Random Assignment under Weak Preferences*

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Abstract

The natural preference domain for many practical settings of the assignment problems is the one in which agents are allowed to be indifferent between objects, the *weak preference domain.* Most of the existing work on assignment problems assumes strict preferences. There are important exceptions, but they provide solutions only to the assignment problems with a social endowment, where agents own objects collectively and there are no private endowments. We consider the general class of assignment problems with private endowments and a social endowment. Our main contribution is a recursive solution for the weak preference domain. Our solution satisfies *individual rationality, ordinal efficiency* and a recently introduced fairness axiom, *no justified-envy.*

Keywords : random assignment, ordinal efficiency, no justified-envy, parametric maximum flow algorithm

Journal of Economic Literature **Classification Numbers:** C71, C78, D71, DF8

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1 Introduction

We consider the assignment problem: a set of objects has to be allocated to a group of agents in such a way that each agent receives at most one object and monetary transfers between the agents are not permitted. Examples include the assignment of campus housing to students, jobs to workers, rooms to housemates, and offices to professors. For convenience, this paper uses language that fits the first example and refers to the objects as *houses*.

A prominent feature of many real-life assignment problems is the presence of agents with private endowments: some agents (existing tenants) own their own houses, whereas others (new applicants) do not, and the houses not owned by the existing tenants are the social endowment. This assignment problem is the focus of the current paper and we refer to it as *an assignment problem with private endowments and a social endowment.* Our goal is to propose an efficient and fair solution defined on the preference domain with indifference permitted, the weak preference domain.

In *assignment problems with a social endowment*, agents own houses collectively. In this class of problems, fairness is essential and to restore ex ante fairness, a lottery mechanism is commonly used in real-life applications: An ordering of agents is randomly drawn from the uniform distribution. For a given ordering, the first agent is assigned the top house in his announced ranking, then the second agent is assigned the top house in his announced ranking of the remaining houses, and so on. This is the *random priority* (RP) solution.[1] The RP solution is *strategy-proof, ex-post efficient;* it *treats equals equally.* However, it does not satisfy *ordinal efficiency,* an efficiency requirement for *ordinal mechanisms* where only individual preferences over sure houses are elicited.[2,3]

Another class of solutions is induced by the parametric family of *eating* algorithms (Bogomolnaia and Moulin (2001)):[4] Each house is imagined as being infinitely divisible. There is one unit of each house. A quantity of house h, given to agent i, represents the probability with which agent i is assigned house h. For each agent, an 'eating' speed is specified. At any time, each agent *eats* the top available house in his announced ranking at the specified speed: if the houses a, b, $c, \dots$ have been entirely eaten (one unit of each has been distributed), and houses x, y, $z, \dots$ have not, each agent starts eating the top house in his announced ranking of x, y, $z, \dots$ This class contains a special solution, the *probabilistic serial* (PS) *solution*, which

[1]The RP solution is equivalent to the 'core from random endowment' solution, a solution that randomly chooses an endowment profile and then selects the core of the induced assignment problem with private endowments. (Abdulkadiroğlu and Sönmez (1998))

[2]For *ordinal efficiency* and its analysis, see Bogomolnaia and Moulin (2001), Abdulkadiroğlu and Sönmez (2003), and McLennan (2002).

[3]This implies that, when agents are equipped with von Neumann-Morgenstern preferences over random allocations (lotteries over assignments of houses), the RP solution is not *ex-ante efficient.* A related result is by Zhou (1990): there is no lottery mechanism that is *ex-ante efficient*, *anonymous*, and *strategy-proof.*

[4]See also Bogomolnaia and Moulin (2002) and Crés and Moulin (2001).

requires agents' eating speeds to be the same. The PS solution improves on the RP solution in terms of efficiency and fairness: it is *ordinally efficient* and *envy-free*. The weakness of the PS solution is that it does not satisfy *strategy-proofness*.

In assignment problems with private endowments and a social endowment, an indispensable property is *individual rationality*: each existing tenant finds his assignment at least as desirable as his endowment. When there is no social endowment, a well-known *individually rational* solution is the *top trading cycles (TTC) solution:* Each house points to its owner; each agent points to his top available house in his announced ranking. Since the number of agents is finite, there is at least one cycle. Each agent in each cycle is assigned the house to which he points. The agents in all cycles are removed with their assigned houses. The procedure is repeated until each agent receives a house. This solution is characterized by *individual rationality*, *efficiency*, and *strategy-proofness* (Ma (1994), Svensson (1999)). It is generalized to the *TTC solution from random orderings*, which reduces to the RP solution when there are no private endowments, and to the TTC solution when there is no social endowment (Abdulkadiroğlu and Sönmez (1999)).[5] The TTC solution from random orderings is *individually rational*, *ex-post efficient* and *strategy-proof*.[6] Also, the eating algorithm is generalized for the assignment problems with private endowments and a social endowment (Yılmaz (2006)). The induced solution, the *individually rational probabilistic serial (*$\text{PS}^{\mathbf{IR}}$*) solution* satisfies *individual rationality*, *ordinal efficiency* and a recently introduced fairness axiom, *no justified-envy*.[7] This last axiom is important: when there are private endowments, *individual rationality* and *no envy* (the central notion of fairness) are incompatible; by weakening *no envy* so that it is compatible with *individual rationality, no justified-envy* is the key in interpreting the fairness aspects of an assignment.

The solutions discussed so far are defined on the strict preference domain, which is fairly restrictive in many practical settings.[8] While most of the existing work assumes strict preferences, there are important exceptions. A deterministic solution on the weak preference domain is the *serially dictatorial solution* and it satisfies *efficiency*, *neutrality* and *strategy-proofness* (Svensson (1994)). Another deterministic solution, the *bi-polar serially dictatorial solution* generalizes the serially dictatorial solution and it is characterized by *strategy-*

[5]Also, Pápai (2000) introduced *hierarchical exchange rules*, which generalize the TTC solution and includes the TTC solution from random orderings as a special class.

[6]Recently, Sönmez and Ünver (2005) generalized the main result in Abdulkadiroğlu and Sönmez (1998): First, construct an endowment structure by assigning each existing tenant his own house and randomly assigning the vacant houses to new applicants with uniform distribution. The core based mechanism chooses the core allocation of the induced housing market. The core based mechanism is equivalent to an extreme case of the TTC mechanism where new applicants are randomly ordered first and existing tenants are randomly ordered next.

[7]This axiom is introduced by Yılmaz (2006); we define it formally in Section 3.3.

[8]See Bogomolnaia et al. (2005) for a discussion on both practical and technical reasons to study the weak preference domain.

proofness, non-bossiness, essential single-valuedness and *Pareto indifference* (Bogomolnaia et al. (2005)). These fixed priority solutions, however, do not satisfy even the weakest fairness axiom. Clearly, fairness considerations lead to randomization. To discuss the random solutions, we first consider a special case of the weak preferences, the case of dichotomous preferences, where each agent views the houses as either acceptable or unacceptable. A well-known result from graph theory, the Gallai (1963, 1964)-Edmonds (1965) Decomposition Lemma, characterizes the set of efficient assignments in the dichotomous domain. The *egalitarian solution* (Bogomolnaia and Moulin (2004)) finds an efficient random assignment that equalizes the agents' probabilities of being assigned to an acceptable house as much as possible in the sense of Lorenz Dominance. The other random solution is the *extended probabilistic serial* (EPS) *solution*, which extends the PS solution for the weak preference domain (Katta and Sethuraman (2006)). The EPS solution is equivalent to the egalitarian solution in the dichotomous domain. While these solutions extend the existing solutions to the assignment problem with a social endowment for the weak preference domain, there is no solution to the assignment problem with private endowments and a social endowment on the weak preference domain.

