Kidney Exchange: An Egalitarian Mechanism

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Abstract

Transplantation is the preferred treatment for the most serious forms of kidney disease; deceased-donor and live-donor kidneys are the two sources for transplantation, and these sources are utilized via two different programs. One of these programs, a two-way *kidney paired donation (KPD)*, involves two patient-donor couples, for each of whom a transplant from donor to intended recipient is not possible due to medical incompatibilities, but such that the patient in each couple could receive a transplant from the donor in the other couple. This pair of couples can then exchange donated kidneys. Another possibility is a *list exchange (LE)*: a living incompatible donor provides a kidney to a candidate on the deceased-donor (DD) waitlist and in return the intended recipient of this donor receives a priority on the DD-waitlist. Recently, thanks to the contributions of the mechanism design literature, several kidney exchange mechanisms are developed. In this work, we explore how to organize such exchanges by integrating KPD and LE, and taking into consideration the fact that transplants from live donors have a higher chance of success than those from cadavers. The fairness implication of this distinction is that if the donor of an intended recipient donates to a patient in the patient-donor couples pool, then that intended recipient should have a priority in receiving live-donor kidney transplant. Our contribution is a new stochastic kidney exchange mechanism involving multiple-way KPD's and LE-chains based on efficiency and egalitarianism.

Keywords : Mechanism design; Matching; Kidney exchange; Random assignment; Lorenz dominance

Journal of Economic Literature **Classification Number:** C71, C78, D02, D63; I10

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1 Introduction

Transplantation is the preferred treatment for the most serious forms of kidney disease. Unfortunately, there is a considerable shortage of deceased-donor kidneys: as of March 2, 2007, there are 70,321 patients waiting for kidney transplants in the U.S., with the median waiting time of over 3 years, and in 2006, there were only 10,659 transplants of deceased-donor kidneys. The cadaveric kidneys are not the only sources for transplantation. Since healthy people have two kidneys and can remain healthy on one, it is also possible for a kidney patient to receive a live-donor transplant. In 2006, there were 6,425 transplants of live-donor kidneys. These two sources of kidneys enable the medical authorities to develop and utilize different programs to increase the number of transplantations.

One of these programs is a *kidney paired donation (KPD)*. A two-way KPD involves two patient-donor couples, for each of whom a transplant from donor to intended recipient is not possible due to medical incompatibilities, but such that the patient in each couple could receive a transplant from the donor in the other couple (Rapaport [29], Ross et al. [30]). This pair of couples can then exchange donated kidneys. Multiple-way exchanges, in which multiple pairs participate, can also be utilized. To expand the opportunity for KPD, optimal matching algorithms have been designed to identify maximal sets of compatible donor/recipient pairs from a registry of incompatible pairs.

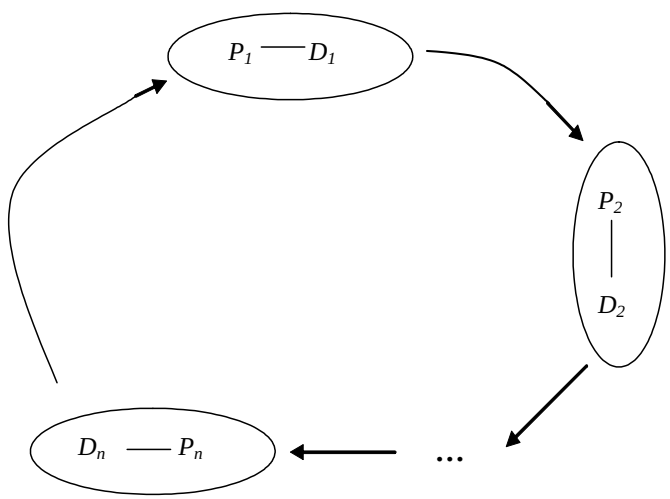

Figure 1: An $n-$ way KPD

Another possibility is a *list exchange (LE)*. In an LE-chain of length two, a living incompatible donor provides a kidney to a candidate on the deceased-donor (DD) waitlist

and in return the intended recipient of this donor receives a priority on the DD-waitlist. This improves the welfare of the patient in the couple, compared to having a long wait for a compatible cadaver kidney, and it benefits the recipient of the live kidney, and other on the DD-waitlist who benefit from the increase in the kidney supply due to an additional living donor. Through April 2006, 24 list exchanges have been performed. LE-chains in which more than one additional pair participates can also be considered. An LE-chain with n pairs is depicted in Figure 1 where R_w denotes the recipient on the waitlist.

