School Choice under Partial Fairness*

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Abstract

A recent trend in school choice suggests that school districts are willing to allow violations of certain types of priorities. Inspired by this, we generalize the school choice problem by allowing such violations. We characterize the set of *constrained efficient* outcomes for a school choice problem in this setting. We introduce a class of algorithms, denoted *Student Exchange under Partial Fairness (SEPF)*, which guarantees to find a *constrained efficient* matching for any problem. Moreover, any *constrained efficient* matching which *Pareto improves* upon a stable matching can be obtained via an algorithm within the SEPF class. We offer two applications of this new framework, each corresponding to a different interpretation of priority violations. For these applications, we propose and characterize a (unique) mechanism (in the SEPF class) satisfying desirable axioms.

Keywords : School choice, stability, efficiency

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1 Introduction

School choice has become an important policy tool for school districts in providing the parents with the opportunity to choose their child's school. School districts adopting public school choice programs allow parents to select schools in other residence areas as well. However, school capacities are limited and it is not possible to enroll each student in her most preferred school, indicating that achieving efficiency is nontrivial. Moreover, the schools' priorities over the students should be respected, which indicates that achieving fairness is nontrivial as well. Therefore, the central issue in school choice is the design of rules for assigning the schools to students so that efficiency and fairness criteria are met. But, unfortunately, it is not possible to meet the requirements of efficiency and fairness at the same time. The current work addresses the school choice problem by focusing on the rules compromising between efficiency and fairness.

In a school choice problem, students submit their preferences over a list of schools to a central placement authority and the authority decides on the assignment based on schools' priorities over the students. A school choice rule is a systematic way of matching students with schools for each school choice problem. However, there are several concerns and it is impossible to design a rule to achieve all of these goals. A major concern is fairness: at the matching chosen by the central authority, there shouldn't be a student who prefers a school, say s, to her assigned school and another student with lower priority at s who is assigned to s. There are rules which always select fair matchings. The well-known student-proposing deferred acceptance (DA) rule is such an example (Gale and Shapley, 1962). The student-proposing DA gives the student-optimal stable matching (SOSM). Actually, the SOSM not only prevents priority violations, but also is the best matching in terms of students' welfare among all the matchings without any priority violation; that is, each student prefers the SOSM to any matching without any priority violation (Gale and Shapley, 1962; Balinski and Sönmez, 1999). Furthermore, the student-proposing DA rule is immune to strategic manipulations by the students: revealing preferences truthfully is a weakly dominant strategy for each student (Dubins and Freedman, 1981; Roth, 1982). However, there is a serious drawback of the DA rule: there might be another matching which is preferred by each student to the SOSM, i.e., it is not *Pareto efficient* (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Although the extent of this inefficiency depends on schools' priorities and students' preferences, from a theoretical perspective, the welfare loss can

be quite large (Kesten, 2010). Moreover, it is observed that, based on the student preference data from the NYC school district, potential welfare gains over the SOSM are significant (Abdulkadiroğlu, Pathak, and Roth, 2009). The welfare loss under the SOSM is actually a deeper issue: fairness and efficiency are incompatible (Gale and Shapley, 1962; Roth, 1982; Balinski and Sönmez, 1999). This incompatibility naturally raises the question of how to compromise either one of these properties to avoid high extents of priority violations or welfare losses. Our approach is to improve students' welfare by allowing certain priority violations. To this end, we begin by generalizing the standard school choice problem such that it allows for violation of certain priorities. The set of allowable priority violations is taken as a premise of the model, and each application considered in the current work is based on a different interpretation of this set. We also extend the standard notions of fairness and stability to define *partial fairness* and *partial stability*, which correspond to the straightforward projections of fairness and stability onto this framework.

Our main contribution in this paper is to propose a general class of rules, the *Student Exchange under Partial Fairness (SEPF)*, for the school choice problem with allowable priority violations. Each rule in our class gives a *partially stable* matching which is not *Pareto dominated* by another *partially stable* matching (that is, *constrained efficient* in the class of *partially stable* matchings). One can begin with any partially stable matching (for instance, SOSM) and SEPF guarantees that the end matching weakly Pareto dominates the initial matching. Moreover, each such matching (constrained efficient and weakly Pareto dominating the initial matching) can be obtained by a member in the class we propose (Theorem 1).

The concept of allowable priority violations is open to several interpretations, and one can consider different applications based on the interpretation one attains. Here, we discuss two applications and suggest some reasons on why a rule in SEPF class, or which rule in SEPF class, should be adopted.

The first interpretation pertains to the case of a school district which believes that certain *types* of priorities can be violated in certain cases, whereas other priorities must always be respected. For instance, Boston Public School System (BPSS) considered to adopt Top Trading Cycles mechanism, which only respects priorities of top q_s students for each school s where q_s is the capacity of school s (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2006). This suggests that the idea of respecting only certain priorities is not a very unusual idea for school districts. Also, recently, BPSS has removed proximity from the priority structure and has started prioritizing students based on sibling status (Dur,

Kominers, Pathak, and Sonmez, 2014). This demonstrates that whereas BPSS considers proximity to be an important factor, the neighborhood priority can be given up, which possibly has efficiency implications (even if the goal of this policy change might not be to improve efficiency). In another example, Recovery School District in New Orleans School replaced TTC with DA mechanism after including the private scholarship schools in the system whose priorities cannot be violated by law (Pathak, 2016), which also indicates to the idea of allowing certain priorities to be violated whereas respecting others. Similarly, some school districts include exam schools, whose priorities determined based on centralized test scores and cannot be violated by law, and regular schools, whose priorities are exogenously determined and respecting priorities is plausible (Abdulkadiroğlu, 2011; Sönmez and Ünver, 2010). All these examples show that school districts may allow the violation of the priorities at some schools (e.g. regular schools whose priorities are not determined by law) or some priority classes (e.g. proximity priority). We propose SEPF as a potential solution that "picks" which priorities to violate in order to obtain efficiency gains. In a model with coarse priorities, a school is typically indifferent among the students in the same priority class, i.e. there are weak priorities. Our basic model has strict priorities, but its extension to an environment in weak priorities is straightforward (Section 7). Moreover, this extension sheds light to an important connection between SEPF and Stable Improvement Cycles algorithm proposed in Erdil and Ergin (2008).

Another solution to the problem of having some priority classes which can be violated is to get rid of these priority classes already. Nevertheless, we show (in Section 4.1) that suppressing a priority class may yield a perverse consequence: it may make each student worse off compared to the SOSM where priority classes are kept. On the other hand, the rules in SEPF class guarantee that SOSM will be weakly improved on, which suggests a reason to favor SEPF over SOSM with fewer priority classes.

Another interpretation of allowing some priorities to be violated requires one to consider the following scenario: the school district can ask students' consent for priority violations and design a rule based on the consent of students (Kesten, 2010). The idea here is to design rules such that students have incentives to consent for their priorities to be violated, and, each student's welfare can be weakly improved thanks to the relaxation of fairness due to students' own consent for violations. This approach has an additional strategic component (compared to the case where the acceptable priority violations are set by school districts) in terms of each student's decision on whether to consent or not (Section 5.3). A rule which is motivated by the same interpretation is the *Efficiency Adjusted*

Deferred Acceptance Mechanism (EADAM) (Kesten, 2010). The EADAM also finds a *partially fair* and *constrained efficient* matching.[1] By our characterization result, the EADAM is in the SEPF class (Proposition 2). We then derive a particular rule, the *Top Priority Rule*, in the SEPF class. This rule satisfies an important property: a consenting student is never better off by instead not consenting. This is an indispensable incentive-compatibility property which assures that the idea of consent is operational. We demonstrate that the Top Priority Rule is the unique *partially fair* and *constrained efficient* rule which gives students incentives to consent (Theorem 2). An immediate corollary to our theorem is the equivalence of the EADAM and Top Priority Rule. Even though the Top Priority Rule is immune to strategic consenting decisions, in general, no rule within the SEPF is immune to violations through misrepresentation of preferences (Proposition 4). This incompatibility is indeed more general: a *constrained efficient* rule can never be *strategy-proof* (Theorem 3).

When we revisit the first interpretation of the allowable priority violations, Top Priority Rule still stands out in the SEPF class because it is the unique rule in that class which guarantees the following: whenever a student's priority is violated, she cannot be better off when her priority is respected. In other words, under Top Priority Rule, a student's priority is violated only when this violation does not make the student worse off.

Related literature

The school choice problem is introduced by Abdulkadiroğlu and Sönmez (2003). Since then, how to decrease the inefficiency loss of the DA rule has been one of the main questions. It has been demonstrated that theoretically, the level of inefficiency can be quite high (Kesten, 2010) and there is empirical support for this insight: in NYC high school match, possible welfare gains over the SOSM is significant (Abdulkadiroğlu, Pathak, and Roth, 2009). Since efficiency and fairness are incompatible in the school choice context, the only remedy for this problem is to relax fairness. One alternative in this direction is to focus on efficiency via Top Trading Cycles (TTC) rule (Abdulkadiroğlu and Sönmez (2003), Hakimov and Kesten (2014), Morrill (2015a), Morrill (2015b)). Another alternative is to weaken the fairness notion. Such a weakening is *reasonable stability*: a matching is *reasonably stable* if whenever a student i's priority is violated at school s then there does not exist a stable matching in

[1] In our model, students can give consent for any set of schools, whereas in Kesten (2010), whenever a student consents, she consents for all schools. We introduce a trivial generalization of the EADAM to capture this extension and this generalized version selects *partially fair* and *constrained efficient* outcome (Appendix C).

which student i is assigned to school s.[2] Another weakening is $\tau-$fairness: a matching with a priority violation is not deemed as unfair if student's objection to that priority violation is counter-objected by another student (Alcade and Romero-Medina, 2015).

A different approach to weaken fairness is to consider certain priority violations as acceptable. Such an example is the NYC school match: motivated by the observation that the efficiency loss of the SOSM is significant, school districts have been considering to allow such violations anywhere but exam schools (Abdulkadiroğlu, 2011). Another interpretation of acceptable priority violations is the proposal by Kesten (2010): ask students to consent for violations of their priorities. The author develops a rule (the EADAM) which guarantees each student that she will not be worse off by consenting. It is also shown that students assigned to certain schools (*underdemanded* schools[3]) are not Pareto improvable at the SOSM and the EADAM can be redefined by taking these schools into account (Tang and Yu, 2014). Moreover, the EADAM outcome is supported as the strong Nash Equilibrium of the preference revelation game under the DA (Bando, 2014). In the affirmative action context, a variant of the EADAM is recently proposed as a *minimally responsive* rule (that is, a rule such that changing the affirmative action parameter in favor of the minorities never results in a matching which makes each minority student weakly worse off) (Doğan, 2015). Yet another example of the approach where certain priorities can be violated (and thus the SOSM can be improved upon) is also seen in Afacan, Alioğullari, and Barlo (2015).[4]

The idea that possible welfare gains can be captured by improvement cycles is first discovered by Erdil and Ergin (2008) in the context of coarse priorities of schools. This idea inspired the rules proposed in some other works (Ehlers, Hafalir, Yenmez, and Yildirim (2014), Abdulkadiroğlu (2011)) and our work as well.

[2]This notion is first discussed in the working paper version of Kesten (2010).

[3]See Section 5.2 for a discussion of underdemanded schools.

[4]The model relies on allowing certain priority violations based on the information that the parents will not appeal these violations. The authors introduce the *Efficiency-Corrected Deferred Acceptance Mechanism (ECDA)* algorithm, which finds a *constrained efficient sticky stable* matching. Besides informational issues, our paper differs from theirs in two major aspects. First, we introduce a class of rules selecting *all* constrained efficient matchings and thus provide a full characterization result. Second, we show that there exists a *unique* rule in this class which satisfies desired strategic properties.

2 The Model

We first present the standard school choice problem, and then introduce the extended model with priority violations.

2.1 School Choice Problem

A **school choice problem** (introduced by Abdulkadiroğlu and Sönmez (2003)) consists of the following elements:

- a finite set of students $I = \{i_1, i_2, ..., i_n\}$,
- a finite set of schools $S = \{s_1, s_2, ..., s_m\}$,
- a strict priority structure of schools $\succ = (\succ_s)_{s \in S}$ where $\succ_s$ is the complete priority order of school s over I,
- a capacity vector $q = (q_s)_{s \in S}$ where q_s is the number of available seats at school s,
- a strict preference profile of students $P = (P_i)_{i \in I}$ such that P_i is student i's preferences over $S \cup \{\emptyset\}$, where $\emptyset$ stands for the option of being unassigned.[5] Let R_i denote the *at-least-as-good-as* preference relation associated with P_i, that is: $s \; R_i \; s' \Leftrightarrow s \; P_i \; s'$ or $s = s'$. Let $R = (R_i)_{i \in I}$ denote the weak strategy profile of all students.

A **matching** $\mu : I \to S \cup \{\emptyset\}$ is a function such that for each $s \in S$, $|\mu^{-1}(s)| \leq q_s$. A **rule** is a systematic procedure which selects a matching for each problem.

A matching μ **violates the priority** of student $i \in I$ at school $s \in S$ if there exists another $j \in I$ such that: (i) $\mu(j) = s$, (ii) $s \; P_i \; \mu(i)$, and (iii) $i \succ_s j$. A matching μ is **fair** if for each $i \in I$ and $s \in S$, it doesn't violate the priority of student i at school s. A matching μ is **individually rational** if for each $i \in I$, $\mu(i) \; R_i \; \emptyset$. A matching μ is **non-wasteful** if there does not exist a student $i \in I$ and a school $s \in S$ such that $s \; P_i \; \mu(i)$ and $|\mu^{-1}(s)| < q_s$. A matching μ is **stable** if it is (i) *fair*, (ii) *individually rational* and (iii) *non-wasteful*.

A matching μ **weakly Pareto dominates** matching μ' if for each $i \in I$, $\mu(i) \;\; R_i \;\; \mu'(i)$. A matching μ **Pareto dominates** μ' if μ *weakly Pareto dominates* μ' and for some $j \in I$, $\mu(j) \; P_j \; \mu'(j)$. A matching μ is **Pareto efficient** if there does not exist another matching μ' which Pareto dominates μ.