Our contribution is to construct a recursive solution to the assignment problem with private endowments and a social endowment, which is defined on the weak preference domain. The algorithm presented here is a natural extension of the algorithm due to Katta and Sethuraman (2006) for this more general problem; we exploit the graph-theoretical results, and the tools and techniques from *network flow theory* that they use. Our solution satisfies *individual rationality*, *ordinal efficiency* and *no justified-envy.* [9]

2 The model

A non-empty finite set of houses H has to be allocated to a non-empty finite set of agents I in such a way that each agent receives at most one house. Being unassigned to any of the houses in H is denoted by h_0.

An **endowment profile** is a function $\mu^0 : I \to H \cup \{h_0\}$ such that $\mu^0(i) = \mu^0(j)$ implies $\mu^0(i) = h_0$. Let $\mathcal{M}^0$ denote the set of all endowment profiles. Given an endowment profile $\mu^0 \in \mathcal{M}^0$, the sets $H_O \equiv \{\mu^0(i) : i \in I\} \backslash \{h_0\}$ and $H_V \equiv H \backslash H_O$ are the sets of **occupied** and **vacant houses**, respectively. Also, $I_E \equiv \{i \in I : \mu^0(i) \in H\}$ and $I_N \equiv \{i \in I : \mu^0(i) = h_0\}$ are the sets of **existing tenants** and **new applicants**, respectively. Each existing tenant i has the right of living in the house he occupies, $\mu^0(i) \in H_O$.

[9] Recently, Athanassoglou and Sethuraman (2007) generalized our solution for the fractional endowments case where agents are allowed to own fractions of different houses summing to an arbitrary quantity. Their solution satisfies *individual rationality*, *ordinal efficiency* and *no justified-envy.*

Each agent i has a transitive and complete (but not necessarily strict) preference relation R_i on H. We denote this domain of preferences by $\mathcal{D}$. Each agent prefers each house to h_0 and also, $|I| = |H|$.[10] Let $R = (R_i)_{i \in I}$ be a preference profile. Also, for each $S \subseteq I$, let $R_S = (R_i)_{i \in S}$.

An **assignment problem with private endowments and a social endowment**, or simply a **problem**, is a quadruple (I, H, μ^0, R). Since the sets I and H are fixed throughout the paper, we use (μ^0, R) instead of (I, H, μ^0, R) to denote a problem.

Given a problem (μ^0, R) and a house $h \in H$, let $U(R_i, h) \equiv \{h' \in H : h' \; R_i \; h\}$ be the **upper contour set of $\mathbf{R}_i$ at h**. We denote $U(R_i, \mu^0(i))$ by U_i. Let $U_S \equiv \bigcup\limits_{i \in S} U_i$. Also, let

$$Top(R_i, H') \equiv \{h \in H' : h \; R_i \; h' \quad \forall h' \in H'\} \text{ and } Top(R_S, H') \equiv \bigcup_{i \in S} Top(R_i, H').$$

A **deterministic assignment** is a bijection μ from I into H; it is represented as a permutation matrix, that is, a $|I| \times |H|$ matrix with entries 0 or 1, and exactly one nonzero entry per row and one per column. Let $\mathcal{M}$ denote the set of all deterministic assignments. We extend the preference of agent i to the set of deterministic assignments in the following natural way: **Agent i prefers μ to μ'** if and only if he prefers $\mu(i)$ to $\mu'(i)$. A deterministic assignment is **Pareto efficient** if no other deterministic assignment makes each agent at least as well off and at least one agent better off. It is **individually rational** if each existing tenant finds his assignment at least as desirable as his endowment.

A **random consumption** is a probability distribution over H. Let $\triangle H$ denote the set of all random consumptions. A **lottery** is a probability distribution over deterministic assignments. Let $\triangle\mathcal{M}$ denote the set of all lotteries. Each lottery induces a **random assignment** $Q = [q_{ih}]_{i \in I, h \in H}$, where $q_{ih} \in [0, 1]$ is the probability that agent i receives house h. Let Q_i denote the resulting random consumption for agent i. A random assignment is represented as a bistochastic matrix. Let $\mathcal{Q}$ denote the set of all random assignments. A **solution** is a function $\varphi : \mathcal{M}^0 \times \mathcal{D}^{|I|} \to \mathcal{Q}$.

Given $i \in I$, $R_i \in \mathcal{D}$, and a pair of random consumptions Q_i and T_i, **$\mathbf{Q}_i$ stochastically dominates $\mathbf{T}_i$** for agent i, written as **$\mathbf{Q}_i$ sd($\mathbf{R_i}$) $\mathbf{T}_i$**, if and only if

$$\forall h \in H, \qquad \sum_{h' \in U(R_i, h)} q_{ih'} \geq \sum_{h' \in U(R_i, h)} t_{ih'}.$$

Given a pair of distinct random assignments Q, and T, **Q stochastically dominates T**, if and only if, for each $i \in I$, Q_i stochastically dominates T_i. A random assignment is **ordinally efficient** if and only if it is not stochastically dominated by any other random

[10] These assumptions are without loss of generality.

assignment.

A solution is **individually rational**, if for each agent, the support of his random consumption is contained in the upper contour set of his preferences at his endowment. A solution is **ordinally efficient** if it always selects *ordinally efficient* random assignments.

A solution is **strategy-proof** if truth-telling is a dominant strategy in its associated preference revelation game. A solution $\varphi : \mathcal{M}^0 \times \mathcal{D}^{|I|} \rightarrow \mathcal{Q}$ is **weakly strategy-proof** if, for each $R \in \mathcal{D}^{|I|}$, each $i \in I$, and each $R_i^* \in \mathcal{D}$,

$$\varphi_i(R_i^*, R_{-i})\ sd(R_i)\ \varphi_i(R) \Rightarrow \varphi_i(R_i^*, R_{-i}) = \varphi_i(R).$$

3 A new solution on the weak preference domain

The PS solution is defined only for the strict preferences. It is extended to the weak preference domain by using tools and techniques from network flow theory (Katta and Sethuraman (2006)): A parametric network (we define it in the next sub-section) is constructed, in which each agent points to his best available houses. However, when there are private endowments, the associated solution is not *individually rational.* By using network flow theory, we introduce a new solution, which satisfies *individual rationality.*

3.1 Flows and cuts: Preliminaries

A **directed graph,** or **digraph** is a pair $G = (N, A)$, consisting of a set of **nodes** N and a set of *ordered* pairs of nodes, A, called **arcs.** A digraph $G = (N, A)$ is **bipartite** if N can be partitioned into two disjoint non-empty sets, N_1, and $\overline{N}_1$, such that each arc is directed out of N_1 into $\overline{N}_1$. Let $G = (N, A)$ be a digraph with two distinguished points **s** the **source** and **t**, the **sink**. Let $k : A \rightarrow \Re_+$ be a function, which associates each arc $a = (x, y)$ of G with a non-negative real number $k(x, y)$ called the **capacity** of the arc. A **network** is defined by a quadruple (G, k, s, t).