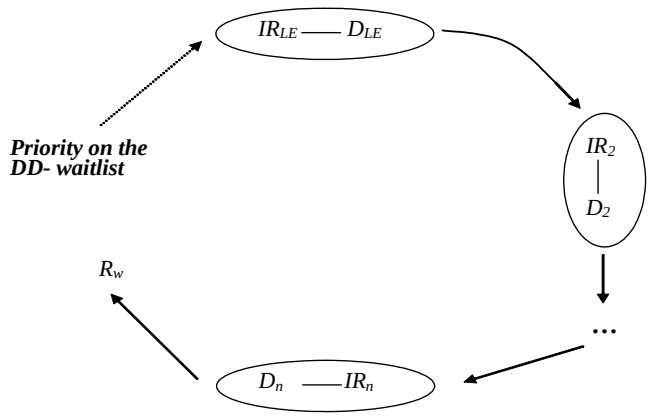

Figure 2: An LE-chain with n pairs

In utilizing these two protocols, KPD and LE, an important distinction is that transplants from live donors have a higher chance of success than those from cadavers. This fact is underlined by medical authorities and is supported by the data. The figure below shows the difference between the patient survival rate for live-donor transplants and for cadaveric transplants performed between 1997 and 2004 in US.

Donor Type	**Years Post Transplant**	**Number Alive/ Functioning**	**Survival Rate**
Cadaveric	1 Year	23,735	94.5
Living	1 Year	18,026	97.9
Cadaveric	3 Year	24,436	88.3
Living	3 Year	18,198	94.3
Cadaveric	5 Year	19,042	82.0
Living	5 Year	12,642	90.2

Figure 3: Kidney Patient Survival Rates

As the data shows, the gap between the patient survival rates increases over the years post transplant. The comparison between the graft survival rates is actually more striking.

Donor Type	**Years Post Transplant**	**Number Alive/ Functioning**	**Survival Rate**
Cadaveric	1 Year	22,753	89.0
Living	1 Year	17,649	95.0
Cadaveric	3 Year	23,075	77.8
Living	3 Year	17,556	87.9
Cadaveric	5 Year	17,629	66.5
Living	5 Year	12,033	79.7

Figure 4: Kidney Graft Survival Rates

Our goal is to explore how to organize kidney exchange by integrating KPD and LE, and taking into consideration the distinction between the success rates of transplants from live donors and those from cadavers. Since, the incompatible patient-donor pairs register the centralized clearinghouse with the expectation of receiving a live-donor kidney transplant for the patient,[1] the fairness implication of this distinction is that if the donor of an intended recipient donates to a patient in the patient-donor couples pool, then that intended recipient should have a priority in receiving live-donor kidney transplant. In random matchings, this wisdom is elevated to equating the difference between the probability of a patient's receiving a live-donor kidney transplant and the probability of his donor's donating her kidney to someone in the patient-donor couples pool, as much as possible among all patient-donor couples in the pool. Our mechanism constructs multiple-way KPD's and LE-chains based on efficiency, and egalitarianism in the sense we have just defined.

While transplants from live donors have a higher chance of success than those from cadavers, the experience of American surgeons suggests that patients should be indifferent among kidneys from healthy donors that are blood type and immunologically compatible with the patient. This is because, in the US, transplants of compatible live kidneys have about equal graft survival probabilities, regardless of the closeness of tissue types between patient and donor (Gjertson and Cecka [20] and Delmonico [10]). In accordance with this medical findings, we assume that while patients' preferences over the set of live-donor kidneys are such 0-1 preferences (following Bogomolnaia and Moulin [8], we refer to such

[1] Note that a patient-donor couple can always go to the DD-waitlist to obtain a priority in receiving a deceased-donor kidney.

preferences as *dichotomous*), they prefer a live-donor kidney transplant to a cadaveric-kidney transplant.

Our work builds on the closely related paper by Bogomolnaia and Moulin [8]. They considered two-sided matching such that an agent on one side of the market can only be matched with an agent on the other side, modelled as a bipartite graph, with dichotomous preferences. Kidney exchange can be interpreted as a special case of this matching problem, that is, as an assignment problem with donors being the resources to be allocated to the patients, and finding an efficient exchange in this model reduces to finding the maximal cardinality matching in the corresponding bipartite graph. The maximum cardinality matching problem is well analyzed in the graph theory literature. More specifically, the Gallai [18, 19]-Edmonds [12] Decomposition Lemma characterizes the set of maximum cardinality matchings. We make use of this result in constructing an efficient exchange. In order to achieve egalitarianism, we need another elegant result from graph theory: Gale's Theorem (Gale [17], Schrijver [40]).

2 Related Literature

While the transplantation community approved the use of KPD and LE programs to increase kidney donations, it has provided little guidance about how to organize such exchanges. Roth, Sönmez, and Ünver [35] suggested that, by modelling kidney exchange as a mechanism design problem, integrating KPD and LE may benefit additional candidates.[2] This approach turns out to be very successful and is supported by the medical community. Since then, a centralized mechanism for kidney exchange based on these two protocols has been used in the regional exchange program in New England (The United Network for Organ Sharing-UNOS-Region 1). In terms of integrating KPD and LE programs, their paper is closest to the present work.