[5] Alternatively, one can think of $\emptyset$ as a school with $q_\emptyset = \infty$.

The properties of *Pareto efficiency* and *fairness* are clearly desirable from a normative point of view. Unfortunately, for some problems, a *Pareto efficient* and *fair* matching may not exist (Balinski and Sönmez, 1999). But, for each problem there always exists a *fair* matching which *Pareto dominates* all other *fair* matchings (Gale and Shapley, 1962; Abdulkadiroğlu and Sönmez, 2003). This matching is called the *Student Optimal Stable Matching (SOSM)* and it is is determined through the well-known **Student-Proposing Deferred Acceptance (DA)** algorithm (Gale and Shapley, 1962).

Suppose that our primary aim is to improve students' welfare over a fair matching, and we encounter a case where SOSM is not *Pareto efficient*. Since the SOSM *Pareto dominates* any other *fair* matching, we immediately conclude that each fair matching is Pareto inefficient. Yet, because the (inefficient) SOSM is the *most efficient* one among the fair matchings, in any fair matching, a students' welfare can be improved only if some priorities are violated. Throughout the rest of the paper, we focus on the incompatibility between *Pareto efficiency* and *fairness* and we follow the idea of compromising *fairness* by allowing violations of some priorities. The set of allowable priority violations, which will be made to obtain efficiency improvements, is obviously not a part of the standard school choice problem. We next present the necessary extension to capture this added structure.

2.2 School Choice Problem with Priority Violation

A **school choice problem with priority violation** (or simply a **problem**) is a school choice problem where acceptable priority violations are given by a correspondence $C : S \rightrightarrows I$, where $C(s)$ denotes the set of students for whom a priority violation at school s is acceptable.

Throughout the paper, we will fix $I, S, \succ$ and q for expositional simplicity. Thus, a problem is defined by a preference profile R and a correspondence C. For problem (R, C), we denote the matching selected by rule ψ with $\psi_{(R,C)}$ and the assignment of student i in $\psi_{(R,C)}$ with $\psi_{(R,C)}(i)$.

As discussed before, there are several interpretations of the correspondence C, such as considering them to be the set of priorities that the central authority can sacrifice, or coming from the consenting decisions of students as in Kesten (2010). We discuss these interpretations in detail in the following section; yet, for now, we'll be agnostic about this object and give the most general treatment we can. For the rest of this section and Section 3.1, we treat this correspondence as a premise of the model.

The addition of this new object relaxes the fairness constraint that the mechanism designer is subject to in the following manner: priorities for which violations are acceptable (according to C)

do not have to be taken into account when fairness is considered. By ignoring these priorities, one can define weaker notions of fairness and stability. These are as follows: A matching μ **violates the priority** of student $i \in I$ at school $s \in S$ if there exists another $j \in I$ such that: (i) $\mu(j) = s$, (ii) $s \; P_i \; \mu(i)$, (iii) $i \succ_s j$ and (iv) $i \notin C(s)$.[6] A matching μ is **partially fair** if for each $i \in I$ and $s \in S$, it doesn't *violate the priority* of student i at school s. A matching μ is **partially stable** if it is (i) *partially fair*, (ii) *individually rational* and (iii) *non-wasteful*.

A matching μ is **constrained efficient** if (i) it is *partially stable*, and (ii) it is not *Pareto dominated* by any other *partially stable* matching. We are interested in rules generating a *constrained efficient* matching for each instance of a school choice problem.

The introduction of acceptable priority violations and the notion of *partial stability* extends the standard school choice model as to compromise between *stability* and *Pareto efficiency* – the two notions which are incompatible with each other. Indeed, one can easily see how the interpolation between the two ends works by investigating the extremes. In one extreme, when no priority violation is acceptable (i.e. when $C(s) = \emptyset$ for all $s \in S$), the notion of *partial stability* collapses to that of *stability*. In this case, the only *constrained efficient* matching is the SOSM. In the other extreme, when each priority violation is acceptable, (i.e. when $C(s) = I$ for each $s \in S$), *partial fairness* is vacuously satisfied by each matching. In this case, *constrained efficiency* is equivalent to *Pareto efficiency*, and one can implement a *constrained efficient* matching through the Top Trading Cycles (TTC) while preserving *strategy-proofness* (Shapley and Scarf, 1974; Abdulkadiroğlu and Sönmez, 2003). Thus, one can easily see that the standard school choice problem can be embedded within this framework.

3 The Student Exchange under Partial Fairness

We present a class of algorithms to characterize the set of *constrained efficient* matchings, which improve the students' welfare upon a stable matching.

3.1 The Algorithm

We introduce notions that we use in the definition of this class. Given a matching μ, for each $s \in S$, let

[6] Throughout the rest of the paper, whenever we say a matching *violates the priority* of a student at a school, we refer to this definition.

- $D_\mu(s) = \{i \in I : s \; P_i \; \mu(i)\}$ (the set of students who prefer school s to the school to which they are assigned under μ)

- $X_\mu(s) = \{i \in D_\mu(s) : \forall j \in D_\mu(s) \setminus (C(s) \cup \{i\}), \, i \succ_s j\}$ (the set of students who are eligible for a *partially fair* exchange involving school s).[7]

Let $G = (V, E)$ be a directed graph with the set of vertices V, and the set of directed edges E, which is a set of ordered pairs of V. A **trail** is a set of edges $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ in E. A trail $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is

- a **path** if the vertices $i_1, i_2 \ldots, i_{n+1}$ are distinct,

- a **cycle** if the vertices $i_1, i_2 \ldots, i_n$ are distinct and $i_1 = i_{n+1}$,

A path $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\}$ is a **chain** if for each $j \in V$, $j i_1, i_{n+1} j \notin E$. Given a chain $\{i_1 i_2, i_2 i_3, \ldots, i_n i_{n+1}\} \subseteq E$, vertex i_1 is called the **tail**.

For each matching μ, let $G(\mu) = (I, E(\mu))$ be the **(directed) application graph** associated with μ where the set of directed edges $E(\mu) \subseteq I \times I$ is defined as follows: $ij \in E(\mu)$ (that is, i *points to* j) if and only if $s = \mu(j)$ and $i \in X_\mu(s)$.

Remark 1 *In the graph $G(\mu)$, if i points to j, then i points to each student who is assigned to the school $\mu(j)$ at the matching μ.*[8]

We say that **cycle** $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_k i_1\} \subseteq E(\mu)$ **is solved** when for each $ij \in \phi$, student i is assigned to $\mu(j)$ towards a new matching. Formally, we denote the *solution of a cycle* by the operation $\circ$; that is, $\eta = \phi \circ \mu$ if and only if for each $ij \in \phi$, $\eta(i) = \mu(j)$, and for each $i' \notin \{i_1, i_2, \ldots, i_k\}$, $\eta(i') = \mu(i')$. The following class of algorithms are defined by *solving* cycles iteratively in the appropriately defined graph.

The SEPF (Student Exchange under Partial Fairness) Algorithm:[9]

Step 0 Let μ_0 be a partially stable matching.

[7] Note that this set is always well-defined. In particular, when $D_\mu(s) \setminus C(s) = \emptyset$, we have $X_\mu(s) = D_\mu(s)$, and when $D_\mu(s) \setminus C(s) = D_\mu(s)$, then $X_\mu(s)$ is a singleton.

[8] This is a consequence of the following observation: i points to j if and only if $i \in X_\mu(\mu(j))$ and thus, i also points to student i' if i' is assigned to school $\mu(j)$.

[9] An example to demonstrate how the SEPF Algorithm works can be seen in Dur, Gitmez, and Yılmaz (2015).

Step k Given a matching μ_{k-1},

(k.1) if there is no cycle in $G(\mu_{k-1})$, stop: μ_{k-1} is the matching obtained;

(k.2) otherwise, *solve* one of the cycles in $G(\mu_{k-1})$, say ϕ_k and let $\mu_k = \phi_k \circ \mu_{k-1}$.

The SEPF is a *class of* algorithms, rather than a particular algorithm. First, the matching which initiates the algorithm, μ_0, is not predefined (beyond the requirement that it needs to be *partially stable*). Consequently, each choice of μ_0 results in a different set of outcomes. Second, Step k.2 of the algorithm requires "one of the cycles" to be solved without specifying which one. Consequently, one can obtain different matchings by choosing different cycles at each stage of the algorithm; thus, the outcome of the algorithm is *a class of* potential matchings, not a matching.[10] Each particular cycle selection in this class generates a matching.

A discussion about the proper choice of μ_0 is in order here: even though the results we present below will be valid when one begins with any *partially stable* matching, we have reasons to favor a particular stable matching, the SOSM, as the initial matching μ_0. One reason for this choice goes back to our starting point: our goal is to improve over a stable matching in terms of efficiency. Since the SOSM already dominates any other stable matching in terms of efficiency, it seems reasonable to start with the SOSM. Another reason why the SOSM is appealing, particularly for school districts using the DA, is that: it gives them a guarantee to argue for (weakly) improving *each* student's welfare compared to the school assignment given by the existing rule, the student-proposing DA. In this sense, one can interpret the SOSM as the status quo outcome, and imagine that the market designer is constrained by giving each agent at least what she gets under the status quo. Due to these reasons, in the main text, we focus on the SEPF algorithm starting with SOSM and present the results by using SOSM as starting point; yet, in the Appendix, we prove our results for the general case in which SEPF starts with any *partially stable* matching.

3.2 A characterization result

We are now ready to present our main characterization result: if the initial matching is the SOSM, each matching obtained by the SEPF is *constrained efficient* and *weakly Pareto dominates* the SOSM;

[10] In the example in the appendix of Dur, Gitmez, and Yılmaz (2015), there are two matchings obtained by the SEPF.

moreover, each *constrained efficient* matching which *weakly Pareto dominates* the SOSM is attainable through some cycle selection rule within the SEPF.

Let (R, C) be a school choice problem with priority violations. Let $\Psi(R, C)$ denote the set of all *constrained efficient* matchings which *weakly Pareto dominate* SOSM with respect to this problem. Let $\Pi(R, C)$ denote the set of all matchings that can be obtained via SEPF algorithm starting with SOSM for (R, C).

Theorem 1 *For each school choice problem with priority violation (R, C), a matching is constrained efficient and weakly Pareto dominates the SOSM if and only if it is obtained by the SEPF. That is, $\Psi(R, C) = \Pi(R, C)$.*

In Appendix A we provide the proof for the general case in which μ_0 can be any *partially stable* matching. This result states that SEPF indeed characterizes the set of matchings which satisfy certain normative properties.[11]

An important remark is that the characterization result concerns the set of *constrained efficient* matchings which *weakly Pareto dominate* the SOSM, rather than any *constrained efficient* matching. In general, there exist *constrained efficient* matchings which do not *weakly Pareto dominate* the SOSM.[12] These matchings are excluded in Theorem 1. We remove this restriction in Section 6 and seek for a *constrained efficient* rules, by imposing an additional restriction of incentive compatibility. The impossibility result we obtain suggest that the restriction of *weak Pareto dominance* over the SOSM is not costly at all as far as the main properties of fairness, efficiency and incentive compatibility are concerned.

Our model is based on allowable priority violations while Theorem 1 is a general result on *constrained efficient* matchings without any reference on the nature of these violations. There are different interpretations for why and how priorities are violated. In the next two sections, we study two such models. First, we show that our approach can be adopted for the case where a school district is willing

[11] One can easily verify that for each C, each matching given by the SEPF is *reasonably stable* (see the introduction for the definition of *reasonable stability*). Moreover, when $C(s) = I$ for each $s \in S$, each member of the SEPF selects a *Pareto efficient* matching. As an immediate corollary to Theorem 1, we obtain a characterization of *reasonably stable* and *Pareto efficient* matchings: the SEPF starting with the SOSM (with $C(s) = I$ for each $s \in S$) finds a *reasonably stable* and *Pareto efficient* matching, and conversely, each *reasonably stable* and *Pareto efficient* matching can be found by a member of the SEPF (with $C(s) = I$ for each $s \in S$). Similarly, the special case of the SEPF starting with the SOSM and $C(s) = I$ for each $s \in S$ characterizes the set of all $\tau-$*fair* matchings (see the introduction for the definition of $\tau-$*fairness*).

[12] See Dur, Gitmez, and Yılmaz (2015) for more on this point.

to violate certain types of priorities, and compare our approach to an alternative where priorities are suppressed by school districts (e.g. Boston School Public System suppressing walk-zone priorities). Second, we analyze the model where priority violations are possible only if students consent for it.

4 Application 1: Violating Certain Types of Priorities

Most of the school districts are using priority classes when they rank the students for schools. For instance, Boston Public School System (BPSS) was using 4 priority classes for each school: Sibling+Walk-Zone, Sibling, Walk-Zone and Others. Each school was giving the highest priority to the students in the the Sibling+Walk-Zone priority class, the second highest priority to the students in the Sibling priority class, and third highest priority to the students in the Walk-Zone priority class. The ranks of the students within each priority class was determined according to a random draw. Recently, BPSS decided to suppress walk-zone priority class and now they are using only two priority classes: Sibling and Others. This demonstrates that although walk-zone priority was important for the Boston School District, it was not something they cannot give up. The school choice problem with priority violation captures this phenomenon: this (weak) relaxation of priorities is equivalent to $C(s)$ being the set of students who only have walk zone priority at school s. Therefore, instead of getting rid of a priority class altogether, the mechanism designer may consider modeling these priority violations as acceptable and using the SEPF.

An important clarification needs to be made at this point. Our motivation for proposing the SEPF, as discussed before, is to improve the SOSM in terms of efficiency. On the other hand, a school district's motivation in suppressing certain priority classes (such as walk-zone priority) is **not** necessarily due to a concern for efficiency, so this comparison of two alternative policies may not capture the whole set of concerns behind suppressing a priority class. Yet, our purpose here is to point out to a side effect of this policy: *suppressing priorities may have unintended perverse efficiency implications.* Below, we compare the SEPF to the approach of suppressing some priority classes on the efficiency dimension alone to point out these unintended results. Our main finding is that any efficiency gain potentially obtained by suppressing a priority class can also be obtained via the SEPF (by Theorem 1), whereas *the SEPF avoids the welfare losses that may occur when a priority class is suppressed.*

4.1 Suppressing a Priority Class vs Allowing Violation of Priorities

Motivated by the Boston Public School System (BPSS) suppressing walk-zone priority, we compare the efficiency consequences of suppressing a priority class completely and allowing the priority violations only when it leads to Pareto improvement.