A **flow in** (G, k, s, t) is a function $f : A \rightarrow \Re_+$, satisfying the following properties:

$$\begin{array}{ll} (i) & \sum\limits_{x} f(x,y) = \sum\limits_{z} f(y,z) \text{ for each } y \text{ in } N \setminus \{s,t\} \text{ and,} \\ (ii) & f(x,y) \leq k(x,y) \text{ for each } (x,y) \text{ in } A. \end{array}$$

The **value of** f, denoted by $v(f)$ is defined by $\sum\limits_{x} f(s,x) - \sum\limits_{y} f(y,s)$.

Let $G = (N, A)$ be a digraph and (G, k, s, t) be a network. Given a set of nodes $N_1 \subseteq N$, let $A(N_1)$ denote the set of all arcs (x, y) such that $x \in N_1$ and $y \in N \setminus N_1$. A **separator** in G is a set of nodes $N_1 \subseteq N \setminus \{s, t\}$. The $\mathbf{s - t}$ **cut determined by the separator** $\mathbf{N'}$ is the set of arcs $A(N' \cup \{s\})$. The capacity of an $s-t$ cut determined by N' is the sum of the capacities of all the arcs in $A(N' \cup \{s\})$; it is denoted by $k(N')$.

The maximum value of any flow cannot be more than the capacity of any $s-t$ cut; thanks to the Max-Flow Min-Cut Theorem (Ford and Fulkerson (1956)), we have a better understanding of the maximum value of the flows.

Theorem 1 *(Max-Flow Min-Cut Theorem) Let (G, k, s, t) be a network. The maximum value of any flow equals the minimum capacity of any $s-t$ cut.*

The straightforward implication of this result is that, to show that a flow is optimal, it is enough to find an $s-t$ cut in the network with the same capacity as the value of the flow.

3.2 The generalized probabilistic serial (GPS) solution: algorithm and its analysis

We introduce a new solution, the *generalized probabilistic serial (GPS) solution.* The GPS solution is induced by our algorithm which is based on a recursive application of the *parametric maximum flow algorithm*; at each step, the algorithm assigns the agents equal amounts of their favorite available houses, as long as there is an assignment of the remainders of the houses such that the resulting assignment is *individually rational.* At some point, it could be that if each agent is assigned a positive amount, the remainders of the houses are not enough to guarantee the *individual rationality* (IR) for at least one of the existing tenants. At this point, IR dictates that the remainders of some houses should be assigned to a set of existing tenants. These houses and agents constitute a bottleneck set, and they induce a sub-problem, in which the same algorithm so far is applied recursively. Before we define the GPS solution, we illustrate how to apply the parametric maximum flow algorithm to capture the IR constraints. Let $(\mu^0, (R_i)_{i \in I})$ be a problem.

Step 1: Constructing the $\lambda-$parametric network
Let $\widetilde{i}$ be a **replica of existing tenant i**. Given $I' \subseteq I_E$, let $\widetilde{I}' = \{\widetilde{i} : i \in I'\}$. First, we construct a bipartite digraph $G = (I \cup \widetilde{I}_E \cup H, A)$, where arcs are directed out of $I \cup \widetilde{I}_E$ into H. Each agent i points to the houses in $Top(R_i, H)$ and each replica tenant $\widetilde{i}$ points to the houses in U_i. Second, we augment the graph G by adding a source node s, and a sink node t, with arcs directed out of node s into each node in $I \cup \widetilde{I}_E$, denoted by $A_{s \to I \cup \widetilde{I}_E}$, and out of each house in H into node t, denoted by $A_{H \to t}$.

Next, we define the capacity function, k. Let the capacity of each arc in $A_{s \to I}$ and $A_{s \to \widetilde{I}_E}$ be equal to $1 \geq \lambda \geq 0$ and $1 - \lambda$, respectively. Let the capacity of each arc in $A_{H \to t}$ be equal to 1, and each arc in A be equal to infinity.

We have now constructed the network $((I \cup \widetilde{I}_E \cup H \cup \{s, t\}, A \cup A_{s \to I \cup \widetilde{I}_E} \cup A_{H \to t}), k, s, t)$.

Example 1 *Constructing the $\lambda-$parametric network.* Consider the following preferences: (A boxed house stands for the private endowment.)

$\underline{R_1}$	$\underline{R_2}$	$\underline{R_3}$	$\underline{R_4}$
$\{h_1, h_2\}$	h_3	$\{h_1, h_4\}$	$\{h_3, h_4\}$
$\boxed{h_3}$	$\{h_2, \boxed{h_4}\}$	h_2	$\{h_1, h_2\}$
		h_3	

Agents 1 and 2 are the existing tenants. Thus, the set of replica tenants is $\{\widetilde{1}, \widetilde{2}\}$ and the associated $\lambda-$parametric network is shown in Figure 1.

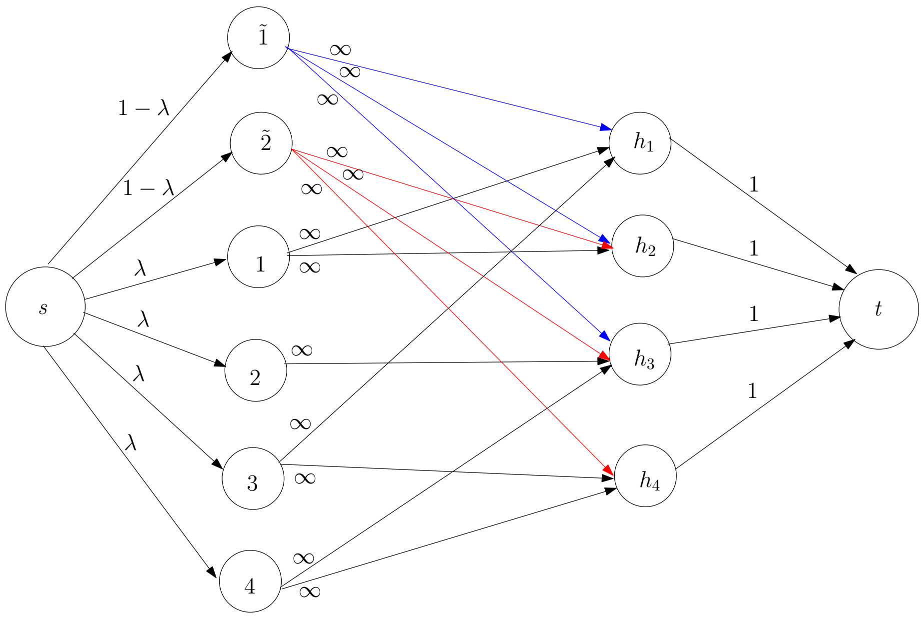

Figure 1: Constructing the $\lambda-$parametric network

Step 2: Solving for the break point λ^*

The parameter λ in the network constructed in Step 1 represents the probability that agents receive one their best available houses, and the arcs from node s to the nodes of the replica tenants are due to *individual rationality.* Since fairness considerations imply that the parameter λ should have the same value for each agent (up to the *individual rationality* constraint), the flow in a $\lambda-$network should be equal to the capacity of the $s-t$ cut determined by the empty set. Thus, at some critical value λ^* of the λ parameter, called the *break point*, it is impossible to increase λ without violating fairness, and this critical level is due to a bottleneck set of nodes. Next, we analyze how the bottleneck sets are determined.