Roth, Sönmez, and Ünver [36] suggested an alternative mechanism which involves only two-way KPD's and no LE's, and assumes that each patient is indifferent between all compatible kidneys. They characterize the maximal cardinality matchings under the constraint that only pairwise exchanges be conducted. They show that, in the constrained problem, efficient and strategy-proof mechanisms exist. These mechanisms include a deterministic mechanism based on the priority setting that organ banks currently use for the allocation of cadaver kidneys, and a stochastic mechanism based on elementary notions of justice. While Bogomolnaia and Moulin [8] considered two-sided matching such that

[2] The kidney exchange problem has some common features with the assignment problem with private endowments and/or a social endowment. (See for example Abdulkadiroğlu and Sönmez [1, 2], Hylland and Zeckhauser [22], Shapley and Scarf [41], Sönmez and Ünver [42], Yılmaz [46, 47])

an agent on one side of the market can only be matched with an agent on the other side, modelled as a bipartite graph, with dichotomous preferences, and characterized the egalitarian and efficient (random) solution, the results of Roth, Sönmez, and Ünver [36] on the egalitarian mechanism generalize the corresponding results of Bogomolnaia and Moulin [8] to general, not necessarily bipartite graphs. One of the crucial aspects of these two papers is finding an efficient matching which reduces to finding a maximum cardinality matching in the corresponding graphs that derive from the dichotomous preferences of agents. The solution to this latter problem is the Gallai-Edmonds Decomposition Lemma [12, 18, 19], and technical aspects of our contribution build on this result.

Roth, Sönmez, and Ünver [37] also explore that, when multiple-way KPD's are feasible, three-way KPD's as well as two-way KPD's will have a substantial effect on the number of transplants that can be arranged, and larger than three-way KPD's have less impact on efficiency.

We use the Lorenz dominance as the criterion for distributive justice as Bogomolnaia and Moulin [8], and Roth, Sönmez, and Ünver [36]. Another work that uses the same criterion for an egalitarian allocation is by Dutta and Ray [11]. They show that, for convex cooperative games, the egalitarian allocation is unique and it is in the core.

3 The model

Let P be a finite set of patients each of whom has an incompatible donor, and D be the set of these donors. We denote the donor of patient $p \in P$ by d_p, and the patient whose donor is $d \in D$ by p_d. For expositional convenience, we assume that all patients are male and all donors are female.

For each $p \in P$, $D_p \subseteq D$ denotes the set of compatible donors for patient p. For each $S \subseteq P$, we write $D_S = \bigcup_{p \in S} D_p$ for the set of donors compatible with at least one patient in S. Also, for each $d \in D$, P_d denotes the set of patients for whom donor d is compatible. For each $F \subseteq D$, we write $P_F = \bigcup_{d \in F} P_d$ for the set of patients for each of whom there is at least one compatible donor in F.

Each patient evaluates each donor as compatible or incompatible and is indifferent between all compatible donors and between all incompatible donors. He prefers each compatible donor to the waitlist option w, and w to each incompatible donor. Thus, for each patient p,

$$d, d' \in D_p \text{ and } d'', d''' \notin D_p \text{ imply } d \sim_p d' \succ_p w \succ_p \ d'' \sim_p d'''.$$

Note that the set D_p fully describes the preferences of patient p.

A **kidney exchange problem**, or simply a **problem** is a triple $(P, D, (D_p)_{p\in P})$.

Let

$$c_{p,d} = \begin{cases} 1 & \text{if } d \in D_p \\ 0 & \text{otherwise} \end{cases}$$

Each problem $(P, D, (D_p)_{p\in P})$ induces a $|P|\times|D|$ **compatibility matrix** $C = [c_{p,d}]_{p\in P, d\in D}$. We refer to the triple (P, D, C) as the **reduced problem of** $(P, D, (D_p)_{p\in P})$. Throughout the paper, we fix a problem $(P, D, (D_p)_{p\in P})$, and the reduced problem (P, D, C) of $(P, D, (D_p)_{p\in P})$.

A **deterministic matching** is an *injective partial function* μ from P into D, that is, for each $d \in D$, there is at most one patient p such that $\mu(p) = d$. An unmatched patient receives high priority on the cadaver queue. By definition of a deterministic matching, the number of unmatched patients is equal to the number of unmatched donors. Thus, for each patient p receiving high priority on the cadaver queue, there is a donor d (not necessarily d_p) who donates her kidney to someone on the queue. A deterministic matching is represented as a $|P| \times |D|$ matrix with entries 0 or 1, and at most one nonzero entry per row and one per column. A deterministic matching μ is **individually rational** if, for each patient $p \in P$, $\mu(p) = d$ implies $d \in D_p$. Let $\mathcal{M}$ denote the set of all individually rational deterministic matchings.[3] Let $P_\mu \equiv \{p \in P : \mu(p) \in D\}$, the set of patients matched by μ. We call $|P_\mu|$ as the **cardinality of matching** μ.