One might think that when a priority class is suppressed, it reduces the number of rejection chains which are the reason of inefficiency under the DA, and therefore, it leads to efficiency gains. However, this is not true. We show that, when a priority class is suppressed, all students could be worse off compared to the DA outcome, whereas this problem does not occur under the SEPF.

Proposition 1 *Let us consider a school choice problem with tie-breaking. Let μ_0 be the SOSM under the original priorities and μ_{supp} be the SOSM under a priority class being suppressed, both after ties are broken. (i) Each student may prefer μ_0 over μ_{supp}. (ii) Suppose μ_{supp} is constrained efficient and Pareto improves μ_0. Then, μ_{supp} is obtained by some cycle selection in the SEPF class.*

Proof. (i) There are 2 schools, $S = \{a, b\}$, and 3 students $I = \{i, j, k\}$. Each school has one available seat. The preferences of students are: $P_i : b - a$, $P_j : a - b$, $P_k : a - b$. Each school uses 4 priority classes: Sibling+Walk Zone-Sibling-Walk Zone-Other. Student i has a sibling attending at school a and she lives in the walk-zone of school b. Student j and k belong to the "other" priority class for both schools. Suppose the random draw favors j most and i least. In this problem, the DA mechanism assigns i to b and j to a. However, when the walk-zone priority is suppressed, DA assigns i to a and j to b.

(ii) This follows from Theorem 1. ■

Thus, the approach of suppressing certain priorities might result in a stable (with respect to relaxed priorities) matching which is *Pareto inferior* to the DA outcome (under the original priorities). On the other hand, the SEPF class always gives stable and *constrained efficient* matchings (with respect to relaxed priorities).

Moreover, taking the approach of suppressing all the walk-zone priorities and using the DA outcome may yield an undesirable allocation for two other reasons:

1. Ignoring some priorities might weaken the fairness aspect of the outcome (with respect to the suppressed priorities) without improving efficiency, and decrease the welfare of students with these priorities.

2. Ignoring walk-zone priorities increases busing costs, which constitute a considerable part of school budget (Russell and Ebbert, 2011).

Thus, the current approach is debatable if school districts care about the *balance* between different objectives (efficiency improvements, fairness, welfare of students with walk-zone priorities and keeping busing costs under control). On the other hand, under our approach, this balance is already embedded since walk-zone priorities are violated only if they lead to inefficiencies, hence the notion of *constrained efficiency* in our framework.

To sum up, suppressing priority classes (e.g. removing walk-zone priorities) has the disadvantage of decreasing the students' welfare. At best, it removes other *constrained efficient* matchings (which respect the walk-zone priorities as well) from the menu and focuses on a particular point on the *constrained efficiency* frontier (via the DA mechanism). This is exactly where the proposed model performs superior: the SEPF characterizes all the matchings on this frontier, including the ones possibly respecting the walk-zone priorities as well. As discussed earlier, we do not propose the SEPF over the approach of suppressing priorities (and then applying the DA), since there might be other concerns for suppressing certain priorities (as it is the case for the Boston Public School System). Our sole purpose here is to emphasize that the approach underlying the SEPF is superior from the efficiency perspective.

5 Application 2: Priority Violation via Students' Consent

We now move to the second interpretation regarding the set of acceptable priority violations: that it is a result of consenting decisions of students as in Kesten (2010). Before making observations pertinent to this interpretations, we will emphasize the relationship between SEPF and EADAM, which is the solution proposed in Kesten (2010).

5.1 The SEPF and EADAM

By Theorem 1, the SEPF class gives the set of all *constrained efficient* matchings which *weakly Pareto dominate* the SOSM. Another rule which attains the same properties is the **Efficiency Adjusted Deferred Acceptance (EADAM)** algorithm (Kesten, 2010). The motivation behind the EADAM is to explore the source of inefficiency of the DA due to fairness constraints and improve the SOSM

on the efficiency dimension by asking students for consent for the violation of their priorities. An important observation is that the priority of a student i at a school s might not help her to get a better school at all. If this is the case, giving i the lowest priority at s instead of her current priority would not change her assignment and the DA would possibly select a matching which *Pareto improves* the matching selected by the DA under the original priorities. Motivated by this observation, Kesten (2010) introduces the EADAM in a setting that allows students to *consent* for the violation of their own priorities that do not affect their assignment. This would correspond to student i consenting for a priority violation at s. This discussion demonstrates that there is a clear connection between our setup and the one considered in Kesten (2010).

Proposition 2 *The EADAM belongs to the SEPF class; that is, for each problem, the matching obtained by EADAM can also be obtained by a particular selection of cycles in the SEPF class (and SOSM as the initial matching μ_0.)*

Proof. This is a direct consequence of Theorem 1 and Proposition 7 in Appendix B. ■

It needs to be pointed out that the setup considered in Kesten (2010) is slightly different from ours: Kesten (2010) assumes that a student can either consent for priority violation at each school, or not consent for it at any of the schools (in other words, either $i \in C(s)$ for each $s \in S$ or $i \notin C(s)$ for each $s \in S$). We introduce the ***generalized EADAM***, a trivial generalization of the EADAM, to accommodate our setup (see Appendix B). Since the EADAM is based on students making their consenting decisions in a specific manner and the generalized EADAM belongs to the SEPF class (Proposition 7), the EADAM belongs to the SEPF class as well.

Proposition 2 is a starting point in analyzing the relationship between EADAM and SEPF. It demonstrates that EADAM belongs to the SEPF class, but our claim is that EADAM is more than that: it is a special member of SEPF class. Indeed, EADAM can be characterized as the only member of SEPF class which satisfies certain desirable properties which are related to the consenting interpretation. Before focusing on this interpretation discussing the uniqueness of EADAM, we provide some insights which will illuminate the ways in which EADAM operates.

5.2 The Concept of Underdemanded Schools

The SEPF class is defined by an algorithm based on iterative selection of cycles and at each iteration, only the welfare of each student in the selected cycle improves. Clearly, a necessary condition for a student to be in a cycle is that her current school is demanded by other students. If this does not hold, then a welfare improvement for this student is not possible under the SEPF class. To formalize this idea, we introduce the concept of *underdemanded* schools.[13]

A school, say s, has **no demand** at μ if there does not exist a student i who prefers s to $\mu(i)$. A school is **underdemanded** at μ if it has *no demand* at μ or each path to a student assigned to that school starts with a student assigned to a school with *no demand* at μ.

If student i is not pointed by another student in the graph $G(\mu_k)$, then school $\mu_k(i)$ has *no demand* at $\mu_{k'}$ for each $k' \geq k$.[14] Consequently, a student assigned to an *underdemanded* school at step k is not part of any cycle at any step $k' \geq k$. This implies that the students assigned to *underdemanded* schools at the μ_0 are not part of any cycle throughout the SEPF algorithm. Thus, by Theorem 1, a student assigned to an *underdemanded* school at μ_0 is assigned to the same school at each *constrained efficient* matching which *weakly Pareto dominates* the μ_0.

We say that a student is **permanently matched at** μ if she is assigned to an *underdemanded* school at μ, and **temporarily matched at** μ if she is not *permanently matched*. Let I^u_μ denote the set of *permanently matched* students at μ.

5.3 No-Consent-Proofness

We now focus on the second interpretation of priority violation discussed in Section 2.2: the violation of student i's priority at school s is acceptable if i has consented for this violation (Kesten, 2010). The decision of consenting (or not consenting) for priority violations is a strategic one and the main issue for a school choice rule based on the idea of consent is whether students have incentives for *consenting*. The mechanism designer would like to give as many incentives to the students to consent, because each additional consent relaxes the partial fairness constraint and brings the mechanism designer closer to the efficiency frontier. Consequently, one should look for the rules which gives each student incentives

[13] See also Kesten and Kurino (2013) and Tang and Yu (2014) for a discussion of the same concept. The terminology in Tang and Yu (2014) is different: the authors call a school with *no demand* as a **tier-0 underdemanded** school and an *underdemanded* school as a **tier-k underdemanded** school for $k > 0$.

[14] This easily follows from Remark 2 in Appendix A.

to consent, i.e. which guarantee that the students will not be made worse off if they consent for the violation of her priority in a school. The following notion formalizes this.

Definition 1 *A rule ψ is* **no-consent-proof** *if for any problem, for each $i \in I$ and $s \in S$, student i who consents for s does not get a better assignment by not consenting for s. That is: take any (R, C) and define C' as the profile of acceptable priority violations where $C'(s) = C(s) \setminus \{i\}$ and $C'(s') = C(s')$ for each $s' \in S \setminus \{s\}$. A rule ψ is no-consent-proof if $\psi_{(R,C)}(i)\ R_i\ \psi_{(R,C')}(i)$ for each i and s.*

Our goal is to search for *no-consent-proof* rules which satisfy *constrained efficiency* and *weak Pareto dominance* over a partially stable matching. Our characterization result (Theorem 1) reduces this search to the specific cycle selection rules within the SEPF class. But, a cycle selection rule within the SEPF class may not satisfy *no-consent-proofness.*[15] Next, we introduce a *no-consent-proof* rule in the SEPF class.

5.4 The Top Priority Rule

For each matching μ, let $G^T(\mu) = (I, E^T(\mu))$ be the **Top Priority (TP) graph** associated with μ, a subgraph of $G(\mu) = (I, E(\mu))$ where the set of directed edges $E^T(\mu) \subseteq E(\mu)$ is defined as follows: $ij \in E^T(\mu)$ if and only if, among the students who are *temporarily matched* at μ and point to j in $G(\mu)$, student i has the highest priority for school $\mu(j)$. The **Top Priority (TP) algorithm** is based on iterative selection of specific cycles in the TP graph.

The TP Algorithm:[16]

Step 0 Let μ_0 be the SOSM.

Step k Given a matching μ_{k-1},

(k.1) if there is no cycle in $G(\mu_{k-1})$, stop: μ_{k-1} is the matching obtained;

(k.2) otherwise, *solve* one of the cycles in $G^T(\mu_{k-1})$, say ϕ_k and let $\mu_k = \phi_k \circ \mu_{k-1}$.

[15] Such a cycle selection rule is given in the appendix of Dur, Gitmez, and Yılmaz (2015).
[16] An example to demonstrate how the TP algorithm works can be seen in Dur, Gitmez, and Yılmaz (2015).

Since the TP algorithm is based on randomly selecting one of the (in general) multiple cycles in Step k.2, it is not clear whether this algorithm defines a rule. We argue that the outcome of the TP algorithm does not depend on the order of cycles solved. Thus, the TP algorithm defines a rule.

Proposition 3 *The TP algorithm defines a rule in the SEPF class.*

The proof of this result is provided in Appendix C. We call the rule defined by the *TP algorithm* as the **Top Priority (TP) rule**. The TP rule satisfies *no-consent-proofness*. Moreover, it is the *unique no-consent-proof* rule within the SEPF class.

Theorem 2 *A rule is constrained efficient, no-consent-proof and improves the SOSM if and only if it is the TP rule.*

The proof of this result is provided in Appendix D.

By Theorem 1, a matching is *constrained efficient* and *weakly Pareto dominates* the SOSM if and only if it is an outcome of the SEPF. By definition, the TP rule is in the SEPF class. Another such rule is EADAM: it is *constrained efficient* and *weakly Pareto dominates* the SOSM (Proposition 2). Moreover, EADAM also satisfies the following property: for any student, *consenting* for all schools weakly dominates *not consenting* for any of the schools (this follows from Proposition 3 by Kesten (2010)). The same property holds for the generalized EADAM as well: it is *no-consent-proof*.[17] This implies that the generalized EADAM is outcome-equivalent to the TP rule. The following important implication of this result is immediate: one cannot do better than the EADAM without sacrificing *no-consent-proofness*. This provides a strong justification for using EADAM in a school choice problem with consent.

5.5 Revisiting Allowable Priority Violations by School Districts

Given the discussion of no-consent-proofness in hand, we can reinterpret Theorem 2 under the first interpretation of allowable priority violations, where school districts consider that certain types of priorities can be violated. Note that the TP has the following property: it is a constraint efficient rule

[17] The argument follows from the proof of Proposition 3 in Kesten (2010), which relies on the consent of a student at a particular school only when that consent is relevant. Consequently, the same argument in the proof applies here as well: under the generalized EADAM, the placement of a student does not change whether she consents or not. This implies that generalized EADAM is *no-consent-proof*.

Pareto dominating SOSM such that $TP_i(C, R) = TP_i(C', R)$ for all $i \in I$ where $C'(s) = C(s) \setminus i$ for all $s \in S$. Moreover, Theorem 2 demonstrates that the TP is the unique such rule. That is, *the TP is the only rule in the SEPF class in which a student's assignment does not depend on whether her priorities are respected or not.* This outlines why the TP stands out even under the first interpretation of priority violations.