We study the minimum capacity $s-t$ cuts as λ varies. Let $\lambda \geq 0$. Each cut is determined by a separator of the form $I_1 \cup \widetilde{I_2} \cup H'$ where $I_1 \subset I$, $I_2 \subset I_E$, and $H' \subset H$. Let $I_1 \cup \widetilde{I_2} \cup H'$ be a separator that determines a minimum capacity $s-t$ cut. Then, since the capacity of each arc in A is infinity,

$$H' \supseteq U_{I_2} \bigcup Top(R_{I_1}, H).$$

Suppose the set $H' \setminus (U_{I_2} \bigcup Top(R_{I_1}, H))$ is non-empty. Let $h \in H' \setminus (U_{I_2} \bigcup Top(R_{I_1}, H))$. Since there is no arc directed from $I_1 \cup \widetilde{I_2}$ to node h, the capacity of the $s-t$ cut determined by the separator $I_1 \cup \widetilde{I_2} \cup (H' \setminus \{h\})$ is $k(I_1 \cup \widetilde{I_2} \cup H') - 1$. This contradicts that the separator $I_1 \cup \widetilde{I_2} \cup H'$ determines a minimum capacity $s-t$ cut. Thus,

$$H' = U_{I_2} \bigcup Top(R_{I_1}, H).$$

Suppose $\widetilde{I_2}$ is non-empty and there is an agent i in $I_2 \setminus I_1$. Since

$$Top(R_i, H) \subseteq U_i \subseteq U_{I_2} \bigcup Top(R_{I_1}, H),$$

the capacity of the $s-t$ cut determined by the separator $I_1 \cup \{i\} \cup \widetilde{I_2} \cup H'$ is $k(I_1 \cup \widetilde{I_2} \cup H') - \lambda$. Thus, the separator $I_1 \cup \widetilde{I_2} \cup H'$ cannot determine a minimum capacity $s-t$ cut. Similarly, if, for some $j \in I$, $Top(R_j, H) \subseteq U_{I_2}$, then $j \in I_1$.

The capacity of the $s-t$ cut determined by a separator $S = I_1 \cup \widetilde{I_2} \cup (U_{I_2} \bigcup Top(R_{I_1}, H))$ is

$$k(S)(\lambda) = \lambda \left(|I_N| + \left|\widetilde{I_2}\right| - |I_1| \right) + \left(\left|U_{I_2} \bigcup Top(R_{I_1}, H)\right| - \left|\widetilde{I_2}\right| \right) + |I_E|.$$

Also, the capacity of the $s-t$ cut determined by the separator $\emptyset$ is

$$k(\emptyset)(\lambda) = \lambda |I_N| + |I_E|.$$

Note that, since each house is occupied by at most one agent, $|U_{I_2}| \geq \left|\widetilde{I_2}\right|$. Thus, $k(S)(0) \geq k(\emptyset)(0)$. This holds with equality only if $|U_{I_2}| = \left|\widetilde{I_2}\right|$. In this case, *individual rationality* implies that the houses in U_{I_2} have to be allocated *only* to the agents in I_2.

Suppose, for each nonempty separator S, $k(S)(0) > k(\emptyset)(0)$. Thus,

$$\left|U_{I_2} \bigcup Top(R_{I_1}, H)\right| - \left|\widetilde{I_2}\right| > 0.$$

For λ close enough to zero, the empty set is the unique separator that determines the minimum capacity $s-t$ cut. Both $k(S)$ and $k(\emptyset)$ are linear functions of λ, and since $I_1 \supseteq I_2$, $k(\emptyset)(\lambda)$

is at least as steep as $k(S)(\lambda)$.

Let $S = I_1 \cup \widetilde{I_2} \cup (U_{I_2} \bigcup Top(R_{I_1}, H)) \subsetneq I \cup \widetilde{I}_E \cup H$ be a nonempty separator such that $k(S)$ crosses $k(\emptyset)$ first:

$$S \in Arg \underset{S \subsetneq I \cup \widetilde{I}_E \cup H}{Min} \{\lambda : k(S)(\lambda) = k(\emptyset)(\lambda)\}.$$

We call S a **bottleneck set**.[11] To see why S is a *bottleneck set,* let λ^* be such that $k(S)(\lambda^*) = k(\emptyset)(\lambda^*)$. We call λ^* the **break point**. For $\lambda < \lambda^*$, the empty set uniquely determines the minimum capacity $s-t$ cut. At λ^*, both S and the empty set determine a minimum capacity $s-t$ cut. Then, the maximum flow in λ^*- network is $\lambda^* |I_N| + |I_E|$. Since it is equal to the sum of the capacities of the arcs in $A_{s \to I \cup \widetilde{I}_E}$, the flow of each arc in $A_{s \to I}$ is λ^* and of each arc in $A_{s \to \widetilde{I}_E}$ is $1 - \lambda^*$. There exists $\varepsilon > 0$ such that in the $(\lambda^* + \varepsilon)-$ network, $k(S)(\lambda^*+\varepsilon) < k(\emptyset)(\lambda^*+\varepsilon)$. Thus, at $\lambda^*+\varepsilon$, the maximum flow is less than $(\lambda^*+\varepsilon)|I_N|+|I_E|$. Then, in any maximum flow f, the flow of at least one of the arcs in $A_{s \to I \cup \widetilde{I}_E}$ is less than its capacity: either $f(s, \widetilde{i}) < 1 - \lambda^* - \varepsilon$ for some $i \in I_E$, or $f(s, j) < \lambda^* + \varepsilon$ for some $j \in I$. Thus, either *individual rationality* is violated or there are at least two agents who are assigned different amounts of their favorite houses. Thus, S identifies the bottleneck set of agents and houses. *At the break point λ^*, the set S is a bottleneck if and only if, at each maximum flow f,*

$$\sum_{i \in I_1 \cup \widetilde{I}_2} \sum_{h \in U_{I_2} \bigcup Top(R_{I_1}, H)} f(i,h) = \left| U_{I_2} \bigcup Top(R_{I_1}, H) \right|.$$

Case 1: $\widetilde{I}_2 = \emptyset$

The maximum flow is $(1 - \lambda^*) |I_E| + \lambda^* (|I| - |I_1|) + |Top(R_{I_1}, H)|$; *individual rationality* implies that the houses in $Top(R_{I_1}, H)$ are allocated *only* to the agents in I_1. To see this, suppose that in a maximum flow, there is $i \in \left(I \cup \widetilde{I}_E\right) \setminus I_1$ and $h \in Top(R_{I_1}, H)$ such that $f(i,h) > 0$. But then, the total flow is no more than $(1 - \lambda^*) |I_E| + \lambda^*(|I| - |I_1|) - f(i,h) + |Top(R_{I_1}, H)|$, which is less than the maximum flow. Thus, each agent in I_1 is assigned a total of λ^* units from his favorite houses and the houses in $Top(R_{I_1}, H)$ are allocated. The other agents are *pledged* λ^* units.