Let $\lambda = (\lambda_\mu)_{\mu\in\mathcal{M}}$ be a **lottery** that is, a probability distribution over $\mathcal{M}$. Let $\Delta\mathcal{M}$ denote the set of all lotteries. Each lottery $\lambda \in \Delta\mathcal{M}$ induces a **random matching (matrix)** $Z(\lambda) = [z_{p,d}(\lambda)]_{p\in P, d\in D}$, where $z_{p,d}(\lambda)$ is the probability that patient p is matched to donor d, that is, the probability that λ selects a deterministic matching μ such that $\mu(p) = d$. Thus, for each $\lambda \in \Delta\mathcal{M}$, the $|P| \times |D|$ matrix $Z(\lambda)$ is substochastic, that is to say it is nonnegative and the sum of each row (each column) is at most 1. Let Z be a non-negative and substochastic matrix such that $z_{p,d} > 0$ implies $d \in D_p$. The set of all such random matching matrices is denoted by $\mathcal{Z}$.

For patient $p \in P$, the aggregate probability that he receives a live-donor transplant, is the canonical utility representation of his preferences over random matchings. Thus, given a random matching $Z \in \mathcal{Z}$, the utility of patient p is defined as the sum of the entries in the $p-th$ row of Z :

$$u_p(Z) = \sum_{d\in D} z_{p,d},$$

and the utility profile is defined as the non-negative real vector $\mathbf{u}(Z) = (u_p(Z))_{p\in P}$. We

[3] Throughout the rest of the paper, we consider only individually rational matchings.

denote by $\mathcal{U}$ the set of all feasible utility profiles. That is, $\mathcal{U} = \{\mathbf{u}(Z) : Z \in \mathcal{Z}\}$.

Given a random matching $Z \in \mathcal{Z}$, the probability that kidney of donor d is transplanted to someone in the exchange pool, $t_d(Z)$, is the sum of the entries in the $d-th$ column of Z :

$$t_d(Z) = \sum_{p \in P} z_{p,d},$$

and the transplantation probability profile is defined as the non-negative real vector $\mathbf{t}(Z) = (t_{d_p}(Z))_{p \in P}$.

A variant of the Birkhoff-von Neumann Theorem [4, 48], implies that each substochastic matrix $Z \in \mathcal{Z}$ obtains as a (in general not unique) lottery $\lambda \in \Delta\mathcal{M}$.[4] Since, for each patient, two lotteries resulting in the same random matching yield the same aggregate probability of receiving a live-donor transplant, we do not distinguish them. Thus, a **random solution to** (P, D, C) is a matrix $Z \in \mathcal{Z}$.

4 Efficiency

A deterministic matching $\mu \in \mathcal{M}$ is **Pareto efficient** if there exists no other matching $\eta \in \mathcal{M}$ such that $P_\eta \supsetneq P_\mu$, i.e. if P_μ is inclusion maximal. Let $\mathcal{E}$ be the set of Pareto efficient matchings. A well-known property of matchings states that each Pareto efficient matching matches the same number of patients. For the sake of completeness, we repeat a result from abstract algebra which implies this property:[5]

A *matroid* is a pair $(X, \mathcal{I})$ such that X is a set and $\mathcal{I}$ is a collection of subsets of X such that

M1. if I is in $\mathcal{I}$ and $J \subseteq I$ then J is in $\mathcal{I}$; and

M2. if I and J are in $\mathcal{I}$ and $|I| > |J|$ then there exists an $i \in I \setminus J$ such that $J \cup \{i\}$ is in $\mathcal{I}$.

Proposition 1 *Let $\mathcal{I}$ be the sets of simultaneously matchable patients, i.e. $\mathcal{I} = \{I \subseteq P : \exists \mu \in \mathcal{M}$ such that $I \subseteq P_\mu\}$. Then, $(P, \mathcal{I})$ is a matroid.*

The following property follows immediately from the second property of matroids:

Lemma 1 *For each pair of Pareto efficient matchings $\mu, \eta \in \mathcal{E}$, $|P_\mu| = |P_\eta|$.*

[4] The Birkhoff-von Neumann Theorem holds for bistochastic matrices. This result is generalized to substochastic matrices by Bogomolnaia and Moulin [7].

[5] This result is also stated by Roth, Sönmez, and Ünver [36].