6 Strategy-proofness

For the model based on the idea of students' consent, *no-consent-proofness* is a notion regarding incentives for consenting. This does not fully capture the strategic component in school choice with priority violation based on students' consent. Students reveal not only consent decisions, but also preferences. This is a complicated game and we argue that it is not possible to prevent manipulation in this game. Actually, we obtain stronger (negative) results by considering the first interpretation where the acceptable priority violations are determined by the school districts and students reveal only preferences. We say that a rule is **strategy-proof** if, given a profile of acceptable priority violations C, for each preference profile R, truthful revelation of preferences R_i is a dominant strategy for each $i \in I$. Clearly, *strategy-proofness* is a desirable property, but it's not always satisfied. Indeed, it is already well established in the literature that *strategy-proofness* and Pareto dominance over the SOSM are incompatible properties. In particular, Theorem 1 of Kesten and Kurino (2013) and Theorem 1 of Abdulkadiroğlu, Pathak, and Roth (2009) demonstrate that "there is no *strategy-proof* rule that dominates the SOSM."[18] Since any cycle selection rule within the SEPF class necessarily improves over the SOSM, the following is obtained as a corollary of these theorems.[19]

Proposition 4 *There is no cycle selection rule within the SEPF class which satisfies strategy-proofness.*

This result, combined with Theorem 1, demonstrates that *constrained efficiency*, *weakly Pareto dominating* the SOSM and *strategy-proofness* are incompatible. Given this incompatibility result, since *constrained efficiency* is an indispensable property in our model, the only possible way to gain

[18] Both papers consider the environments where schools have indifferences among students and some tie-breaking rule is needed. Nevertheless, out setup can easily be considered as a special case where each indifference class consists of one student only. Kesten and Kurino (2013)'s result is more general than that of Abdulkadiroğlu, Pathak, and Roth (2009), as it does not require the existence of a null school. Since we allow for the null school, both theorems are applicable in our setup.

[19] Dur, Gitmez, and Yılmaz (2015) provides a separate proof for this Proposition.

strategy-proofness is to consider all the *constrained efficient* matchings instead of only the ones which *weakly Pareto dominate* the SOSM. We show that the impossibility extends.

Theorem 3 *In the school choice problem with priority violation, there is no strategy-proof rule which always yields a constrained efficient matching.*

The proof of this result is provided in Appendix E. An alternative is to relax the dominant-strategy incentive compatibility requirement, and consider the Nash Equilibrium. In this case, one can adopt the information setting offered in Section V.B. of Kesten (2010), which is an intermediate between the "complete information" and "symmetric incomplete information" setting. In this setup, the set of schools are partitioned into quality classes. Each student unambiguously prefers a school in a higher quality class over a school in a lower class; yet, the comparison of schools within the same class are not common knowledge, and each student has symmetric information about these schools. Here, *symmetric* has a particular meaning: it means that for any two schools s and s' in the same quality class, encountering a student who prefers s over s' is equally likely as encountering a student who prefers s' over s. Moreover, students' information about the set of acceptable priority violations are also uniform across the schools in the same quality class. Note that the extreme case where each quality class consists of only one school corresponds to the the complete information-common preferences setting, whereas the other extreme with only one quality class corresponds to the symmetric incomplete information setting.

One can now analyze the preference revelation game in this setup. Given the preferences R_i of a student, we say that a strategy R_i' *stochastically dominates* another strategy R_i'' if the probability distribution over the outcomes induced by R_i' stochastically dominates the probability distribution induced by R_i'', where the comparison is based on preferences R_i. The following is an adaptation of Theorem 2 of Kesten (2010), which demonstrates that the EADAM has truthful revelation in an ordinal Nash Equilibrium under such a setting. Since the generalized EADAM is equivalent to the TP rule (Section 5.4), the following result is not surprising.

Proposition 5 *Suppose that the following is common knowledge among students: The set of schools is partitioned into quality classes as follows: Let $\{S_1, S_2, \ldots, S_m\}$ be a partition of S. Given any $k, l \in 1, \ldots, m$ such that $k < l$, each student prefers any school in S_k to any school in S_l. Moreover, each student's information is symmetric for any two schools s and s' such that $s, s' \in S_r$ for some $r \in 1, \ldots, m$. Then for any student the strategy of truth telling stochastically dominates any other*

strategy when other students behave truthfully under the TP rule. Thus, truth telling is an ordinal Bayesian Nash equilibrium of the preference revelation game under the TP rule.

Proposition 5 is almost the same as Theorem 2 of Kesten (2010), and unsurprisingly, its proof follows the proof of Proposition 2 in Kesten (2010) very closely, too. It most critically uses Theorem 3.1 of Ehlers (2008),[20] which demonstrates that a sense of *strategy-proofness* which is very reminiscent of the one hypothesized in the proposition is achieved by any rule which satisfies the two basic properties: *anonymity* and *positive association*. *Anonymity* is simply the requirement that the rule should treat the schools equally, up to the permutation of their names. *Positive association* is the requirement that the rule should be invariant to certain types of transformations in an agent's preferences. The TP rule satisfies these two properties.[21]

7 Extension: The SEPF for Weak Priorities

School districts usually rank students using some predetermined criteria such as proximity and sibling status. Since in practice the number of applicants is more than the number of the priority classes, many students end up being grouped under the same priority classes. However, the student assignment rules used by the school districts are defined under the strict priority orders. Therefore, school districts use random lottery numbers to order students within priority classes. Erdil and Ergin (2008), Abdulkadiroğlu, Pathak, and Roth (2009), and Kesten (2010) point out that the random tie breaking between the students in the same priority classes causes efficiency loss, and that the particular tie-breaking rule may have dramatic effects on the outcome.[22] In order to overcome the efficiency loss caused by single random tie breaking, Erdil and Ergin (2008) and Kesten (2010) propose two solutions which are built on the DA rule. In particular, Erdil and Ergin (2008) propose a class of rules called Stable Improvement Cycles (SIC) algorithm. The SIC algorithm takes the SOSM for a given tie breaking rule and then improves the assignment by utilizing trade cycles between students, where solving these cycles do not cause any priority violation. In any step of the SIC algorithm, there may exist more than one trade cycles and there is no certain rule for the selection of the cycle that will be solved in

[20] This theorem is a generalization of Roth and Rothblum (1999).

[21] See the appendix of Dur, Gitmez, and Yılmaz (2015) for the proof of this result.

[22] See the appendix of Dur, Gitmez, and Yılmaz (2015) for an example which illustrates the welfare loss caused by the DA rule with single tie-breaking rule.

that step. On the other hand, Kesten (2010) modifies the EADAM[23] to deal with the efficiency losses caused by single tie-breaking rule. Different from Erdil and Ergin (2008), Kesten's algorithm for weak priorities selects a unique outcome.

We extend our model by allowing each school s to have coarse priority order over students denoted by $\succsim_s$. We denote the strict priority order of school s on set of students by $\succ_s$ and the associated indifference relation by $\sim_s$. By slightly changing the SEPF algorithm introduced in Section 3.1, we are able to propose an alternative way to improve the DA rule with single tie-breaking.

Given a weak priority order $\succsim$ and a random draw π over students, denote the strict priority profile attained from $\succsim$ by using π with $\succ'$. Let $I(i, s, \succsim_s)$ be the set of students who have same priority with i for school s, i.e., $I(i, s, \succsim_s) = \{j \in I : j \sim_s i\}$. Note that $i \in I(i, s, \succsim_s)$ for all $i \in I$. Given a matching μ, for each $s \in S$, let

- $D_\mu(s) = \{i \in I : s \; P_i \; \mu(i)\}$
- $X_\mu(s) = \{i \in D_\mu(s) : \forall j \in D_\mu(s) \setminus (C(s) \cap \{i\}), \; i \succ'_s j\}$
- $\tilde{X}_\mu(s) = \{k \in I(i, s, \succsim_s) \cap D_\mu(s) : i \in X_\mu(s)\}$

Note the basic difference: in the case with weak priorities, if student $i \in D_\mu(s)$ has the highest priority among the students in $D_\mu(s)$ according to $\succ'$, then all students in $I(i, s, \succsim_s) \cap D_\mu(s)$ point take part in $\tilde{X}_\mu(s)$ regardless of whether i's priority at school s can be violated or not.

Similar to the SEPF under strict priorities, for each matching μ, let $G(\mu) = (I, E(\mu))$ be the **(directed) application graph** associated with μ where the set of directed edges $E(\mu) \subseteq I \times I$ is defined as follows: $ij \in E(\mu)$ (that is, i *points to* j) if and only if $s = \mu(j)$ and $i \in \tilde{X}_\mu(s)$.

The **SEPF** for weak priorities is class of algorithms defined by iteratively with the following steps:

Step 0 Let μ_0 be the SOSM for $\succ'$.

Step k Given a matching μ_{k-1},

(k.1) if there is no cycle in $G(\mu_{k-1})$, stop: μ_{k-1} is the matching obtained;

(k.2) otherwise, *solve* one of the cycles in $G(\mu_{k-1})$, say ϕ_k and let $\mu_k = \phi_k \circ \mu_{k-1}$.

[23] We discuss EADAM in detail in Section 5.1.

The SEPF for weak priorities aims to overcome the inefficiencies caused by the random tie-breaking and rejection cycles caused due to priorities which do not have any role on the assignment of the students. It inherits *constrained efficiency* of the SEPF algorithm defined in Section 3.2. When our particular focus is to recover the efficiency losses due to the single tie-breaking rule, we do not need priority violations. Thus, in this case, we can exclude $C(s)$ from the calculation of $X_\mu(s)$. Or alternatively, we can set $C(s) = \emptyset$ for each $s \in S$ and in that case the SEPF for weak priorities and SIC algorithm of Erdil and Ergin (2008) are equivalent, i.e. for the same tie-breaking rule and chain selection rule they select the same matching. We formally state these results in the following proposition.

Proposition 6 *For each school choice problem with random tie-breaking rule π,*

(1) the SEPF for weak priorities selects a constrained efficient matching which Pareto dominates the SOSM obtained under tie breaking rule π, and

(2) it is equivalent to the SIC algorithm when $C(s) = \emptyset$ for all $s \in S$.

Proof. For the proof of the part (1), we refer to the proof of the "if part" of Theorem 1. For the second part, when $C(s) = \emptyset$ for all $s \in S$ in each step k of the SEPF for weak priorities and SIC we have the same directed graph as long as the same cycle is selected in step $k-1$. Thus, for the same cycle selection order, both algorithms select the same outcome. ■

The connection between the SEPF and SIC indicates that the SEPF can be used in order to overcome the inefficiencies caused by tie-breaking. The inefficiencies caused by the tie-breaking is not the only reason for why having priority classes generates welfare losses, though. In general, having an additional priority class may tighten the fairness constraint of mechanism designer and cause inefficiencies and welfare losses. We now investigate how the SEPF can be adopted to resolve this problem.

8 Conclusion

This study introduces the *school choice problem with priority violation*. The main result is a characterization of a class of algorithms, each of which always yields a *constrained efficient* matching *weakly Pareto dominating* the SOSM. One could adopt different interpretation of priority violations, each of which would suggest a member of the SEPF class (and in some cases, a particular member) to be used.

The framework of school choice problem with priority violation is easily applicable to settings where some priority violations are deemed feasible. We demonstrate that in this case, marking such violations as allowable and using SEPF has superior welfare properties compared to getting rid of these priority classes altogether. Another example where some violations are allowable is a setting where the centralization of assignments to exam and regular schools is possible. Whereas the priorities to exam schools are legal constraints which cannot be violated, the regular schools are more flexible in terms of their priorities. One can then simply adopt the framework offered in Section 2.2 and specify that priority violation in private schools are allowed. Each cycle selection rule within SEPF, including the "uniform cycle selection rule" which solves each cycle at each step with equal probabilities,[24] is guaranteed to produce a *constrained efficient* matching in this case.

Another case in which the SEPF is applicable is the setup where school district officials ask for the consents of students (Kesten, 2010). Clearly, this setup generates the need to incentivize (or at least avoid punishing) students for consenting and *no-consent-proofness* is indispensable in this setting. Our proposal, the TP rule, is the unique rule satisfying this property within the SEPF class. Indeed, the mechanism designer can also attempt to provide more incentives. One such rule may be the one which favors consenting students in tie breaking or giving the consenting students higher priorities when they apply to the school system for the higher grades. The characterization of such rules is left for future work.

One deficiency of the SEPF is that no rule in this class satisfies *strategy-proofness*, which is indeed the deficiency of any *constrained efficient* rule (Theorem 3). Consequently, the school choice problem with priority violations is in general prone to manipulations via misrepresentation of preferences. One could perhaps follow some "large market" results (as in Kojima and Pathak (2009)) and characterize the extent to which the students can gain by such manipulations. The incentive-compatibility properties in large markets of school choice problem with priority violation remains an open question.

Another interesting side result of our findings is the connection between Erdil and Ergin (2008)'s Stable Improvement Cycles (SIC) mechanism and Kesten (2010)'s Efficiency Adjusted Deferred Acceptance Mechanism (EADAM). Both mechanisms are proposed to overcome the inefficiencies caused by the stability requirement in different contexts. Proposition 6 and Proposition 2 combined suggests

[24] One may also expect better incentive-compatibility properties from the uniform cycle selection rule, at least under the symmetric incomplete information setup in Proposition 5, which is left for future work.

that these too solutions are not too distant from each other: both mechanism fall into the SEPF class, thus, it is also possible to interpret SEPF as a bridge connecting two popular mechanisms.

Appendix A Proof of Theorem 1

Instead of proving the characterization specific to the case in which μ_0 is the SOSM, we provide the proof of the general case in which μ_0 can be any *partially stable* matching. We begin by introducing a few remarks which will be useful in the following argument.

A cycle is *solved* at each step of the SEPF algorithm, which implies that the students in the cycle are better off and no student is worse off at the new matching obtained by solving a cycle. Thus, the matching achieved at each step *Pareto dominates* the matching in the previous step. This implies that for a student i, if a school s is better than $\mu_k(i)$, then it is also better than $\mu_{k-1}(i)$.

Remark 2 *For each $k \geq 1$ and each $s \in S$, $D_{\mu_k}(s) \subseteq D_{\mu_{k-1}}(s)$.*

A consequence of this remark is that, if student i points to student j in the graph $G(\mu_{k-1})$ and i is not better off at step k, then in the graph $G(\mu_k)$, i points to the students who are assigned at μ_k to school $\mu_{k-1}(j)$. In particular, if i points to j and both are not better off at a step, then i points to j in the next step as well.

