{k-1})$ when $i \notin D_{\mu_{k-1}}(s')$.[32] Therefore, there must be an $\underline{i}_0 i_1 \in \phi_k$ such that: $\mu_{k-1}(i_1) = s'$ and $i \succ_{s'} \underline{i}_0$. Moreover, the solution of ϕ_k allows i to receive a better school eventually. This means that solution of ϕ_k initiates the formation of a sequence of cycles that appear in the later steps, where the last cycle in the sequence contains i. More formally, there exists steps $k, \ldots, k+l-1$ and cycles $\phi_{k+1}, \ldots, \phi_{k+l}$ such that: (i) for any $m \in \{1, \ldots, l\}$, ϕ_{k+m} appears

[32] Lemma 9 implies that a cycle that appears remains when it's not solved, so ϕ_k will eventually be solved. Without loss of generality, we let k be the step it is solved.

in $G^T(\mu_{k+m-1})$, (ii) for any $m \in \{1, \ldots, l\}$, the appearance of ϕ_{k+m} requires the solution of ϕ_{k+m-1}, and (iii) ϕ_{k+l} contains i. The critical thing here is condition (ii). It implies that for any $m \in \{1, \ldots, l\}$, there exists an agent in ϕ_{k+m-1} who prevents ϕ_{k+m} from appearing. In particular, there exists an edge $i_{m-1}\bar{i}_m \in \phi_{k+m-1}$, and another edge $\underline{i}_m i_{m+1} \in \phi_{k+m}$ such that: $\bar{i}_m \in D_{\mu_{k+m-1}}(\mu_{k+m-1}(i_{m+1}))$ and $\bar{i}_m \succ_{\mu_{k+m-1}(i_{m+1})} \underline{i}_m$. Note that such agents can be found for each $m \in \{1, \ldots, l\}$. Also, condition (iii) implies that there exists an edge $i_l i \in \phi_{k+l}$. Now, we can take the paths $i_m i_{m+1}, \ldots, i_{m-1}\bar{i}_m \subset \phi_{k+m-1}$ for each $m \in \{1, \ldots, l\}$ and add them up to construct the cycle: $ii_1, \underbrace{i_1 i_2, \ldots, i_0 \bar{i}_1}_{\subset \phi_k}, \bar{i}_1 i_2, \underbrace{i_2 i_3, \ldots, i_1 \bar{i}_2}_{\subset \phi_{k+1}}, \bar{i}_2 i_3, \ldots, \underbrace{i_{l+1} i_{l+2}, \ldots, i_l i}_{\subset \phi_{k+l}}$. This cycle must appear at step $k-1$, indicating that i must be a part of a cycle in $G^T(\mu_{k-1})$. But remember that the first $k-1$ steps are common under both cases, so i takes part in a cycle at step $k-1$ when she submits P_i. This contradicts with i being assigned to s when she submits P_i.

The construction of cycle in this case can perhaps be better illustrated via the following figure.

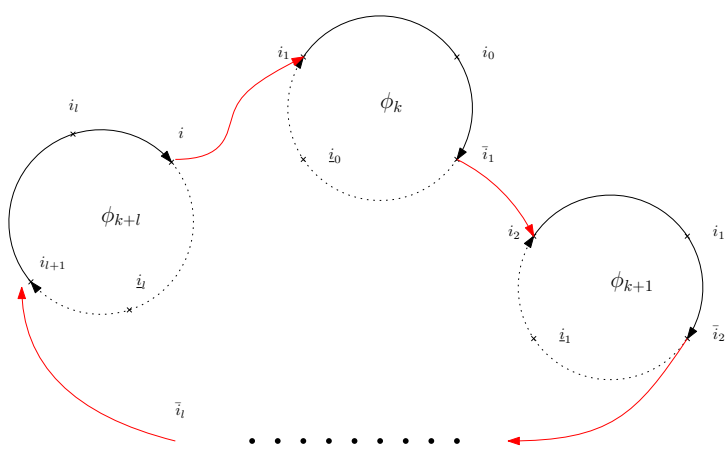

Figure 17: Construction of cycle

In the case of the second possibility, i receives a worse school than s at Round 0 and then improves in the later steps. Once again, strategy-proofness of SOSM implies that Round 0

allocation must be the same in both cases. Because the only difference between both problems is that i points to one less school, any cycle that appears when i submits P_i must also appear when i submits $P_i^{s \leftrightarrow s'}$. Lemma 9 implies that these cycles must be solved in both problems, so the cycles which contain i when she submits P_i will also appear, and i will receive s at some step when she submits $P_i^{s \leftrightarrow s'}$ as well. At this point, one can repeat the argument in the previous paragraph to demonstrate that i cannot be contained in any further cycles when she submits $P_i^{s \leftrightarrow s'}$. The result follows. ■

As discussed before, Lemma 15 combined with Corollary 2 takes care of "simple manipulations". The rest of the argument essentially takes care of cases where s and s' does not belong to the same quality class. We first present a simple and useful lemma.