4.1 Gallai-Edmonds Decomposition

The *Gallai-Edmonds decomposition* (GED) of bipartite graphs, a well-known result in graph theory, further clarifies the structure of Pareto efficient deterministic matchings.[6]

Lemma 2 *(Gallai-Edmonds decomposition) Given a reduced problem* (P, D, C)*, there is a unique pair of partitions* $\{P^o, P^f, P^u\}$ *of* P *and* $\{D^u, D^f, D^o\}$ *of* D *such that:*
(i) D^u *is only compatible with* P^o*, and* D^u *is underdemanded by* P^o *:*

$$P_{D^u} = P^o$$

and

$$\text{for each } S \subseteq P^o : |D_S \cap D^u| > |S|;$$

(ii) there is a full match between P^f *and* D^f*, that is, all patients in* P^f *can be matched with all donors in* D^f *:*

$$\text{for each } S \subseteq P^f : \left|D_S \cap D^f\right| \geq |S|;$$

(iii) P^u *is only compatible with* D^o*, and* D^o *is overdemanded by* P^u *:*

$$D_{P^u} = D^o$$

and

$$\text{for each } F \subseteq D^o : |P_F \cap P^u| > |F|.$$

Note in particular that $|P^o| < |D^u|$, $\left|P^f\right| = \left|D^f\right|$, and $|P^u| > |D^o|$. The GED Lemma states that it is possible to match the patients in P^o with the donors in D^u such that each patient in P^o receives a live-donor transplant from the set D^u. In this case, there are $|D^u| - |P^o|$ donors in D^u, each of whom donates her kidney to someone on the queue. Also, the patients in P^u can be matched only with the donors in D^o. But, there are not enough donors in D^o such that each patient in P^u receives a live-donor transplant. Thus, if each patient in P^u is matched with a donor in D^o, then there are $|P^u| - |D^o|$ patients in P^u, each of whom receives high priority on the cadaver queue rather than a live-donor transplant. Note that $|D^u| - |P^o| = |P^u| - |D^o|$.

As shown before, finding a Pareto efficient deterministic matching reduces to finding a maximum cardinality matching. The GED Lemma characterizes the set of maximum cardinality matchings.

[6] All the results in this section are also stated by Bogomolnaia and Moulin [8].

Lemma 3 *A deterministic matching $\mu \in \mathcal{M}$ is Pareto efficient if and only if exactly $|P^o| + \left|P^f\right| + |D^o|$ patients are matched by μ.*
In each Pareto efficient matching, patients in P^o are matched to donors in a proper subset of D^u, patients in a proper subset of P^u are matched to donors in D^o, and there is a full match between P^f and D^f.

We now turn our attention to random matchings. A lottery λ is **ex post efficient** if its support is a subset of the set of Pareto efficient deterministic matchings, that is, if $\lambda_\mu > 0$ implies $\mu \in \mathcal{E}$. A random matching Z is **ex ante efficient** if there exists no other random matching Z' such that $\mathbf{u}(Z') \geq \mathbf{u}(Z)$ and for some $p \in P$, $u_p(Z') > u_p(Z)$. We denote the set of ex ante efficient random matchings by $\mathcal{Z}^e$. A utility profile $\mathbf{u} \in \mathcal{U}$ is **efficient** if there exists no other utility profile $\mathbf{v} \in \mathcal{U}$ such that $\mathbf{v} \geq \mathbf{u}$ and for some $p \in P$, $v_p > u_p$. We denote the set of efficient utility profiles by $\mathcal{U}^e$.

The GED Lemma is also key to the characterization of the efficient utility profiles.

Lemma 4 *(i) A lottery is ex post efficient if and only if, with probability one, it matches exactly $|P^o| + \left|P^f\right| + |D^o|$ patients.*
(ii) A random matching is ex ante efficient if and only if the sum of its entries is $|P^o| + \left|P^f\right| + |D^o|$.
(iii) A random matching is ex ante efficient if and only if $z_{p,d} > 0$ implies $(p, d) \in (P^o, D^u) \cup (P^f, D^f) \cup (P^u, D^o)$, and its restriction to (P^o, D^u) is row-stochastic, to (P^f, D^f) is bistochastic, and to (P^u, D^o) is column-stochastic.

Throughout the rest of the paper, we consider only efficient matchings.

5 Stochastic Exchange

Given a random matching Z, and a patient p, the difference between his utility and the probability that the kidney of his donor d_p is transplanted to someone in the exchange pool (we call it as the **u-t difference for patient p**) is important in the sense of fairness: if the donor d_p donates her kidney to someone in the exchange pool, then it is plausible to think that patient p should have the priority in receiving a live-donor kidney transplantation in exchange for his donor's contribution to the pool. But, there may be several patients whose donors donate their kidneys to the pool, yet there are not enough compatible donors in the pool to donate their kidneys to these patients. Thus, for a random matching Z, the vector $\mathbf{u}(Z) - \mathbf{t}(Z) = (u_p(Z) - t_{d_p}(Z))_{p \in P}$, is key to evaluating its fairness; equalizing the u-t differences as much as possible is very plausible from an equity perspective. We use the

Lorenz criterion as the partial ordering of the matchings. The *Lorenz dominance* is the following partial orderings of vectors in $\mathbb{R}^{|P|}$: $\mathbf{v}$ *Lorenz dominates* $\mathbf{y}$ if upon rearranging their $|P|$ coordinates increasingly as $\mathbf{v}^*$ and $\mathbf{y}^*$, we have

$$\text{for each } k = 1, \ldots, |P| : \quad \sum_{i=1}^{k} (v_i^* - y_i^*) \geq 0.$$