Case 2: $\widetilde{I}_2 \neq \emptyset$

The houses in $U_{I_2} \bigcup Top(R_{I_1}, H)$ are allocated *only* to the agents in I_1 and the replica tenants in $\widetilde{I}_2$. Each agent $i \in I_2$ is *pledged* a total of λ^* units of his favorite houses, and his replica $\widetilde{i}$ is *pledged* $1-\lambda^*$ units of the houses in U_i. But, at this point, the assignments of these agents are not determined; they are obtained recursively by solving the same parametric network flow

[11] The separator $I \cup \widetilde{I_E} \cup H$ determines an $s-t$ cut with capacity $|H|$. For $\lambda \in [0,1]$, $k(\emptyset)(\lambda) \leq |H| = k(I \cup \widetilde{I_E} \cup H)(\lambda)$. Thus, we can exclude the separator $I \cup \widetilde{I_E} \cup H$. If $I_N \neq \emptyset$, then it holds with equality for $\lambda = 1$. If $I_N = \emptyset$, then it holds with equality for $\lambda \in [0,1]$.

problem with the agents in I_1, the replica tenants in $\widetilde{I_2}$ and the houses in $U_{I_2} \bigcup Top(R_{I_1}, H)$.

The GPS solution. Each agent i is connected to his favorite available houses, and each replica tenant $\widetilde{j}$ is connected to the available houses in U_j. The capacity of each of these arcs is infinity. Each available house is connected to the sink node through an arc of unit capacity. The capacity of each arc from node s to an agent will be specified at each step during the algorithm.

Step 1: *Initialization.*
Let $m = 0$, $I^0 = I$, $I^0_E = I_E$, $H^0 = H$. For each $i \in I$, let $k^0(s, i) = 0$. For each $i \in I^0_E$, let $k^0(s, \widetilde{i}) = 1$.

Step 2: *Network construction.*
Construct the network as follows:

(i) each agent $i \in I^m$ points to the houses in $Top(R_i, H^m)$, and each replica tenant $\widetilde{i} \in \widetilde{I^m_E}$ points to the houses in $U_i \cap H^m$,

(ii) each agent $i \in I^m$ is connected to node s through an arc with capacity $k^m(s, i) + \lambda$,

(iii) each replica tenant $\widetilde{i} \in \widetilde{I^m_E}$ is connected to node s through an arc with capacity $k^m(s, \widetilde{i}) - k^m(s, i) - \lambda$,

(iv) each house in H^m is connected to node t through an arc with capacity 1.

Step 3: *Identifying the bottleneck.*
Solve the corresponding parametric max-flow problem. Let λ^*_m be the break point.

- If there is a bottleneck set S such that $S \cap \widetilde{I_E} = \emptyset$, then,

  (i) let $I' = \bigcup\{S \cap I : S$ is a bottleneck set and $S \cap \widetilde{I_E} = \emptyset\}$,

  (ii) for an agent $i \in I'$, let $k^{m+1}(s, i) = 0$ and assign him an amount $k^m(s, i) + \lambda^*_m$ from the set $Top(R_i, H^m)$; if $i \in I_E$, let $k^{m+1}(s, \widetilde{i}) = k^m(s, \widetilde{i}) - k^m(s, i) - \lambda^*_m$,

  (iii) for an agent $i \in I^m \setminus I'$, let $k^{m+1}(s, i) = k^m(s, i) + \lambda^*_m$; if $i \in I_E$, let $k^{m+1}(s, \widetilde{i}) = k^m(s, \widetilde{i})$,

  (iv) let $H^{m+1} = H^m \setminus Top(R_{I'}, H^m)$,

  (v) if $H^{m+1} \neq \emptyset$, then increase m by 1 and go to step 2. Otherwise, terminate the algorithm.

- Let $S = I_1 \cup \widetilde{I_2} \cup H' \subsetneq I^m \cup \widetilde{I^m_E} \cup H^m$ be a bottleneck set such that $\widetilde{I_2} \neq \emptyset$. Then,

  (i) for an agent $i \in I^m \setminus I_1$, let $k^{m+1}(s, i) = k^m(s, i) + \lambda^*_m$; if $i \in I_E$, let $k^{m+1}(s, \widetilde{i}) = k^m(s, \widetilde{i})$,

  (ii) for an agent $i \in I_1 \setminus I_2$, $k^{m+1}(s, i) = 0$; if $i \in I_E$, let $k^{m+1}(s, \widetilde{i}) = k^m(s, \widetilde{i}) - k^m(s, i) - \lambda^*_m$,

  (iii) allocate the houses in H' to the agents in I_1 via the Sub-algorithm S.

Sub-algorithm S : Let $I^0 = I_1$, $I_E^0 = I_2$, $H^0 = H'$. For each $i \in I^0$, let $k^0(s,i) = k^m(s,i) + \lambda_m^*$, and for each $i \in I_E^0$, let $k^0(s,\widetilde{i}) = k^m(s,\widetilde{i})$. The algorithm is applied from step 2 onwards, except that while each agent $i \in I_E^0$ is connected to node s through an arc with capacity $k^0(s,i) + \lambda$, each agent $i \in I^0 \setminus I_E^0$ is connected to node s through an arc with capacity $k^0(s,i)$; this capacity is fixed and is not parametric throughout the sub-algorithm.[12]

(iv) Remove the houses and agents as follows:

$$I^{m+1} = I^m \setminus I_2,$$

$$I_E^{m+1} = I_E^m \setminus I_2,$$
$$H^{m+1} = H^m \setminus H'.$$

(v) if $H^{m+1} \neq \emptyset$, then increase m by 1 and go to step 2. Otherwise, terminate the algorithm.

3.3 The properties of the GPS solution

In each step of the GPS solution, each existing tenant i and his replica $\widetilde{i}$ never point to a house that is worse than $\mu^0(i)$ for agent i. Also, if an existing tenant belongs to a bottleneck set whereas his replica does not, and the bottleneck set involves a nonempty set of replica tenants, then the capacity of the arc from node s to this existing tenant is not parametric in the sub-algorithm of this bottleneck. Moreover, throughout the algorithm, the total flow that goes through the node of existing tenant i and his replica $\widetilde{i}$ is equal to 1. Also, in each sub-algorithm corresponding to a bottleneck set involving replica tenant $\widetilde{i}$, there are two cases: either the break point is such the capacity (and the flow) of the arc connecting node s to replica tenant $\widetilde{i}$ is zero, or there is a bottleneck set. Applying the same argument to this bottleneck set, we conclude that the total amount assigned to replica tenant $\widetilde{i}$ is zero. Thus, the GPS solution is *individually rational.*

In the algorithm describing the GPS solution, each agent points to his favorite available houses. This is the intuition for *ordinal efficiency* of the GPS solution.

Proposition 1 *The GPS solution is ordinally efficient.*

In assignment problems with private endowments and a social endowment, *individual rationality* and *no envy*, the central concept of fairness, are incompatible. Recently, a new fairness notion, *no justified-envy,* is introduced (Yılmaz (2006)); it is weaker than *no envy*

[12]This is due to *individual rationality.*

and compatible with *individual rationality*. It views an assignment as unfair if an agent does not prefer his consumption to another agent's consumption and the assignment obtained by swapping their consumptions respects the *individual rationality* requirement of the latter agent.