Lemma 16 *Suppose that the setup given in the Proposition 5 holds, and assume that every student other than i reports truthfully. Then, under any mechanism within SEPF, i's placement in Round 0 and her final allocation belong to the same quality class.*

Proof. It's easy to see that, given the preference structure in Proposition 5, once a student is placed to a school in S_k, she never points to a school in S_r for any $r > k$ in any step afterwards (because the schools in S_r are strictly worse than schools in S_k). But this means that once a student is placed to a school in S_r at Round 0, she is never pointed by any student who is placed to a school in S_k for $k < r$. This means that she is never involved in a cycle containing students other than those who are placed to S_r. Consequently, once the students receive their assignments at Round 0, the only cycles which appear in any later step contain only schools which are in the same quality class. Therefore a student never leaves her quality class after Round 0, and the result follows. ■

Remember that by Proposition 2 we know that TP Rule is within SEPF class, so the following corollary attains (which parallels Lemma A.3 of Kesten (2010)).

Corollary 3 *Suppose that the setup given in the Proposition 5 holds. Then, under TP Rule, if other students report truthfully, a student i's placement in Round 0 and her final allocation belong to the same quality class.*

Now we can start dealing with the cases where s and s' do not belong to the same quality class. The following is the comprehensive argument covering all cases.

Suppose player i has preferences P_i, and let P_{-i} be a realization of $\tilde{P}_{-i}$. Take any two schools $s, s' \in S$, and without loss of generality assume that sP_is'. Consider the alternative strategy $P_i^{s\leftrightarrow s'}$.[33] We will demonstrate that the strategy of submitting P_i stochastically dominates submitting $P_i^{s\leftrightarrow s'}$. Suppose that student i is placed to school x, which belongs to the quality class S_r, under (P_i, P_{-i}) by the Top Priority rule. Consider the alternatives:

1. If s and s' belong to the same quality class, Lemma 15 combined with Corollary 2 implies that P_i stochastically dominates $P_i^{s\leftrightarrow s'}$.

2. If s and s' belong to different quality classes, and $S_r \cap \{s, s'\} = \emptyset$, we consider three cases:

   (a) If xP_isP_is': by Corollary 3, student i must be assigned to a school within S_r in Round 0 under P_i, which is strictly better than s. Strategy-proofness of SOSM implies that i must receive the same school in Round 0 under $P_i^{s\leftrightarrow s'}$ as well. This means that Round 0 allocation μ_0 and the graph $G^T(\mu_0)$ is the same in both cases. The remaining steps in both problems are identical, and hence i ends up with the same allocation when she submits P_i and $P_i^{s\leftrightarrow s'}$.

   (b) If $sP_is'P_ix$: by Corollary 3, student i must be assigned to a school within S_r in Round 0 under P_i, which is strictly worse than s'. Strategy-proofness of SOSM implies that i must receive the same school in Round 0 under $P_i^{s\leftrightarrow s'}$ as well. This means that Round 0 allocation μ_0 and the graph $G^T(\mu_0)$ is the same in both cases. Since by Corollary 3 i remains in class S_r until the end of the algorithm, in the remaining rounds, i keeps demanding both schools under both strategies, so the remaining rounds are also unchanged. Therefore i ends up with the same allocation when she submits P_i and $P_i^{s\leftrightarrow s'}$.

[33] An induction argument which is along the lines of Theorem 3.1 (b) of Ehlers (2008) demonstrates that this is exhaustive of all possible manipulations.

(c) If sP_ixP_is': now, Round 0 allocations may be different under the two cases.

i. If i is assigned to s' in Round 0 when she submits $P_i^{s\leftrightarrow s'}$, by Corollary 3, she remains in the class of s' until the end of the algorithm, which contains schools which are strictly worse than x. Therefore i ends up with a strictly worse outcome when she submits $P_i^{s\leftrightarrow s'}$.

ii. If i is assigned to some other school in Round 0 when she submits $P_i^{s\leftrightarrow s'}$, the way in which the Student Optimal DA algorithm operates implies that she needs to be assigned to the same school in Round 0 when she submits P_i as well. Therefore Round 0 allocations are the same, and by Corollary 3 i remains in the same class S_r until the end of the algorithm. Clearly swapping the positions of s and s', which are outside S_r, cannot change the cycles i takes place in during the following steps, so the remaining steps are also identical and i ends up with the same allocation when she submits P_i and $P_i^{s\leftrightarrow s'}$.

3. If s and s' belongs to different quality classes, and $S_r \cap \{s, s'\} = \{s\}$, one can repeat the last two points above. If i is assigned to s' in Round 0 when she submits $P_i^{s\leftrightarrow s'}$, she remains in that class, where every school is strictly worse than x. Otherwise i ends up with the same allocation under both cases.