A matching $Z \in \mathcal{Z}^e$ such that $\mathbf{u}(Z) - \mathbf{t}(Z)$ is Lorenz dominant in the set $\{\mathbf{u}(Z') - \mathbf{t}(Z') : Z' \in \mathcal{Z}^e\}$ has a very strong claim to fairness within the set of efficient matchings. It achieves the maximum over $\{\mathbf{u}(Z') - \mathbf{t}(Z') : Z' \in \mathcal{Z}^e\}$ of any collective welfare function averse to inequality in the sense of the Pigou-Dalton transfer principle. Also, it maximizes not only the leximin ordering but also any collective welfare function $\sum_P f(u_p - t_p)$ for each increasing and concave function f. (See Moulin [27] and Sen [41] for these results and more on Lorenz dominance.) This leads to the following definition.

Definition 1 *A random matching $Z \in \mathcal{Z}^e$ is* ***egalitarian*** *if the vector $\mathbf{u}(Z) - \mathbf{t}(Z)$ is Lorenz dominant in the set $\{\mathbf{u}(Z') - \mathbf{t}(Z') : Z' \in \mathcal{Z}^e\}$.*

Let $P^{u;u}$ denote the set of underdemanded patients whose donors are underdemanded, $P^{u;f,o}$ denote the set of underdemanded patients whose donors are fully demanded or overdemanded. Also, let $P^{u;1}(Z)$ denote the set of underdemanded patients who receive live-donor transplantations with probability 1 in the random matching Z. Similarly, $D^{u;u}$ denotes the set of underdemanded donors of underdemanded patients, and $D^{u;f,o}$ denotes the set of underdemanded donors of fully demanded or overdemanded patients. Note that $P^u = P^{u;u} \cup P^{u;f,o}$ and $D^u = D^{u;u} \cup D^{u;f,o}$. Also, let $P^{f,o;u}$ denote the set of fully demanded or overdemanded patients whose donors are underdemanded.

To convey the idea in our characterization result, let us consider the special case where $P^f = D^f = \emptyset$, and $P^{u;f,o} = D^{u;f,o} = \emptyset$, that is, each underdemanded patient has an underdemanded donor. For an overdemanded patient p, at each efficient random matching $Z \in \mathcal{Z}^e$, the probability of both him receiving a live donor kidney and also his donor donating her kidney someone in the exchange pool is one, thus, $u_p(Z) - t_{d_p}(Z) = 0$. For an underdemanded patient, on the other hand, the u-t difference may be negative or positive. Also, for each $Z \in \mathcal{Z}^e$, $\sum_{p \in P} (u_p(Z) - t_{d_p}(Z)) = 0$. (Note that this equality holds for the general case as well.) Since an egalitarian random matching necessarily maximizes the leximin ordering of the u-t differences vectors, first step in finding such a matching is to find the maximum possible first coordinate of the vector upon rearranging their coordinates increasingly.

For each $E \subseteq D$, define $P(E) \equiv \{p \in P : D_p \subseteq E\}$ as the set of patients whose compatible donors are in only E. Let $S \subseteq P^u$ be a set of patients. Our goal is to find a random matching such that the u-t difference for each patient in S is the same and as maximum as possible. For each F' such that $\{d_p : p \in S\} \subseteq F' \subseteq D^u$, $P(F') \subseteq P^o$; and each patient in $P(F')$ receives a live-donor kidney transplantation with probability 1. By Gallai-Edmonds Decomposition, at an efficient random matching, the patients in S can receive at most $|D_S|$ live-donor kidney transplantations, and also, at best, their donors donate only to $|P(F')| - |F' \setminus \{d_p : p \in S\}|$ patients in P^o. Then, if the u-t difference for each patient in S is the same, then its maximum possible value can not be greater than

$$f(S, F') = \frac{|D_S| - (|P(F')| - |F' \setminus \{d_p : p \in S\}|)}{|S|}.$$

Since, given F', this number is an upperbound, to find the maximum possible u-t difference, we need to take the minimum of this function over all such sets. Let

$$F = Arg \min_{F' : F' \supseteq \{d_p : p \in S\}} \frac{|D_S| - (|P(F')| - |F' \setminus \{d_p : p \in S\}|)}{|S|}.$$

But, the problem is that we don't know whether there is a match such that each donor in $F \setminus \{d_p : p \in S\}$ donates to a patient in $P(F)$. It turns out that there is such a match and it follows from Hall's Theorem:

Hall's Theorem: *There exists a matching such that each donor in $F \setminus \{d_p : p \in S\}$ donates to a patient in $P(F)$ if and only if*

$$\textit{for each } E \subseteq F \setminus \{d_p : p \in S\} : |E| \leq |\{p \in P(F) : D_p \cap E \neq \emptyset\}|.$$

Suppose there does not exist a matching such that each donor in $F \setminus \{d_p : p \in S\}$ donates to a patient in $P(F)$. Then, by Hall's Theorem, there is a set $E \subseteq F \setminus \{d_p : p \in S\}$ such that

$$|E| > |\{p \in P(F) : D_p \cap E \neq \emptyset\}|.$$

This is equivalent to

$$|E| > |P(F)| - |\{p \in P(F) : D_p \cap E = \emptyset\}|.$$

Consider now the set $F \setminus E$. Note that $P(F \setminus E) = \{p \in P(F) : D_p \cap E = \emptyset\} > |P(F)| - |E|$.