Definition 1 *i) A solution $\varphi : \mathcal{M}^0 \times \mathcal{D}^{|I|} \to \mathcal{Q}$ satisfies* ***no justified-envy*** *if, for each $\mu^0 \in \mathcal{M}^0$, each $R = (R_i)_{i \in I} \in \mathcal{D}^{|I|}$, and each $i, j \in I$: either*

$$\varphi_i(\mu^0, R)\ sd(R_i)\ \varphi_j(\mu^0, R) \quad \text{or } Support(\varphi_i(\mu^0, R)) \setminus U_j \neq \emptyset.$$

ii) A solution $\varphi : \mathcal{M}^0 \times \mathcal{D}^{|I|} \to \mathcal{Q}$ satisfies ***weak no justified-envy*** *if, for each $\mu^0 \in \mathcal{M}^0$, each $R = (R_i)_{i \in I} \in \mathcal{D}^{|I|}$, and each $i, j \in I$: either*

$$\varphi_j(\mu^0, R)\ sd(R_i)\ \varphi_i(\mu^0, R) \Rightarrow \varphi_i(\mu^0, R) = \varphi_j(\mu^0, R) \text{ or } Support(\varphi_i(\mu^0, R)) \setminus U_j \neq \emptyset.$$

If a bottleneck set involves replica tenant $\widetilde{i}$, then, at the given break point, each maximum flow is such that the total flow that goes through existing tenant i and his replica $\widetilde{i}$ is equal to 1. If there is a flow going through an agent, who is not in the bottleneck set and points to a house in the bottleneck set, then *individual rationality* is violated. Thus, an agent is assigned units from his favorite available houses, but only until this critical point is reached. This is the reason for the fairness of the GPS solution.

Proposition 2 *The GPS solution satisfies no justified-envy.*

The $\text{PS}^{\mathbf{IR}}$ solution does not satisfy *weak strategy-proofness* (Yılmaz (2006)). Since the GPS solution reduces to the $\text{PS}^{\mathbf{IR}}$ solution on the strict preference domain, the GPS solution does not satisfy *weak strategy-proofness* neither. Also, *individual rationality*, *no justified-envy*, and *strategy-proofness* are incompatible (Yılmaz (2006)).

4 Conclusion

We have constructed a recursive solution to the assignment problems with private endowments and a social endowment, the *generalized probabilistic serial (GPS) solution*; it is defined on the weak preference domain. The GPS solution satisfies *individual rationality*, *ordinal efficiency* and *no justified-envy*. The GPS solution extends the *probabilistic serial (PS) solution* in two dimensions: it is defined under weak preferences and it accounts for *individual rationality* requirement. It reduces to the $\text{PS}^{\mathbf{IR}}$ solution on the strict preference domain, and to the EPS solution when there are no private endowments.

5 Appendix

5.1 The GPS solution: An example

Example 2 Consider the following preferences:

R_1	R_2	R_3	R_4	R_5	R_6
h_1	h_1	h_2	$\{h_2, h_3\}$	h_3	h_3
$\boxed{h_2}$	$\{h_4, \boxed{h_5}\}$	$\{\boxed{h_3}, h_5\}$	h_4	h_5	$\{h_4, h_5, h_6\}$
			h_1	$\{h_4, h_6\}$	h_1
			h_5	$\{h_1, h_2\}$	h_2
			h_6		

Step 1: We construct the network shown in Figure 2.

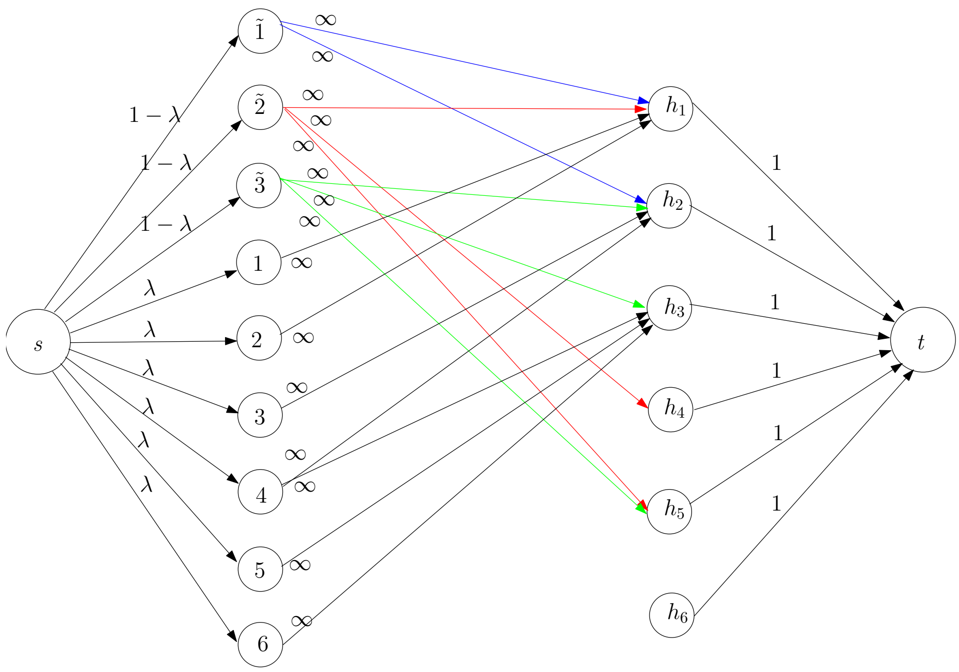

Figure 2: Step 1

At $\lambda_1 = \frac{2}{5}$, the separator $\{\widetilde{1}, 1, 2, 3, 4, 5, 6\} \cup \{h_1, h_2, h_3\}$ determines an $s-t$ cut with capacity $\frac{21}{5}$. Since there is a flow of value $\frac{21}{5}$, it is a bottleneck set:

$$f(1, h_1) = f(2, h_1) = f(3, h_2) = f(5, h_3) = f(6, h_3) = \frac{2}{5},$$

$$f(4,h_2)=f(4,h_3)=\frac{1}{5};\ f(\widetilde{1},h_1)=\frac{1}{5};\ f(\widetilde{1},h_2)=\frac{2}{5};\ f(\widetilde{2},h_4)=f(\widetilde{3},h_5)=\frac{3}{5}.$$

We proceed with the sub-algorithm to allocate houses $\{h_1,h_2,h_3\}$ to agents $\{1,2,3,4,5,6\}$.

Step 2: *Sub-algorithm* $\{\widetilde{1},1,2,3,4,5,6\}\cup\{h_1,h_2,h_3\}$. The network is shown in Figure 3. At $\lambda_2=\frac{1}{5}$, the separator $\{1,2\}\cup\{h_1\}$ determines an $s-t$ cut with capacity 3. Since there is a flow of value 3,

$$f(1,h_1)=\frac{3}{5};\ f(2,h_1)=f(3,h_2)=f(5,h_3)=f(6,h_3)=\frac{2}{5},$$

$$f(4,h_2)=f(4,h_3)=\frac{1}{5};\ f(\widetilde{1},h_1)=0;\ f(\widetilde{1},h_2)=\frac{2}{5},$$

the set $\{1,2\}\cup\{h_1\}$ is bottleneck set, and house h_1 is allocated to agents 1 and 2; agent 1 is assigned $\frac{3}{5}$ units, and agent 2 is assigned $\frac{2}{5}$ units. Since the capacity of the arc $(s,2)$ is fixed at $\frac{2}{5}$, agent 2 is assigned $\frac{2}{5}$ units and leaves.