4. If s and s' belongs to different quality classes, and $S_r \cap \{s, s'\} = \{s'\}$, again by Corollary 3, the Round 0 allocation must be in the same class as s' under P_i. A short argument shows that Round 0 allocation must also be in the the same class under $P_i^{s\leftrightarrow s'}$.[34] But then, i will not be able to receive a better allocation than schools in S_r even when she submits $P_i^{s\leftrightarrow s'}$. This implies that i can move s back to its original place and her allocation will remain the same as $P_i^{s\leftrightarrow s'}$. Similarly, i can move s' to the top of S_r and receive the same allocation as $P_i^{s\leftrightarrow s'}$. To sum up, we've constructed a profile P_i',whose only difference with P_i is that: s' is moved just above the other schools in S_r. By the

[34] If the Round 0 allocation is weakly worse than s' under P_i, the stretegy-proofness of SOSM implies that it must be the same under $P_i^{s\leftrightarrow s'}$. If Round 0 allocation is strictly better than s', by strategy-proofness of SOSM, under $P_i^{s\leftrightarrow s'}$,$i$ may get s' or the same allocation. In either case, she receives a school in the same quality class as s'.

argument above, P_i' yields the same assignment as $P_i^{s \leftrightarrow s'}$ to i. But by construction, P_i' keeps every school in its quality class and by Corollary 2 (combined with Lemma 15), P_i' is dominated by P_i.

Appendix J Proof of Theorem 3

Proof. Consider the following problem: $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$ and $q_s = 1$ for all $s \in S$. The preferences are as given below:

P_{i_1}	P_{i_2}	P_{i_3}
s_1	s_1	s_3
s_2	s_2	s_1
s_3	s_3	s_2

and the priority structure is:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_1	i_1
i_1	i_2	i_2
i_2	i_3	i_3

Assume that $C(s_1) = \{i_1\}$, $C(s_2) = C(s_3) = \emptyset$.

This problem has three partially stable matchings:[35]

$$\mu := \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$$

$$\mu' := \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$$

$$\mu'' := \{(i_1, s_2), (i_2, s_3), (i_3, s_1)\}$$

Among these matchings, μ'' is Pareto dominated by μ. Therefore, μ and μ' are the only constrained efficient matchings.

[35] Any matching where i_1 is assigned to s_3 violates the priority of i_1 for s_2, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1.

Consider any mechanism ψ which is strategy proof and selects a constrained efficient matching. Suppose μ' is the outcome given by ψ in this problem. If i_1 deviates and reports $P'_{i_1} : s_1 P'_{i_1} s_3 P'_{i_1} s_2$, in the new problem, the only constrained efficient matching is μ.[36] Then ψ must select μ for this problem. Hence, i_1 can gain from misreporting if ψ selects μ in the original problem.

Alternatively, suppose μ be the outcome of ψ in the original problem. If i_2 deviates and reports $P'_{i_2} : s_1 P'_{i_2} s_3 P'_{i_2} s_2$, in the new problem, the only constrained efficient matching is μ'.[37] Then ψ must select μ' for this problem. Hence, i_2 can gain from misreporting if ψ selects μ in the original problem. ■

Appendix K An Example with Weak Priorities

Example 5 *Consider the example in Appendix D. We change only the priority order of school s_1: $i_1 \succ_{s_1} i_2 \sim_{s_1} i_3 \sim_{s_1} i_4$. If the ties in the priority orders are broken favoring i_4 over i_2 over i_3, then we obtain the strict priority order for school s_1 as follows: $s_1 : i_1 \succ'_{s_1} i_4 \succ'_{s_1} i_2 \succ'_{s_1} i_3$. The outcome of DA mechanism under this tie breaking rule is: $\mu = \{(i_1, s_1), (i_2, s_2), (i_3, s_3), (i_4, s_4)\}$. It is easy to verify that μ is stable. However, it is not the unique stable matching. In particular, there are two more stable matchings: $\nu = \{(i_1, s_3), (i_2, s_2), (i_3, s_1), (i_4, s_4)\}$ and $\gamma = \{(i_1, s_2), (i_2, s_1), (i_3, s_3), (i_4, 2_4)\}$. Moreover, both ν and γ Pareto dominate the outcome of DA mechanism μ.*

[36] Any matching where i_1 is assigned to s_2 violates the priority of i_1 for s_3, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1. The only partially stable matchings are μ and $\{(i_1, s_3), (i_2, s_2), (i_3, s_1)\}$, but the former Pareto dominates the latter.

[37] Any matching where i_1 is assigned to s_3 violates the priority of i_1 for s_2, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1. Also, under μ, i_2's priority for s_3 is viloated. The only partially stable matchings are μ' and μ'', but the former Pareto dominates the latter.

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