Thus,

$$\begin{aligned} f(S, F \setminus E) &= \frac{|D_S| - |P(F \setminus E)| + |(F \setminus E) \setminus \{d_p : p \in S\}|}{|S|} \\ &= \frac{|D_S| - |P(F \setminus E)| + |F \setminus \{d_p : p \in S\}| - |E|}{|S|} \\ &< \frac{|D_S| - (|P(F)| - |E|) + |F \setminus \{d_p : p \in S\}| - |E|}{|S|} \\ &= f(S, F). \end{aligned}$$

Since $F \setminus E \supseteq \{d_p : p \in S\}$, this contradicts with the definition of the set F. Thus, the maximum possible u-t difference for each patient in S can be achieved by matching each donor in $F \setminus \{d_p : p \in S\}$ to a patient in $P(F)$, and the donors in $\{d_p : p \in S\}$ to the remaining patients in $P(F)$.

Since $f(S, F)$ is an upperbound for the set S, and we need to take the minimum of this function over all subsets of P^u, we conclude that the maximum value of the first coordinate of the leximin ordering can not be greater than

$$\lambda^* = \min_{S' : S' \subseteq P^u} \left\{ \min_{F' : F' \supseteq \{d_p : p \in S'\}} \frac{|D_{S'}| - (|P(F')| - |F' \setminus \{d_p : p \in S'\}|)}{|S'|} \right\}.$$

The question is whether there exists a random matching such that the u-t difference for each patient is at least λ^*. Our main result shows that if the value of this upperbound is negative, then there exists such a matching. In the egalitarian random matchings, the characterization of the minimum positive u-t difference is slighlty different than the characterization of the minimum non-positive u-t difference. We will come to this point later. First, we generalize our findings here and present a recursive construction of the egalitarian random matchings.

5.1 The Egalitarian Mechanism: Recursive Construction of the Egalitarian Random Matchings

First, the donors in D^f are matched to the patients in P^f, such that they are fully matched to each other. Let $P_1^{u;u} = P^{u;u}$, $P_1^{u;f,o} = P^{u;f,o}$, $D_1^{u;u} = D^{u;u}$, $D_1^{u;f,o} = D^{u;f,o}$, $D_1^o = D^o$ and $P_1^o = P^o$.

Step 1: For each $S \subseteq P_1^u$, $F \subseteq D_1^u$, define a real-valued function f_1 through

$$f_1(S, F) = \frac{|D_S| - |P_1^o(F)| - |S| + |F|}{|S|}.$$

Let

$$\lambda_1 = \min_S \left\{ \min_{F:\{d_p : p \in S \cap P_1^{u;u}\} \subseteq F} f_1(S,F) \right\}$$

and S_1, and F_1 be the largest sets in the sense of inclusion [7] such that

$$\lambda_1 = f_1(S_1, F_1).$$

Let

$$P_2^{u;u} = P_1^{u;u} \setminus \left(S_1 \cup \{p \in P_1^u : d_p \in (F_1 \cap D_1^{u;u}) \setminus \{d_p : p \in S_1 \cap P_1^{u;u}\}\}\right),$$

$$P_2^{u;f,o} = \left(P_1^{u;f,o} \setminus S_1\right) \cup \{p \in P_1^u : d_p \in (F_1 \cap D_1^{u;u}) \setminus \{d_p : p \in S_1 \cap P_1^{u;u}\}\},$$

$D_2^{u;u} = D_1^{u;u} \setminus F_1$, $D_2^{u;f,o} = D_1^{u;f,o} \setminus F_1$, $D_2^o = D_1^o \setminus D_{S_1}$, and $P_2^o = P_1^o \setminus P_1^o(F_1)$. Let $Z^1 \subseteq \mathcal{Z}^e$ denote the set of all random matchings Z such that for each patient $p \in S_1$, $u_p(Z) - t_{d_p}(Z) = \lambda_1 \leq 0$, and $p \in P \setminus S_1$, $u_p(Z) - t_{d_p}(Z) > \lambda_1$.