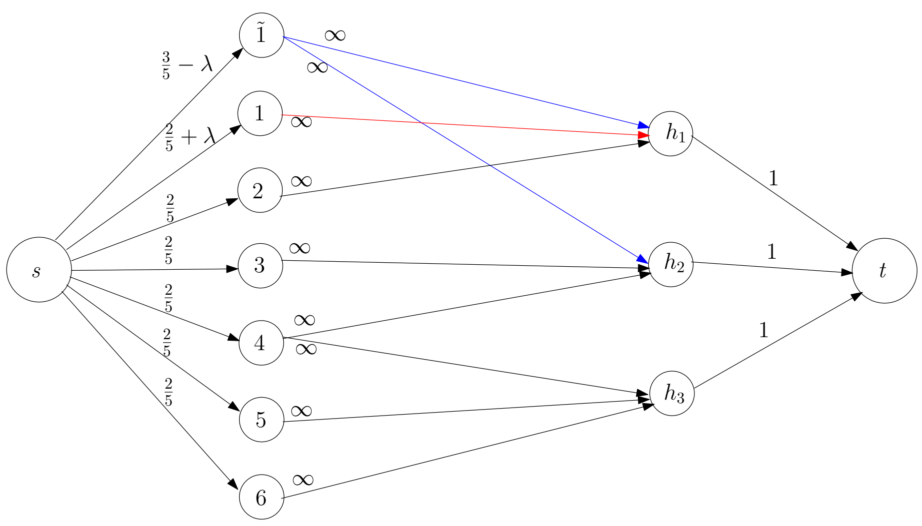

Figure 3: Step 2

Step 3: *Sub-algorithm* $\{\widetilde{1},1,2,3,4,5,6\}\cup\{h_1,h_2,h_3\}$. The network is shown in Figure 4. The break point is $\lambda_3=\frac{2}{5}$. Agent 1 leaves and houses h_1, h_2, h_3 are allocated; the flows are follows:

$$f(1,h_2)=f(3,h_2)=f(5,h_3)=f(6,h_3)=\frac{2}{5};\ f(4,h_2)=f(4,h_3)=\frac{1}{5};\ f(\widetilde{1},h_2)=0.$$

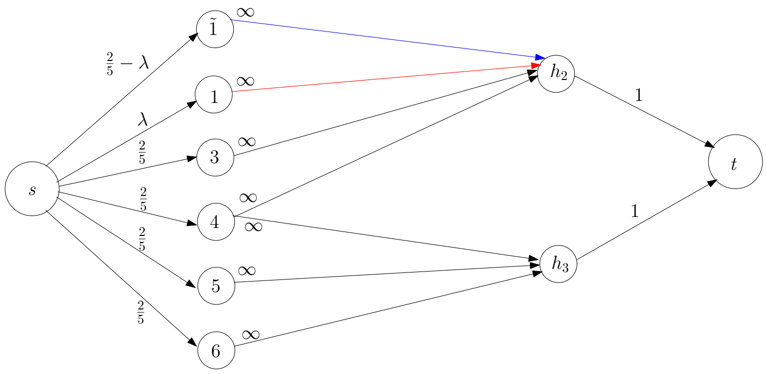

Figure 4: Step 3

Step 4: We construct the network in Figure 5.

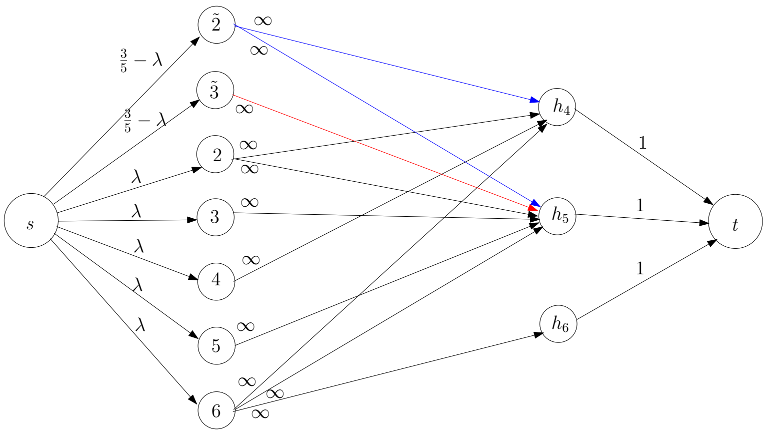

Figure 5: Step 4

At $\lambda_2 = \frac{2}{5}$, the separator $\{\widetilde{2}, \widetilde{3}, 2, 3, 4, 5\} \cup \{h_4, h_5\}$ determines an $s-t$ cut with capacity $\frac{12}{5}$. The set $\{\widetilde{2}, \widetilde{3}, 2, 3, 4, 5\}$ is a bottleneck, since there is a flow of value $\frac{12}{5}$:

$$f(2, h_4) = f(3, h_5) = f(4, h_4) = f(5, h_5) = f(6, h_6) = \frac{2}{5}; \text{ and } f(\widetilde{2}, h_4) = f(\widetilde{3}, h_5) = \frac{1}{5}.$$

Step 5: *Sub-algorithm* $\{\widetilde{2}, \widetilde{3}, 2, 3, 4, 5\} \cup \{h_4, h_5\}$. We construct the network in Figure 6.

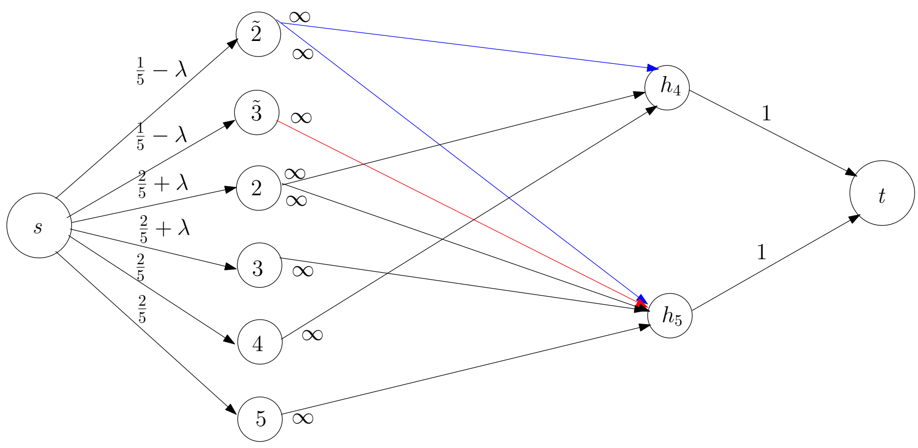

Figure 6: Step 5

At $\lambda_3 = \frac{1}{5}$, the empty set determines an $s - t$ cut with capacity 2; the houses h_4 and h_5 are allocated, and $f(2, h_4) = f(3, h_5) = \frac{3}{5}$; $f(4, h_4) = f(5, h_5) = \frac{2}{5}$.

Step 6: Agents 2 and 3 leave. The final network is shown in Figure 7.