Remark 3 *If i points to j in $G(\mu_{k-1})$ and both students' assignment do not change at step k, then i points to j in $G(\mu_k)$.*

To see this, let cycle $\phi_k = \{i_1 i_2, i_2 i_3 \ldots, i_n i_1\}$ be *solved* in the graph $G(\mu_{k-1})$ such that $\mu_k = \phi_k \circ \mu_{k-1}$. Suppose i points to j in $G(\mu_{k-1})$ and $i, j \notin \{i_1, i_2, \ldots, i_n\}$. By definition of the graph $G(\mu_{k-1})$, $i \in X_{\mu_{k-1}}(s)$ where $s = \mu_{k-1}(j)$. Since $\mu_k(i) = \mu_{k-1}(i)$, $i \in D_{\mu_k}(s)$. Let $i' \in D_{\mu_k}(s)$ be such that $i' \succ_s i$. By Remark 2, $i' \in D_{\mu_{k-1}}(s)$. Thus, since $i \in X_{\mu_{k-1}}(s)$ and $i' \succ_s i$, we have $i' \in C(s)$. Thus, each student in $D_{\mu_k}(s)$ with a higher priority than student i at school s is in the set $C(s)$. Thus, $i \in X_{\mu_k}(s)$. Since $s = \mu_k(j) = \mu_{k-1}(j)$, i points to j in the graph $G(\mu_k)$.

Now we can start with the proof.

(Proof of the "if" part)

Lemma 1 *Each matching obtained by the SEPF algorithm is partially stable.*

Proof. (i) *Partial fairness.* Let $\mu_0, \mu_1, \ldots, \mu_k, \ldots, \mu_K$ be the matchings obtained by SEPF at each step of the algorithm. We prove this statement by induction on k. The initial matching, μ_0, is required to be partially stable, thus it is *partially fair*.

As an inductive hypothesis, suppose μ_{k-1} is *partially fair*. Suppose there is a student i and school s such that $s \; P_i \; \mu_k(i)$ and $i \notin C(s)$. At each step of the algorithm, each student is either better off (she is in the selected cycle) or she is assigned to the same school as in the previous step. Thus, for each $\ell \in I$, $\mu_k(\ell) \; R_\ell \; \mu_{k-1}(\ell)$. Since $s \; P_i \; \mu_k(i)$, this implies that $s \; P_i \; \mu_{k-1}(i)$ and $i \in D_{\mu_{k-1}}(s)$. Take any $j \in \mu_k^{-1}(s)$. If $j \in \mu_{k-1}^{-1}(s)$, then by *partial fairness* of μ_{k-1}, $j \succ_s i$. Alternatively, suppose $j \notin \mu_{k-1}^{-1}(s)$. Since $j \in \mu_k^{-1}(s)$, student j is in the cycle selected in step k of the algorithm. Thus, $j \in X_{\mu_{k-1}}(s)$. By assumption, $i \notin C(s)$, thus, $i \in D_{\mu_{k-1}}(s) \setminus C(s)$. Since $j \in X_{\mu_{k-1}}(s)$, by definition, $j \succsim_s i$. Since the priorities are strict and $j \neq i$, we obtain that $j \succ_s i$. Thus, μ_k does not violate the priority of student i at school s and it is *partially fair*. The induction follows.

(ii) *Individual rationality.* Since μ_0 is *individually rational* and each student is weakly better off at each step of the SEPF algorithm, its outcome is *individually rational.*

(iii) *Non-wastefulness.* By the definition of the SEPF algorithm, for each school s, the number of students assigned to s remains at each step the same as it is under μ_0. Thus, if $|\mu_0^{-1}(s)| = q_s$, then each matching obtained by the SEPF assigns q_s students to s and it does not violate *non-wastefulness* for s. Suppose $|\mu_0^{-1}(s)| < q_s$. Since μ_0 is *non-wasteful*, the set $D_{\mu_0}(s)$ is empty. By Remark 2, at each step k, $D_{\mu_k}(s)$ is empty. Thus, each matching obtained by the SEPF satisfies *non-wastefulness.* ■

Lemma 2 *For a matching μ and $s \in S$, $X_\mu(s) = \emptyset$ if and only if $D_\mu(s) = \emptyset$*

Proof. *(Only if)* Let $X_\mu(s) = \emptyset$. Suppose $D_\mu(s) \neq \emptyset$. Then, for each $i_m \in D_\mu(s)$, there exists $i_{m+1} \in D_\mu(s) \setminus C(s)$ such that $i_{m+1} \succ_s i_m$. Note that $i_{m+1} \in D_\mu(s)$ as well. Let $i_1 \in D_\mu(s)$. Then, there is a sequence of students $(i_1, i_2, \ldots, i_n, \ldots)$ such that each student is in $D_\mu(s)$. Since the problem is finite, the sequence repeats at least one student, without loss of generality, say $i_1 = i_n$. This contradicts with the binary relation $\succ_s$ being a *strict linear order. (If)* It follows directly from the definition of the set $X_\mu(s)$. ■

Lemma 3 *Each matching obtained by the SEPF algorithm is constrained efficient.*

Proof. Let μ_k be a matching obtained by the SEPF algorithm. We will show that there does not exist a partially stable matching which Pareto dominates μ_k. Suppose there exists a *partially stable* matching $\widetilde{\mu}$ and it *Pareto dominates* μ_k.

We know that, by definition of the SEPF, there is no cycle in the graph $G(\mu_k)$. There are two possible cases.

Case 1: *There is no chain in* $G(\mu_k)$. Then, for each $s \in S$, $X_{\mu_k}(s) = \emptyset$. By Lemma 2, this implies that $D_{\mu_k}(s) = \emptyset$. Thus, at μ_k, each student is assigned to her best school. Thus, μ_k is *Pareto efficient* and $\widetilde{\mu}$ cannot Pareto dominate μ_k.

Case 2: *There is a chain in* $G(\mu_k)$. Let I_1 be the set of students who is the tail of some chain in $G(\mu_k)$. Let ϕ be a chain in $G(\mu_k)$ with the tail $i_1 \in I_1$ such that $\mu_k(i_1) = s_1$. Since i_1 is not pointed by any student, by definition of the graph $G(\mu_k)$, $X_{\mu_k}(s_1) = \emptyset$. By Lemma 2, this implies that $D_{\mu_k}(s_1) = \emptyset$. Then, since $\widetilde{\mu}$ *Pareto dominates* μ_k, the following must hold: there does not exist $i \in I$ such that $\mu_k(i) \neq s_1$ but $\widetilde{\mu}(i) = s_1$. Thus, $\widetilde{\mu}^{-1}(s_1) \subseteq \mu_k^{-1}(s_1)$. Suppose first that $\mu_k^{-1}(s_1) \setminus \widetilde{\mu}^{-1}(s_1) \neq \emptyset$. Then, there exists a school s such that $(q_s \geq)|\widetilde{\mu}^{-1}(s)| > |\mu_k^{-1}(s)|$. Since $\widetilde{\mu}$ *weakly Pareto dominates* μ_k, the second inequality implies that there exists $j \in \widetilde{\mu}^{-1}(s)$ such that $s \; P_j \; \mu_k(j)$. Since $q_s > |\mu_k^{-1}(s)|$, this violates *non-wastefulness* of μ_k. Therefore, we must have $\mu_k^{-1}(s_1) = \widetilde{\mu}^{-1}(s_1)$. Since i_1 is chosen arbitrarily, this holds for each $s \in S$ such that $\mu_k^{-1}(s) \subseteq I_1$. Let S_1 denote the set of these schools. That is, for each $s \in S_1$, $\mu_k^{-1}(s) = \widetilde{\mu}^{-1}(s)$.

There exists at least one student in $I \setminus I_1$ such that she is pointed only by students in I_1. (Otherwise there is a cycle in $G(\mu_k)$, a contradiction.) Let I_2 be the set of such students and take some $i_2 \in I_2$, $s_2 = \mu_k(i_2)$. We first show the following: there does not exist $j \in I$ such that $\mu_k(j) \neq s_2$ but $\widetilde{\mu}(j) = s_2$. To see why, suppose there is such a j. Since $\widetilde{\mu}$ *Pareto dominates* μ_k, this implies that $s_2 \; P_j \; \mu_k(j)$ and thus $j \in D_{\mu_k}(s_2)$. Nevertheless, we must have $j \notin X_{\mu_k}(s_2)$. This is because otherwise $j \in I_1$ (recall that i_2 is pointed only by students in I_1), thus, by the above paragraph, we must have $\mu_k(j) = \widetilde{\mu}(j)$, a contradiction. We conclude that $j \notin X_{\mu_k}(s_2)$ and it implies that for some $j' \in D_{\mu_k}(s_2) \setminus C(s_2)$, $j' \succ_{s_2} j$. Note that i_2 is pointed by student $i \in D_{\mu_k}(s_2) \setminus C(s_2)$, who, among the students in $D_{\mu_k}(s_2) \setminus C(s_2)$, has the top priority at s_2. Moreover, since i_2 is pointed only by students in I_1, $i \in I_1$. Since student i is assigned to the same school both under μ_k and $\widetilde{\mu}$, matching $\widetilde{\mu}$ violates the priority of student i at school s_2, which contradicts with *partial fairness* of $\widetilde{\mu}$. Thus, there does not exist a student j such that $\mu_k(j) \neq \; s_2$ but $\widetilde{\mu}(j) = s$. By *non-wastefulness* of μ_k (repeating the same argument in the previous paragraph), $\mu_k^{-1}(s_2) = \widetilde{\mu}^{-1}(s_2)$. Let S_2 denote the set of the schools such that for each $s \in S_2$, $\mu_k^{-1}(s) \subseteq I_2$.

Now we can continue in the same manner. If there is a student in $I \setminus (I_1 \cup I_2)$, who is pointed by a student in $G(\mu_k)$, then at least one of them, say i_3, is pointed only by a student in $I_1 \cup I_2$. By same argument above, the same students are assigned to school $\mu_k(i_3)$ both under μ_k and $\widetilde{\mu}$. Once

again, each student in a chain is assigned to the same school both under μ_k and $\widetilde{\mu}$. Repeating the same argument, we take care of all the students in a chain in $G(\mu_k)$.

Now, consider the students who are not in a chain in $G(\mu_k)$. If such a student assigned to school s, then $X_{\mu_k}(s) = \emptyset$, and by Lemma 2, $D_{\mu_k}(s) = \emptyset$. Thus, under μ_k, each student in $I \setminus \mu_k^{-1}(s)$ prefers her assignment to s. Since μ_k and $\widetilde{\mu}$ coincide on the students who are in a chain, if a student (not in a chain) is assigned to a different school under $\widetilde{\mu}$, then she is assigned to a school s' such that $X_{\mu_k}(s') = \emptyset$, and by Lemma 2, $D_{\mu_k}(s') = \emptyset$. Since she prefers s to s' and $\widetilde{\mu}$ *Pareto dominates* μ_k, this is a contradiction. Since s is chosen arbitrarily, this holds for each such school which is assigned to a student who is not in a chain. Therefore, the matchings μ_k and $\widetilde{\mu}$ coincide also on the students who are not in a chain. Thus, $\mu_k = \widetilde{\mu}$ and $\widetilde{\mu}$ cannot Pareto dominate μ_k. ■

Since the SEPF is such that the matching achieved at each step *improves* the matching in the previous step, with the initial step being the SOSM, clearly it *improves* the SOSM. This completes the proof of the "if" part of the theorem.

(Proof of the "only if" part)

Definition 2 *An* ***improvement cycle*** ϕ ***over a matching*** μ *is a set of ordered pairs of students* $\phi = \{i_1 i_2, i_2 i_3, \ldots, i_n i_1\}$ *such that for each* $ij \in \phi$, $\mu(j)\ P_i\ \mu(i)$.

Lemma 4 *Let* μ *and* η *be partially stable matchings such that* η Pareto dominates μ. *Then, there exists a set of distinct improvement cycles* $\Phi = \{\phi_1, \ldots, \phi_m\}$ *such that* $\eta = \phi_m \circ \ldots \circ \phi_1 \circ \mu$.

Proof. We first claim that the number of students who are assigned to each school is the same under μ and η. That is, for each $s \in S$, $|\eta^{-1}(s)| = |\mu^{-1}(s)|$. Take a school $s \in S$. To show that $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$, assume the contrary: $(q_s \geq)|\eta^{-1}(s)| > |\mu^{-1}(s)|$ and i be a student who is assigned to this school under η but not under μ. Since η *Pareto dominates* μ and preferences are strict, this implies that $s\ P_i\ \mu(i)$. But since $|\mu^{-1}(s)| < q_s$, this violates *non-wastefulness* of μ. This implies we must have $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$ for each $s \in S$. To show that $|\eta^{-1}(s)| \geq |\mu^{-1}(s)|$, again assume the contrary, i.e. assume that $|\eta^{-1}(s)| < |\mu^{-1}(s)|$. Adding over all schools and using the previous finding that $|\eta^{-1}(s)| \leq |\mu^{-1}(s)|$, we have: $\sum_{s \in S} |\eta^{-1}(s)| < \sum_{s \in S} |\mu^{-1}(s)|$. However, since η *Pareto dominates* μ and since both matchings are partially stable, if a student is assigned to a school under μ, then she is also assigned to a school under η. This means we have $\sum_{s \in S} |\eta^{-1}(s)| \geq \sum_{s \in S} |\mu^{-1}(s)|$, a contradiction.

Let N be the set of students who are better off under η. Let $G(\mu, \eta)$ be the graph with the set of vertices N and the set of edges, where student $i \in N$ points to a unique student in $N \cap \mu^{-1}(\eta(i))$ such that each student in N is pointed by a unique student. We claim that the graph $G(\mu, \eta)$ is well-defined. Since for each school s, $|\mu^{-1}(s)| = |\eta^{-1}(s)|$, if $\mu^{-1}(s) \neq \eta^{-1}(s)$, then clearly, $|\mu^{-1}(s) \backslash \eta^{-1}(s)| = |\eta^{-1}(s) \backslash \mu^{-1}(s)|$. Moreover, each $i \in \mu^{-1}(s) \backslash \eta^{-1}(s)$ is pointed by one of the students in $\eta^{-1}(s) \backslash \mu^{-1}(s)$. Thus, it is possible to construct the graph $G(\mu, \eta)$ as defined. Since each student in N is pointed by a unique student and points to a unique student, each student is in a cycle and no two cycles intersect. Each of these distinct cycles is an *improvement cycle* over μ, and the matching η is obtained by *solving* these cycles in any order, so that the numbering of these cycles is not important. ■

We next prove that each *constrained efficient* matching which Pareto dominates μ_0 can be obtained by the SEPF algorithm. For each k, a cycle in the graph $G(\mu_k)$ of the SEPF algorithm is called a **SEPF-cycle**. The previous lemma states that each *constrained efficient* matching which *improves* the SOSM can be obtained by *solving* a sequence of *improvement cycles*. To complete our proof, we prove a similar result using the *SEPF-cycles*.