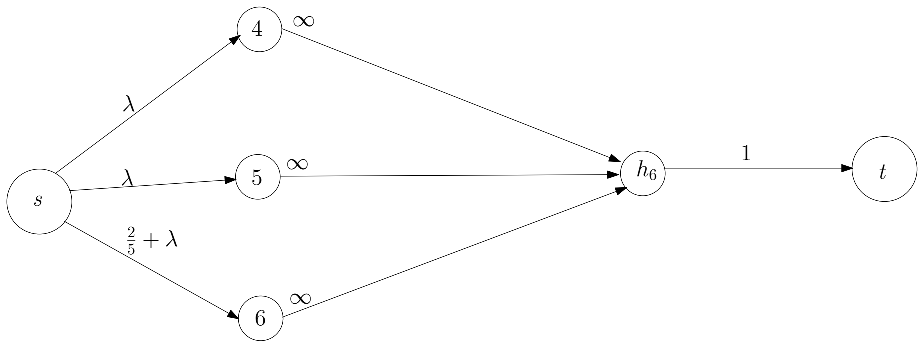

Figure 7: Step 6

The break point is $\lambda_4 = \frac{1}{5}$. The random assignment is as follows:

$$Q = \begin{array}{ccccccc} & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 \\ 1 & \frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 & 0 \\ 2 & \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 & 0 \\ 3 & 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} & 0 \\ 4 & 0 & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 & \frac{1}{5} \\ 5 & 0 & 0 & \frac{2}{5} & 0 & \frac{2}{5} & \frac{1}{5} \\ 6 & 0 & 0 & \frac{2}{5} & 0 & 0 & \frac{3}{5} \end{array}$$

5.2 Proofs

PROOF OF PROPOSITION 1

We use the following characterization result:

Lemma 1 *(Katta and Sethuraman (2006)) Given a preference profile $R \in \mathcal{D}$ and a random assignment Q, define a binary relation in H as follows:*

$$\text{for each } h, h' \in H : h\ \tau(Q,R)\ h' \Leftrightarrow \{\text{there is } i \in I : h\ R_i\ h' \text{ and } q_{ih'} > 0\}.$$

*The binary relation is **strict** if in the above expression $h\ P_i\ h'$. The relation $\tau(Q,R)$ is **strictly cyclic** if it is cyclic and at least one of the binary relations in the cycle is strict.*

The random assignment $Q \in \mathcal{Q}$ is ordinally efficient at profile R if and only if the relation $\tau(Q,R)$ is not strictly cyclic.

Let (μ^0, R) be a problem. Suppose GPS(R, μ^0) is not *ordinally efficient*. Then, by Lemma 1, there is a set of houses $\{h_1, h_2, ..., h_K\}$ and a function $\beta : \{h_1, h_2, ..., h_K\} \to I$ such that

$$h_1\ R_{\beta(h_1)}\ h_2\ R_{\beta(h_2)}\ h_3\ R_{\beta(h_3)}\ ...\ h_K\ \ R_{\beta(h_K)}\ h_1$$

with at least one of the binary relations in the cycle is strict, and for $i = 1, ..., K-1$,

$$GPS_{\beta(h_i),i+1}(R, \mu^0) > 0$$

and

$$GPS_{\beta(h_K),1}(R, \mu^0) > 0.$$

Without loss of generality, assume that $k^*\ th$ binary relation is strict for some $k^* \in \{1, ..., K\}$: $h_{k^*}\ P_{\beta(h_{k^*})}\ h_{k^*+1}$. For each $k \leq K$, let s^k be the step of the algorithm that describes the GPS solution, during which h_k is in the bottleneck set. Since for agent $\beta(h_k)$, h_k is at least as good as h_{k+1}, agent $\beta(h_k)$ is assigned a positive amount of h_{k+1}, h_{k+1} could not have been in the bottleneck set before h_k. Thus, $s^k \leq s^{k+1}$. Thus, $s^1 \leq s^2 \leq \ldots \leq s^K \leq s^1$. Thus, each house in $\{h_1, \ldots, h_K\}$ is allocated in the same step. Let S be a bottleneck set such that $\{h_1, \ldots, h_K\} \subseteq S$. If $S \cap \widetilde{I_E} = \emptyset$, then, since house h_{k^*} is available and agent $\beta(h_{k^*})$ does not point to house h_{k^*+1}, he is not assigned any parts of house h_{k^*+1}. But it contradicts $GPS_{\beta(h_k^*),k^*+1}(R, \mu^0) > 0$. Suppose $S = I_1 \cup \widetilde{I_2} \cup H_1$, where $I_2 \neq \emptyset$. If there is an agent $\beta(h_k) \notin I_1$, then, since, by definition of the algorithm, the houses in H_1 are allocated to the agents in I_1, he is not assigned any parts of house h_{k+1}. But it contradicts $GPS_{\beta(h_k),k+1}(R, \mu^0) > 0$. Thus, $\{\beta(h_1), \beta(h_2), \ldots, \beta(h_K)\} \subseteq I_1$.

At the beginning of the sub-algorithm S, each house in H_1 is available. The same argument above applies here: Each house in $\{h_1, \ldots, h_K\}$ is allocated in the same step. Let $S' = I'_1 \cup \widetilde{I'_2} \cup H'_1$ be a bottleneck such that $\{h_1, \ldots, h_K\} \subseteq H'_1$, and $I'_2 \neq \emptyset$. Then, in the sub-sub-algorithm S', each house in $\{h_1, \ldots, h_K\}$ is allocated in the same step. Let S'' be a bottleneck. ... But this contradicts the finiteness of the problem. Thus, the GPS solution is *ordinally efficient.*

PROOF OF PROPOSITION 2

Let (μ^0, R) be a problem. Let $i, j \in I$. Let $k(i)$ be the step at which agent i leaves. Suppose $k(j) < k(i)$. Since the GPS solution allocates each house in U_j in or before step $k(j)$, and agent i is assigned positive amounts after $k(j)$, the set $Support(\text{GPS}_i)\backslash U_j$ is nonempty. Thus, agent i does not envy agent j. Suppose $k(j) > k(i)$. Let $h \in U_i$. Let k be the first step in which each house in $U(R_i, h)$ is allocated. In the first k steps, agent i is assigned houses only from the set $U(R_i, h)$. Thus, the amount of the houses in $U(R_i, h)$, which are allocated to agent i, is $\sum_{h' \in U(R_i,h)} q_{i,h'} = \lambda_1^* + \lambda_2^* + \ldots + \lambda_k^* + \Delta$. The last term is positive only if $k = k(i)$. Since agent j could be assigned some houses outside the set $U(R_i, h)$, $\sum_{h' \in U(R_i,h)} q_{j,h'} \leq \lambda_1^* + \lambda_k^* + \ldots + \lambda_k^*$. Thus, $\sum_{h' \in U(R_i,h)} q_{i,h'} \geq \sum_{h' \in U(R_i,h)} q_{j,h'}$. Thus, GPS_i stochastically dominates GPS_j at R_i. Thus, agent i does not envy agent j. Suppose $k(j) = k(i)$. Let $S = I_1 \cup \widetilde{I_2} \cup H_1$ be the bottleneck at step $k(i)$. Note that either $I_2 = \emptyset$ or $i, j \in I_2$. If $I_2 = \emptyset$, the same argument in the case $k(j) > k(i)$ applies. If $i, j \in I_2$, all the arguments above apply recursively to the sub-algorithm S. Thus, the GPS solution satisfies *no justified-envy.*

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