Lemma 5 *Let μ and η be partially stable matchings such that η* Pareto dominates *μ. Then, there exists a sequence of* SEPF-cycles *$(\gamma_1, \ldots, \gamma_n)$ such that:*

- *γ_1 appears in $G(\mu)$;*
- *for each $i \in \{2, \ldots, n\}$, γ_i appears in $G(\gamma_{i-1} \circ \ldots \circ \gamma_1 \circ \mu)$;*
- *$\gamma_n \circ \ldots \circ \gamma_1 \circ \mu = \eta$.*

Proof. By Lemma 4, there is a set of distinct *improvement cycles* $\Phi = \{\phi_1, \ldots, \phi_m\}$ such that $\eta = \phi_m \circ \ldots \circ \phi_1 \circ \mu$. The proof is trivial for the case where the matching η is achieved by *solving* a *SEPF-cycle* at each step. To prove the other case, we assume that none of the cycles in $\Phi = \{\phi_1, \ldots, \phi_m\}$ is a *SEPF-cycle*. This assumption is without loss of generality because of the following: If some of these cycles are *SEPF-cycles* at μ, then first a *SEPF-cycle* is *solved*. At the matching obtained, if some of the remaining cycles are *SEPF-cycles*, then first a *SEPF-cycle* is *solved*. This continues until none of the remaining *improvement cycles* is a *SEPF-cycle* at the matching obtained.

Let $\phi \in \Phi$. Since ϕ is not a *SEPF-cycle*, there exists a student i with $ij \in \phi$ such that $i \notin X_\mu(\eta(i))$. We call student i a **prevented** student. We claim that there exists $i_p \in X_\mu(\eta(i))$ such that i_p is in an

improvement cycle in Φ. Since $i \notin X_\mu(\eta(i))$, there exists a student i' such that $i' \in D_\mu(\eta(i)) \setminus C(\eta(i))$ and $i' \succ_{\eta(i)} i$. Let i_p be the student with the highest priority for the school $\eta(i)$ among such students. Clearly, $i_p \in X_\mu(\eta(i))$. If $\eta(i_p) = \mu(i_p)$, then, since $\eta(i)\ P_{i_p}\ \eta(i_p)$, $i_p \notin C(\eta(i))$ and $i_p \succ_{\eta(i)} i$, there is a priority violation at η, contradicting *partial stability* of η. Thus, $\eta(i_p)\ P_{i_p}\ \mu(i_p)$, which implies that i_p is in an *improvement cycle* in Φ. We call student i_p as the **preventer of** i. By definition of the *preventer*, for each *prevented* student i, there exists a unique *preventer*, denoted by i_p.

Let $ij \in \phi_k$ and i be a *prevented* student. We consider the sequence, which starts with student j and ends with the next *prevented* student in the cycle ϕ_k. If there is no other *prevented* student in this cycle, then this sequence ends at student i. Similarly, if j is a *prevented* student, then the sequence consists only of student j.

Let $G(\mu, \eta)$ be the directed graph defined in Lemma 4 and note that it consists of *improvement cycles* in Φ.[25] We next construct the directed graph $G^{SEPF}(\mu, \eta)$ by using $G(\mu, \eta)$. First, we break each cycle in the graph $G(\mu, \eta)$ into its sequences,[26] each of which starts with a student who is pointed by a *prevented* student and ends with a *prevented* student. Clearly, each student in an *improvement cycle* is in a sequence. Second, for each *prevented* student i with $ij \in \phi_k \in \Phi$, the directed edge $i_p j$ is added to the graph $G(\mu, \eta)$. Thus, since student i is *prevented*, student j is the first member of a sequence and it is pointed by the *preventer* i_p of student i. Since for each *prevented* student there exists a unique *preventer*, the graph $G^{SEPF}(\mu, \eta)$ is such that the first student in each sequence is pointed by a (unique) student, who is also in a sequence. Thus, there exists a cycle γ_1 in this graph.

We claim that γ_1 is a *SEPF-cycle*, that is each edge in γ_1 is in the set of edges $E(\mu)$ of the application graph $G(\mu)$. First note that, the edges in γ_1 are such that either

(i) a student i' points to the next student in the sequence; that is, to the student who is assigned to school $\eta(i')$ under μ, or

(ii) a *preventer* i_p in a sequence points to the first student of a (possibly different) sequence; that is, to the student who is assigned to school $\eta(i)$ under μ.

By definition of a sequence, since only the last student is *prevented*, (i) implies that $i' \in X_\mu(\eta(i'))$.

[25] By the definition of the graph $G(\mu, \eta)$ and by Lemma 4, each cycle in the graph corresponds to an *improvement cycle* in Φ. Thus, we refer to cycles in the graph $G(\mu, \eta)$ and *improvement cycles* in Φ interchangeably.

[26] Note that this is achieved simply by removing each ij from the set of edges of $G(\mu, \eta)$ such that i is a *prevented* student.

Moreover, by definition of a *preventer*, (ii) implies that $i_p \in X_\mu(\eta(i))$. Thus, each edge in the cycle γ_1 is also an edge in the *directed application graph* $G(\mu)$. Thus, γ_1 is a *SEPF-cycle*.

We next show that the matching $\gamma_1 \circ \mu$ *Pareto dominates* μ and is *(weakly) Pareto dominated* by η. First note that under the matching $\gamma_1 \circ \mu$, each student i' who (in γ_1) points to the next student in the sequence is assigned to school $\eta(i')$. Also, in the cycle γ_1, a *preventer* i_p points to a student, who is assigned to school $\eta(i)$ under μ. We claim $\eta(i_p)\ R_{i_p}\ \eta(i)$. Suppose $\eta(i)\ P_{i_p}\ \eta(i_p)$, that is $i_p \in D_\eta(\eta(i))$. By definition of a *preventer*, $i_p \notin C(\eta(i))$ and $i_p \succ_{\eta(i)} i$. Thus, matching η *violates the priority of student* i_p *at school* $\eta(i)$, a contradiction. Thus, under the matching $\gamma_1 \circ \mu$, each student in γ_1 is better off than the matching μ and weakly worse off than the matching η; each remaining student is assigned to the same school to which she is assigned under μ, which implies that the matching $\gamma_1 \circ \mu$ *Pareto dominates* μ and is *weakly Pareto dominated* by η. Moreover, by the same argument in Lemma 1, $\gamma_1 \circ \mu$ is *partially stable*. If the matching $\gamma_1 \circ \mu$ is equivalent to η, the proof is complete. If not, we use the same argument inductively: By Lemma 4, there is a set of distinct *improvement cycles* such that the matching η is obtained by *solving* these cycles over $\gamma_1 \circ \mu$ and one can construct a *SEPF-cycle*.[27] ■

Appendix B The Generalized EADAM

We generalize the idea of *consent* introduced in Kesten (2010) by allowing each student to *consent* for violation of her priorities at *selected* schools, instead of restricting students to *consent* for all schools or not to *consent* for any school. That is, differently from Kesten (2010), we assume that when a student *consents* for a school s, she does not have to *consent* for violation of her priorities at all schools.[28]

The idea behind the (generalized) EADAM relies on an important observation on the DA: a student's (say i) priority at a school (say s) could prevent another student from enrolling to s although i is not enrolled to s at the matching given by the DA. To formalize this point, Kesten (2010) introduces the following definition: If student i is tentatively accepted by school s at some step t and is rejected by s in a later step t' of DA and there exists another student j who is rejected by s in step $t'' \in \{t, t+1, ..., t'-1\}$, then i is called **interrupter** for s and (i, s) is called an **interrupting pair** of step t'.

Each student reveals the set of schools at which she consents for the violation of her priorities. Given

[27] See Dur, Gitmez, and Yılmaz (2015) for an example which illustrates the intuition for the idea of the proof.
[28] This difference is important also for strategic *consenting* which is analyzed in Section 5.3

the consent profile, the generalized EADAM selects its outcome through the following algorithm:

Generalized Efficiency-Adjusted Deferred Acceptance (gEADAM) Algorithm:

Round 0: Run the DA algorithm.

Round $k > 0$: Find the last step of the DA run in Round $k-1$ in which an interrupter is rejected from the school for which she is an interrupter and she consents. Identify all the interrupting pairs of that step with interrupters who consent for the schools in those pairs. For each identified interrupting pair (i, s), remove s from the preferences of i without changing the relative order of the other schools. Rerun DA algorithm with the updated preference profile. If there are no more interrupters who consent for the schools they are an interrupter to, then stop.

We argue that the matching given by the gEADAM is *constrained efficient* and thus, it belongs to the SEPF class. (This result generalizes Theorem 1 of Tang and Yu (2014).)

Proposition 7 *For each problem (R, C), the matching obtained by the gEADAM is constrained efficient and weakly Pareto dominates the SOSM.*

Proof. We first show that the outcome of the gEADAM *weakly Pareto dominates* the SOSM, which is the tentative outcome of gEADAM in Round 0. When the preference profile is updated in Round $k > 0$ at most one school is removed from each student's preferences while keeping the relative order of the other schools fixed and the removed school has rejected her in Round $k-1$. That is, each student's assignment in Round $k-1$ is not removed. The priority order does not change between rounds. Thus, the matching selected in Round $k-1$ is *stable* under the updated preference profile in Round k. Since the DA selects the SOSM under the updated preference profile, in Round k, no student is assigned to a school worse than her assignment in Round k. Hence, matching in Round k weakly Pareto dominates matching in Round $k-1$, and the result follows by transitivity of Pareto dominance relation.

Since the gEADAM outcome *Pareto dominates* the DA outcome, it is *individually rational* and *non-wasteful*. At each step, the DA is run and a student's priorities can be violated only if she *consents* for the violation of those priorities. Thus, the final outcome of gEADAM is *partially stable*.

What remains to be shown is that the outcome is *constrained efficient*. Suppose the gEADAM terminates in Round K. Let $(\tilde{R}, C)$, and $\tilde{\mu}$ be the problem considered in Round K and the matching selected by the DA in Round K, respectively. We need that $\tilde{\mu}$ cannot be *Pareto dominated* by another *partially stable* matching in (R, C).

We first show that $\tilde{\mu}$ is constrained efficient in $(\tilde{R}, C)$. Since $\tilde{\mu}$ is the SOSM for problem $(\tilde{R}, C)$, it is *stable* for problem $(\tilde{R}, C)$, which implies that it is *partially stable* for $(\tilde{R}, C)$. We now show that there does not exist another *partially stable* in problem $(\tilde{R}, C)$ which Pareto dominates $\tilde{\mu}$. Take another *partially stable* matching, $\bar{\mu}$, for $(\tilde{R}, C)$. For any school $s \in S$, let $r_n(s)$ and $t_n(s)$ denote the sets of rejected and tentatively accepted students by s, respectively, in the nth step of the DA run in the last round (Round K) of the gEADAM. By induction we show that for each $n > 0$, students in $r_n(s)$ cannot be assigned to s in $\bar{\mu}$ and thus, $\bar{\mu}$ cannot *Pareto dominate* $\tilde{\mu}$.

Consider Step 1 of the DA run in Round K of the gEADAM. If for each $s \in S$, $r_1(s) = \emptyset$, then each student is assigned to her top-ranked school in $\tilde{R}$ and $\tilde{\mu}$ is *Pareto efficient*, that is, it cannot be *Pareto dominated* by another matching with respect to preference profile $\tilde{R}$. Otherwise, let $s \in S$ be a school such that $r_1(s) \neq \emptyset$. Since the gEADAM terminates at the end of this round (Round K), there does not exist an *interrupting pair* (i, s) such that $i \in C(s)$. Thus, each student in $t_1(s)$ is either (i) permanently accepted by s at $\tilde{\mu}$, or (ii) they are in $t_1(s) \setminus \tilde{\mu}(s)$. But no one in $t_1(s) \setminus \tilde{\mu}(s)$ *consents* for s (Otherwise, the consenting student in $j \in t_1(s) \setminus \tilde{\mu}(s)$ is an *interrupter* for s this contradicts generalized EADAM terminating in Round K.) In either case, students in $r_1(s)$ cannot be assigned to s at any partially stable matching, in particular $\bar{\mu}$. Now suppose that students in $r_{\bar{n}}(s)$ are not assigned to s in $\bar{\mu}$ for each $n \leq \ell$ and $s \in S$. Consider step $\ell+1$. Students who are tentatively accepted by school s in step ℓ and students applying to school s in step ℓ have already been rejected by their better options, and by inductive hypothesis, they cannot be assigned to a better school than s in $\bar{\mu}$. If for each $s \in S$, $r_{\ell+1}(s) = \emptyset$, then for each $s \in S$, $t_{\ell+1}(s) = \tilde{\mu}^{-1}(s)$ and any student i cannot be assigned to a better school than $\tilde{\mu}(i)$ at $\bar{\mu}$. Suppose for some $s \in S$, $r_{\ell+1}(s) \neq \emptyset$. Then, each student in $t_{\ell+1}(s)$ is either (i) permanently accepted by s at $\tilde{\mu}$ or (ii) they are in $t_{\ell+1}(s) \setminus \tilde{\mu}(s)$. Again, no one in $t_{\ell+1}(s) \setminus \tilde{\mu}(s)$ consents for s, otherwise K cannot be the last step. In either case students in $r_{\ell+1}(s)$ cannot be assigned to s at $\bar{\mu}$. The induction follows, and we have the result that any $\bar{\mu}$ which is *partially stable* in $(\tilde{R}, C)$ cannot *Pareto dominate* $\tilde{\mu}$; hence, $\tilde{\mu}$ is *constrained efficient* in $(\tilde{R}, C)$.

Now, our final claim is that *constrained efficiency* in $(\tilde{R}, C)$ implies *constrained efficiency* in (R, C). We prove this by backward induction. Let μ_k be the matching selected in Round k, with $k \in \{1, \ldots, K\}$. By Tang and Yu (2014) (Section 3.2), the students whose preferences are updated in Round k are assigned to *underdemanded* schools in matching μ_{k-1}. That is, only difference between problems considered in Round k and $k-1$ is in the preference profiles of some students who are assigned to

underdemanded schools in Round $k-1$, and these are the students whose assignments can't be made better off in any partially stable matching. Thus, if a matching is *constrained efficient* in problem considered in Round k, then it is also *constrained efficient* in problem considered in Round $k-1$. The induction follows, and we obtain the result. ■

Appendix C Proof of Proposition 3

Lemma 6 *Suppose student i is assigned to an underdemanded school s at step k of the SEPF algorithm. Then, at each step $t \geq k$, she is assigned to s, thus she is not part of any cycle.*

Proof. Let $\mu_k(i) = s$. Remember that if s is underdemanded at step k, either s has no demand at μ_k or each path to i starts with a student assigned to a school with no demand. That is, i is a part of a path in $G(\mu_k)$, and she may be located at any part of this path (including the tail).[29] We prove the result by induction on the location of i within the path.

Initial step: Student i is at the tail: she is assigned to a school with no demand. Since i is not pointed in $G(\mu_k)$, $X_{\mu_k}(s) = \emptyset$. By Lemma 2, $D_{\mu_k}(s) = \emptyset$. Moreover, by Remark 2, $D_{\mu_k}(s) = \emptyset$ implies that, for each $t \geq k$, $D_{\mu_t}(s) = \emptyset$. Since by definition, $X_\mu(s) \subseteq D_\mu(s)$, for each $t \geq k$, $X_{\mu_t}(s) = \emptyset$, and student i is not part of any cycle.

Inductive hypothesis: Student i is pointed only by students who are not part of a cycle at each step $t \geq k$. Suppose $i' \in D_{\mu_k}(s)$ but i' does not point to i. Since $i' \in D_{\mu_k}(s) \backslash X_{\mu_k}(s)$, there exists $j \in D_{\mu_k}(s) \backslash C(s)$ such that $j \succ_s i'$ and j points to i. Since for each $t \geq k$, j is not part of a cycle, at each step $t \geq k$, i' does not point to i. Note that this is true for each student in $D_{\mu_k}(s)$ who does not point to student i at step k. Thus, at each step $t \geq k$, i is pointed only by the students who point to her at step k, and thus she is not part of a cycle. By inductive hypothesis, if i is assigned to an *underdemanded* school at step k, then at each step $t \geq k$, i is pointed only by students who are not part of a cycle. Thus, in each matching defined by the SEPF class, student i is matched to school $\mu_k(i)$. ■

Lemma 6 justifies the language we use for the students assigned to *underdemanded* schools at some matching μ_k: by definition, a *permanently matched* student, say i, at μ_k is assigned to an *underdemanded* school at μ_k and she is not part of any cycle through the end of the SEPF algorithm.

[29] One can interpret a student who is not pointed by anyone and who doesn't point to anyone as a path with one vertex, so this is a comprehensive statement.

Thus, at each *constrained efficient* matching which *weakly Pareto dominates* μ_k, student i is assigned to school $\mu_k(i)$.

Lemma 7 *In the graph $G^T(\mu_{k-1})$, let cycle $\phi_k = \{i_1 i_2, i_2 i_3 \ldots, i_n i_1\}$ be* solved *by TP rule such that $\mu_k = \phi_k \circ \mu_{k-1}$. Then, if i points to j in $G^T(\mu_{k-1})$ where $i \notin \{i_1, i_2, \ldots, i_n\}$ and i is temporarily matched at μ_k, then i points to j in $G^T(\mu_k)$.*

Proof. Note that a student can be pointed by at most one student in $G^T(\mu_{k-1})$. Since i points to j in $G^T(\mu_{k-1})$, $i \notin \{i_1, i_2, \ldots, i_n\}$ implies $j \notin \{i_1, i_2, \ldots, i_n\}$ Since the schools to which i and j are assigned do not change at step k, by Remark 3, i points to j in $G(\mu_k)$. Suppose student $i' \neq i$ points to j in $G^T(\mu_k)$. Thus, $i' \in D_{\mu_k}(\mu_k(j)) \backslash C(\mu_k(j))$ and i' is *temporarily matched* at μ_k. Since $\mu_k(j) = \mu_{k-1}(j)$, by Remark 2, $i' \in D_{\mu_{k-1}}(\mu_{k-1}(j)) \setminus C(\mu_{k-1}(j))$. Moreover since i is *temporarily matched* at μ_k, i' has a higher priority at the school $\mu_{k-1}(j)$ than i. Thus, since i (but not i') points to j in the graph $G^T(\mu_{k-1})$ implies that i' is *permanently matched* at μ_{k-1}. This contradicts Lemma 6, which implies that for each $t \geq k-1$, i' must be *permanently matched* at the matching μ_t. ■

Lemma 8 *If a cycle ϕ in the graph $G^T(\mu_{k-1})$ is not* solved *at step k, then ϕ exists in the graph $G^T(\mu_k)$.*

Proof. Let $ij \in \phi$. By Remark 3, i points to j in $G(\mu_k)$. Since this holds for each edge in ϕ, ϕ is a cycle in $G(\mu_k)$. Thus, i is *temporarily matched* at μ_k. By Lemma 7, this implies that i points to j in $G^T(\mu_k)$. Since this holds for each $ij \in \phi$, the graph $G^T(\mu_k)$ has cycle ϕ. ■

Lemma 9 *Let μ_0 be the SOSM under problem (R, C). Consider a cycle selection order denoted by $\Phi = (\phi_1, \phi_2, ..., \phi_n)$ such that ϕ_1 occurs in $G^T(\mu_0)$, $\mu_k = \phi_k \circ \mu_{k-1}$ and ϕ_k occurs in $G^T(\mu_{k-1})$ for all $k \in \{1, 2, ..., n\}$. Denote the outcome of TP algorithm under Φ with μ. If there exists $\tilde{k}$ such that $\phi_{\tilde{k}+1}$ occurs in $G^T(\mu_{\tilde{k}-1})$, then TP algorithm selects μ in cycle selection order $\hat{\Phi} = (\phi_1, ...\phi_{\tilde{k}-1}, \phi_{\tilde{k}+1}, \phi_{\tilde{k}}, \phi_{\tilde{k}+2}, ..., \phi_n)$.*

Proof. Let ν_k be the matching selected in step k of TP algorithm under $\hat{\Phi}$. Since in the first $\tilde{k}-1$ steps the same cycles are removed under both $\hat{\Phi}$ and Φ, we have $\mu_k = \nu_k$ for all $k \leq \tilde{k}-1$. Hence, $G^T(\mu_{\tilde{k}-1}) = G^T(\nu_{\tilde{k}-1})$. That is, $\phi_{\tilde{k}+1}$and $\phi_{\tilde{k}}$ exist in $G^T(\nu_{k-1})$. Moreover, $\phi_{\tilde{k}+1}$and $\phi_{\tilde{k}}$ are disjoint. When $\phi_{\tilde{k}+1}$ is solved in step k, by Lemma 8 $\phi_{\tilde{k}}$ exists in $G^T(\nu_k)$. We have $\mu_{\tilde{k}+1} = \nu_{\tilde{k}+1}$ and $G^T(\mu_{\tilde{k}+1}) = G^T(\nu_{\tilde{k}+1})$ since the cycles are disjoint and only the students in $\phi_{\tilde{k}+1}$and $\phi_{\tilde{k}}$ improved to the same schools. Then, ϕ_k occurs in $G^T(\nu_k)$ and $\mu_k = \nu_k$ for all $k \geq \tilde{k}+1$. ■

Lemma 10 *The outcome of the TP algorithm is independent of the order of cycles solved in each step.*

Proof. We prove by constructing a cycle selection order, Φ, which generates the same outcome as any other cycle selection order, $\hat{\Phi}$, under the TP algorithm.

Take a given problem (R, C), and let μ_0 be the SOSM under this problem. Denote the set of cycles in $G^T(\mu_0)$ with A_0. The construction of the "universal cycle selection" order Φ first requires a tie-breaker vector. Let $\pi = (\pi_i)_{i \in I}$ be such a tie-breaker vector where π_i is the number assigned to student $i \in I$. Given this, the order Φ is as follows:

"At round $k \geq 0$, given matching μ_k:

1. Let A_k be the set of cycles in $G^T(\mu_k)$.

2. Consider the cycles in $\cup_{\tilde{k} \leq k} A_{\tilde{k}}$ which are not yet solved. Among those, pick the cycle to solve according to the following (lexicographic) cycle selection rule:

   (a) for all m and m' such that $m < m' \leq k$, all cycles in A_m are solved before the cycles in $A_{m'}$;

   (b) for all $m \leq k$, the cycles in A_m are solved according to the highest tie breaker number of the student in the cycle.

3. Solve the cycle given according to this rule and obtain the new matching μ_{k+1}."

Suppose Φ ends at Round K, and yields μ_K. Take any other cycle selection rule $\hat{\Phi}$. We show that $\hat{\Phi}$ also produces the same matching. First realize that all cycles in A_0 necessarily appear under any cycle selection rule. By Lemma 8, they are solved under $\hat{\Phi}$. By using Lemma 9, we can rearrange the order of cycles such that first $|A_0|$ rounds are the same as those of Φ, and the final outcome of $\hat{\Phi}$ is unchanged. This produces, say, $\tilde{\Phi}$, whose final outcome is the same as $\hat{\Phi}$ and whose first $|A_0|$ rounds are the same as Φ. Since first $|A_0|$ rounds are the same, the cycles in $\cup_{\tilde{k} \leq k} A_{\tilde{k}}$ all appear under $\tilde{\Phi}$. By Lemma 8, these cycles are solved under $\tilde{\Phi}$. One can then reapply Lemma 9 and get another cycle selection rule which yields the same outcome as $\hat{\Phi}$, and yields the same matchings in the first $|A_0| + |A_{|A_0|}|$ steps. One can then continue until the cycle selection rule whose final outcome is the same as $\hat{\Phi}$ and whose first K steps are the same as Φ. Thus, $\hat{\Phi}$ and Φ produce the same outcome. ■

Lemma 11 *For each $k \geq 1$, there is a cycle in the graph $G(\mu_k)$ if and only if there is a cycle in the graph $G^T(\mu_k)$.*

Proof. (Only if) Since there is a cycle in $G(\mu_k)$, the set of *temporarily matched* students, $I \setminus I^u_{\mu_k}$, is nonempty. By definition of the graph $G^T(\mu_{k-1})$, each student in $I \setminus I^u_{\mu_k}$ is pointed by a unique student in $I \setminus I^u_{\mu_k}$.[30] Thus, there exists a cycle in $G^T(\mu_k)$. In particular, each cycle in $G^T(\mu_k)$ is formed by the students in $I \setminus I^u_{\mu_k}$. (If) It follows directly from the fact that $G^T(\mu_k)$ is a subgraph of $G(\mu_k)$. ■

Proof of Proposition 3: By Lemma 11, the TP algorithm is in the SEPF class. By Lemma 10, any cycle selection order gives the same matching under the TP algorithm. Thus, the TP algorithm produces a unique matching and it defines a rule in the SEPF class.

Appendix D Proof of Theorem 2

(Proof of the "if" part)

Lemma 12 *Let i be a permanently matched student at μ for (R, C). Then, i is permanently matched at μ for each problem (R, C') where C and C' coincide except i's consent.*

Proof. If i does not point to j in the graph $G(\mu)$, then i's *consent* for $\mu(j)$ is irrelevant in terms of who point to j in $G(\mu)$.[31]

Suppose i is *permanently matched* at μ. Then, by the definition of an *underdemanded* school, either (i) $\mu(i)$ has no demand at μ or (ii) each path to i starts with a student assigned to a school with *no demand* at μ. For case (i), a school having *no demand* depends only on the students' preferences, so there is no way to change it through the consenting behavior. Now assume case (ii). For this case, first realize that the only way to change the underdemanded status is through changing the arrows in the paths leading to i. Nevertheless, consenting behavior for schools where no students in this group (the group of students in a path leading to i) are assigned has no effect on these arrows. This means that we can restrict attention to changes in consents to schools where some students in the paths leading to i are assigned. Let j be a student on a path to i. Clearly, i does not point j (otherwise, i is not *permanently matched* at μ). Thus, since, by the argument in the previous paragraph, i's *consent* for school $\mu(j)$

[30] Note that by Remark 1, the students who are in $I \setminus I^u_{\mu_k}$ and are assigned to the same school at μ_k are pointed in $G^T(\mu_k)$ by the same student.

[31] The following argument clarifies this. Clearly, when $\mu(i) = \mu(j)$ the consent doesn't matter at all, so assume $\mu(i) \neq \mu(j)$. Now consider two cases. (i) If $\mu(i)\ P_i\ \mu(j)$, then $i \notin D_\mu(\mu(j))$, so i's consent is never used in the construction of $G(\mu)$. Therefore it doesn't determine who points to j. (ii) If $\mu(j)\ P_i\ \mu(i)$, there is another student i' pointing to j such that she has a higher priority than i at $\mu(j)$ and does not consent for $\mu(j)$. But then the consent of i is does not determine who points to j, because there is higher priority and non-consenting student.

is irrelevant in terms of which students point to j, each path to i remains the same regardless of the consent of i for the schools which the students on these paths are assigned. This means that i remains permanently matched at μ regardless of her consenting behavior for the schools of students in the paths leading to i. Therefore, $\mu(i)$ remains underdemanded at μ for each problem (R, C') where C and C' coincide except i's consent. Thus, i is *permanently matched* at μ for such a problem. ■

Proposition 8 *Under the TP rule, the placement of a student does not change whether she consents or not. Consequently, the TP rule is no-consent-proof.*

Proof. By the definition of the TP rule, at each step k, the consent of only the *permanently matched* students at μ_{k-1} is relevant for the graph $G^T(\mu_{k-1})$.[32] Moreover, by Lemma 12, a student remains *permanently matched* at μ_{k-1} regardless of her consenting decisions. Also, by Lemma 6, each *permanently matched* student at μ_{k-1} is assigned to the same school under the matching given by the TP rule. That is, whenever a student's *consent* matters at some step k of the TP rule, then that student is already assigned to her school under the matching given by the TP rule at an earlier step $k' < k$, and her consenting decision can't affect this. ■

(Proof of the "only if" part)

Fix a problem (R, C). Let ψ be a *constrained efficient* and *no-consent-proof* rule which gives a matching that *weakly Pareto dominates* the SOSM. The matchings given by the TP rule and ψ for (R, C) are denoted by $TP_{(R,C)}$ and $\psi_{(R,C)}$, respectively. Let μ_k be the matching selected at step k of the TP rule. By the following lemma, we first show that for each k, $\psi_{(R,C)}$ *weakly Pareto dominates* the matching μ_k, which implies that $\psi_{(R,C)}$ *weakly Pareto dominates* $TP_{(R,C)}$. Since both matchings $\psi_{(R,C)}$ and $TP_{(R,C)}$ are *constrained efficient*, this implies that $\psi_{(R,C)} = TP_{(R,C)}$. This completes the proof.

Lemma 13 *For each step k of the TP rule, $\psi_{(R,C)}$ weakly Pareto dominates μ_k.*

Proof. We prove by contradiction. In particular, we will start by assuming the contrary, and then we will generate a consent profile C^* for which $\psi_{(R,C^*)}$ does not produce a constrained efficient matching.

Let $A_0 = \emptyset$. Let ϕ_k be the cycle *solved* in $G^T(\mu_{k-1})$ and μ_k the matching obtained at step k of the TP rule. Suppose the TP rule terminates at step K; that is, $\mu_K = TP_{(R,C)}$.

[32] A *temporarily matched* student can potentially affect the graph $G(\mu_{k-1})$ by her consenting decision, but not $G^T(\mu_{k-1})$. This is because the only way in which a temporarily matched student i affects $G^T(\mu_{k-1})$ by not consenting for s is by being the top priority agent among those who are temporarily matched and who point to $\mu_{k-1}^{-1}(s)$ in $G(\mu_{k-1})$. But in this case i points to $\mu_{k-1}^{-1}(s)$ under $G^T(\mu_{k-1})$ anyway, so her consenting decision is irrelevant.

Suppose that, to get a contradiction, there is a step $\tilde{k} \leq K$ where $\psi_{(R,C)}$ does not weakly Pareto dominate $\mu_{\tilde{k}}$. Let k be the first such step. That is, for all all $k' < k$ and for all $i \in I$, $\psi_{(R,C)}(i)\ R_i\ \mu_{k'}(i)$ and $\mu_k(j)\ P_j\ \psi_{(R,C)}(j)$ for some $j \in I$. Let $\phi_k = \{i_1 i_2, i_2 i_3 \ldots, i_n i_1\}$. Since we chose k to be the *first* step which is not weakly Pareto dominated, there exists a student in $\{i_1, i_2 \ldots i_n\}$ who prefers her assignment under μ_k to $\psi_{(R,C)}$. Without loss of generality, suppose it is i_1. That is, assume: $\mu_k(i_1) P_{i_1} \psi_{(R,C)}(i_1)$. Note that $\mu_k(i_1) = \mu_{k-1}(i_2)$, and denote $\mu_{k-1}(i_2)$ with s_1.

We begin by adding the student-school pair (i_1, s_1) to A_0. Let $A_1 := A_0 \cup \{(i_1, s_1)\}$. Furthermore, consider consent profile C^1 such that $i_1 \notin C^1(s_1)$ and the *consent* profile for the remaining schools/students is the same as C. Now, we consider two possible cases:

Case 1: Suppose i_1 does not *consent* for s_1; that is, $i_1 \notin C(s_1)$. (Note that in this case, $C = C^1$, so we don't change anything on the original consent profile. Moreover, we have: $s_1 P_{i_1} \psi_{(R,C^1)}(i_1)$ by assumption.) i_1 has the highest priority at s among the *temporarily matched* students who prefer s_1 to their assignment at μ_{k-1}. Thus, only i_1 points to students in $\mu_{k-1}^{-1}(s_1)$. Since $\psi_{(R,C^1)}$ is *constrained efficient* and *weakly Pareto dominates* μ_{k-1}, the matching $\psi_{(R,C^1)}$ is obtained by solving of a sequence of *SEPF-cycles* (Theorem 1). Thus, since i_1 is assigned a school worse than s_1 under $\psi_{(R,C^1)}$, she prevents each student not in $\mu_{k-1}^{-1}(s_1)$ from being assigned to s_1 under $\psi_{(R,C^1)}$. Moreover, at each *partially stable* matching *weakly Pareto dominating* μ_{k-1}, the number of students assigned to s_1 is $|\mu_0^{-1}(s_1)|$ (Lemmas 4 and 5). Thus, i_1 prevents each student in $\mu_{k-1}^{-1}(s_1)$ from being better off under $\psi_{(R,C^1)}$ as well. Thus, each student in $\mu_{k-1}^{-1}(s_1)$ is assigned to s_1 under $\psi_{(R,C^1)}$. That is, *if* $i_1 \notin C(s_1)$, *then* i_2 *is assigned a school worse than* $\mu_k(i_2) = \mu_{k-1}(i_3)$ *in* $\psi_{(R,C^1)}$.

Case 2: Suppose $i_1 \in C(s_1)$. By *no-consent-proofness* of ψ, we must have $\psi_{(R,C)}(i_1) R_{i_1} \psi_{(R,C^1)}(i_1)$. Since $s_1 P_{i_1} \psi_{(R,C)}(i_1)$, we have: $s_1 P_{i_1} \psi_{(R,C^1)}(i_1)$. There are two possibilities:

Case 2.1: $\psi_{(R,C^1)}^{-1}(s_1) = \mu_{k-1}^{-1}(s_1)$. In this case, all students in $\mu_{k-1}^{-1}(s_1)$ are assigned to s_1 in $\psi_{(R,C^1)}$. Then, i_2 *is assigned a school worse than* $\mu_k(i_2) = \mu_{k-1}(i_3)$ *in* $\psi_{(R,C^1)}$.

Case 2.2: $\psi_{(R,C^1)}^{-1}(s_1) \neq \mu_{k-1}^{-1}(s_1)$. In this case, there is another student j who is not in $\mu_{k-1}^{-1}(s_1)$ and who is assigned to s_1 in $\psi_{(R,C^1)}$. But by *partial fairness* of ψ, each student assigned to s_1 in $\psi_{(R,C^1)}$ must have higher priority than i_1. This means that $j \succ_{s_1} i_1$.

Consider the two possibilities: under (R, C), either $\mu_{k-1}(j) P_j s_1$ or $s_1 P_j \mu_{k-1}(j)$. In the former case, j *is assigned a school worse than* $\mu_{k-1}(j)$ *in* $\psi_{(R,C^1)}$. In the latter case, we must have: $j \in C(s_1)$ and j is *permanently matched* at μ_{k-1}. Then, the assignment of j to s_1 in $\psi_{(R,C^1)}$ implies that at least one

student is assigned to an underdemanded school in $\psi_{(R,C^1)}$ which is worse than her assignment under μ_{k-1}.[33] Therefore, under both cases there exists $k' < k$ and a student $\tilde{j}$ such that $\mu_{k'}(\tilde{j})\ P_{\tilde{j}}\ \psi_{(R,C^1)}(\tilde{j})$.

Let's summarize everything we have done so far. We began with the first step k where ψ does not Pareto dominate μ_k. Then, we found a student-school pair (i_1, s_1), with the property that $s_1 P_{i_1} \psi_{(R,C)}(i_1)$. Then, we found a consent profile C^1 where $i_1 \notin C^1(s_1)$, and a step $k^1 \leq k$ with the following property: for some $\ell \in I$, $\mu_{k^1}(\ell)\ P_\ell\ \psi_{(R,C^1)}(\ell)$. Remember that at this point $A_1 = \{(i_1, s_1)\}$.

Now, we repeat the whole argument over again. Take step k^1 defined in the previous paragraph, and take the student-school pair $(i_2, s_2) := (\ell, \mu_{k^1}(\ell))$. Realize that by construction this pair satisfies the property that $s_2\ P_\ell\ \psi_{(R,C^1)}(i_2)$. Add this pair to A_1, and let $A_2 := A_1 \cup \{(i_2, s_2)\}$. Consider the C^2 where $i_2 \notin C^2(s_2)$ and the consent profile for the remaining schools/students is the same as C^1. Following the exact same argument, one can find a step $k^2 \leq k^1$ with the following property: for some $\ell \in I$, $\mu_{k^2}(\ell)\ P_\ell\ \psi_{(R,C^2)}(\ell)$.

In general, at each step m, given A_{m-1} and k^{m-1}, take this pair, and let $(i_m, s_m) := (\ell, \mu_{k^{m-1}}(\ell))$. Define $A_m = A_{m-1} \cup \{(i_m, s_m)\}$, find a consent profile C^m where $i_m \notin C^m(s_m)$, and a step $k^m \leq k^{m-1}$ with the following property: for some $\ell \in I$, $\mu_{k^m}(\ell)\ P_\ell\ \psi_{(R,C^m)}(\ell)$.

Realize that k^m is a weakly decreasing sequence, and A_m is expanding at each step. These two facts, combined with the finiteness of student and school sets, implies that eventually the next pair (i_{m+1}, s_{m+1}) will be a pair which is already in A_m. That is, the process will *cycle*. Fix the consent profile C^m and the step k^m at this moment, and denote them C^* and k^*, respectively. Now we have a consent profile C^*, a step k^*, and a cycle of agents $\phi = (i_1 i_2, i_2 i_3, \ldots, i_m i_1)$ which appears in $G^T(\mu_{k^*})$, with the following property: "for each $n \in \{1, \ldots, m\}$, $\mu_{k^*}(i_n)\ P_{i_n}\ \psi_{(R,C^*)}(\ell)$." Since the solution of this cycle ϕ does not violate *partial fairness* of $\psi_{(R,C^*)}$ and does not make any student worse off, $\psi_{(R,C^*)}$ cannot be *constrained efficient*. ■

[33] The heuristics is as follows: Given that j is assigned to s_1, now, by constrained efficiency of ψ, someone must fill the seat that j left in $\mu_{k-1}(j)$ under $\psi_{(R,C^1)}$. Call this student j'. j' may prefer $\mu_{k-1}(j)$ to her assignment under $\mu_{k-1}(j')$, in which case we found the student. Alternatively, j' may prefer $\mu_{k-1}(j')$ to $\mu_{k-1}(j)$. But then, since j is permanently matched under μ_{k-1}, either (i) j' is permanently matched under μ_{k-1} too, (ii) j' is temporarily matched under μ_{k-1}, but $j \notin X_{\mu_{k-1}}(\mu_{k-1}(j)$ because she is blocked by a higher-priority, non-consenting student (say, j''). In case (i), we continue with the seat that j' left. In case (ii), by partial stability, j'' must also be assigned to a better school, and we continue with the seat that j'' left. Because we always continue with seats in underdemanded schools, the process can't cycle and will eventually end up with such a student.

Appendix E Proof of Theorem 3

Proof. Consider the following problem: $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$ and $q_s = 1$ for all $s \in S$. The preferences and the priorities are: $s_1 P_{i_1} s_2 P_{i_1} s_3$, $s_1 P_{i_2} s_2 P_{i_2} s_3$, $s_3 P_{i_3} s_1 P_{i_3} s_2$, $i_3 \succ_{s_1} i_1 \succ_{s_1} i_2$, $i_1 \succ_{s_2} i_2 \succ_{s_2} i_3$, and $i_1 \succ_{s_3} i_2 \succ_{s_3} i_3$.

Assume that $C(s_1) = \{i_1\}$, $C(s_2) = C(s_3) = \emptyset$. This problem has three *partially stable* matchings:[34] $\mu := \{(i_1, s_1), (i_2, s_2), (i_3, s_3)\}$, $\mu' := \{(i_1, s_2), (i_2, s_1), (i_3, s_3)\}$, $\mu'' := \{(i_1, s_2), (i_2, s_3), (i_3, s_1)\}$.

Among these matchings, μ'' is *Pareto dominated* by μ. Thus, μ and μ' are the only *constrained efficient* matchings.

Let ψ be a *strategy proof* rule which selects a *constrained efficient* matching. Suppose μ' is the outcome given by ψ in this problem. If i_1 deviates and reports $P'_{i_1} : s_1 \; P'_{i_1} \; s_3 \; P'_{i_1} \; s_2$, in the new problem, the only *constrained efficient* matching is μ.[35] Then ψ must select μ for this problem. Hence, i_1 can gain from misreporting if ψ selects μ in the original problem.

Alternatively, suppose μ be the outcome of ψ in the original problem. If i_2 deviates and reports $P'_{i_2} : s_1 \; P'_{i_2} \; s_3 \; P'_{i_2} \; s_2$, in the new problem, the only *constrained efficient* matching is μ'.[36] Then ψ must select μ' for this problem. Thus, i_2 is better off by misrepresenting if ψ selects μ in the original problem.[37] ■

[34] Any matching where i_1 is assigned to s_3 violates the priority of i_1 for s_2, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1.

[35] Any matching where i_1 is assigned to s_2 violates the priority of i_1 for s_3, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1. The only *partially stable* matchings are μ and $\{(i_1, s_3), (i_2, s_2), (i_3, s_1)\}$, but the former *Pareto dominates* the latter.

[36] Any matching where i_1 is assigned to s_3 violates the priority of i_1 for s_2, and any matching where i_3 is assigned to s_2 violates the priority of i_3 for s_1. Also, under μ, i_2's priority for s_3 is violated. The only *partially stable* matchings are μ' and μ'', but the former *Pareto dominates* the latter.

[37] One can easily modify the problem used in the proof of Proposition 1 of Ehlers and Westkamp (2009) in order to construct a counterexample in which a school is consented by either all students or no student.

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