Step k: For each $S \subseteq P_k^u$, $F \subseteq D_k^u$, define a real-valued function f_k through

$$f_k(S,F) = \frac{|D_S \cap D_k^o| - |P_k^o(F)| - |S| + |F|}{|S|}.$$

Let

$$\lambda_k = \min_S \left\{ \min_{F:\{d_p : p \in S \cap P_k^{u;u}\} \subseteq F} f_k(S,F) \right\}$$

and S_k, and F_k be the largest sets in the sense of inclusion such that

$$\lambda_k = f_k(S_k, F_k).$$

Let

$$P_{k+1}^{u;u} = P_k^{u;u} \setminus \left(S_k \cup \{p \in P_k^u : d_p \in \left(F_k \cap D_k^{u;u}\right) \setminus \{d_p : p \in S_k \cap P_k^{u;u}\}\}\right),$$

$$P_{k+1}^{u;f,o} = \left(P_k^{u;f,o} \setminus S_k\right) \cup \{p \in P_k^u : d_p \in \left(F_k \cap D_k^{u;u}\right) \setminus \{d_p : p \in S_k \cap P_k^{u;u}\}\},$$

$D_{k+1}^{u;u} = D_k^{u;u} \setminus F_k$, $D_{k+1}^{u;f,o} = D_k^{u;f,o} \setminus F_k$, $D_{k+1}^o = D_k^o \setminus D_{S_k}$, and $P_{k+1}^o = P_k^o \setminus P_k^o(F_k)$. Let $Z^k \subseteq Z^{k-1}$ denote the set of all random matchings Z such that for each patient $p \in S_k$, $u_p(Z) - t_{d_p}(Z) = \lambda_k \leq 0$, and $p \in P \setminus \bigcup_{i=1}^{k} S_i$, $u_p(Z) - t_{d_p}(Z) > \lambda_k$.

Let step K be such that $\lambda_K \leq 0$ and $\lambda_{K+1} > 0$. For each $Z \in Z^K$, let $P^{u;1}(Z) \equiv P_{K+1}^{u;1} \equiv P_{K+1}^{u;f,o}$.

Step K+1: For each $T \subseteq P_{K+1}^{u;u} \cup P_{K+1}^{u;1}$, and $H \subseteq D_{K+1}^u$, define a real-valued function

[7] As we show in the Appendix, these largest sets are well defined.

g_1 through

$$g_1(T,H) = \frac{\left|D_T \cap D^o_{K+1}\right| - \left|P^o_{K+1}(H)\right| - |T| + |H|}{|H|}.$$

Let

$$\beta_1 = \min_H \left\{ \min_{T:\{d_p : p \in T \cap P^{u;u}_{K+1}\} \subseteq H} g_1(T,H) \right\}$$

and T^1, and H^1 be the largest sets in the sense of inclusion such that

$$\beta_1 = g_1(T^1, H^1).$$

Let $P^{u;u}_{K+2} = P^{u;u}_{K+1} \setminus \{p : d_p \in H^1\}$,

$$P^{u;1}_{K+2} = (P^{u;1}_{K+1} \cup \{p : d_p \in D^{u,u}_{K+1} \cap H^1\}) \setminus T^1,$$

$D^u_{K+2} = D^u_{K+1} \setminus H^1$, $P^o_{K+2} = P^o_{K+1} \setminus P^o_{K+1}(H^1)$, and $D^o_{K+2} = D^o_{K+1} \setminus D_{T^1}$. Let $Z^{K+1} \subseteq Z^K$ denote the set of all random matchings Z such that for each patient $p \in T^1$, $u_p(Z) - t_{d_p}(Z) = \beta_1 > 0$, and $p \in P \setminus \left(\bigcup_{i=1}^{K} S_i \cup T^1 \right)$, $u_p(Z) - t_{d_p}(Z) > \beta_1$.

Step K+m: For each $T \subseteq P^{u;u}_{K+m} \cup P^{u;1}_{K+m}$, and $H \subseteq D^u_{K+m}$, define a real-valued function g_m through

$$g_m(T,H) = \frac{\left|D_T \cap D^o_{K+m}\right| - \left|P^o_{K+m}(H)\right| - |T| + |H|}{|H|}.$$

Let

$$\beta_m = \min_H \left\{ \min_{T:\{d_p : p \in T \cap P^{u;u}_{K+m}\} \subseteq H} g_m(T,H) \right\}$$

and T^m, and H^m be the largest sets in the sense of inclusion such that

$$\beta_m = g_m(T^m, H^m).$$

Let $P^{u;u}_{K+m+1} = P^{u;u}_{K+m} \setminus \{p : d_p \in H^m\}$,

$$P^{u;1}_{K+m+1} = (P^{u;1}_{K+m} \cup \{p : d_p \in D^{u,u}_{K+m} \cap H^m\}) \setminus T^m,$$

$D^u_{K+m+1} = D^u_{K+m} \setminus H^m$, $P^o_{K+m+1} = P^o_{K+m} \setminus P^o_{K+m}(H^m)$, and $D^o_{K+m+1} = D^o_{K+m} \setminus D_{T^m}$. Let $Z^{K+m} \subseteq Z^{K+m-1}$ denote the set of all random matchings Z such that for each patient $p \in T^m$, $u_p(Z) - t_{d_p}(Z) = \beta_m > 0$, and $p \in P \setminus \left(\bigcup_{i=1}^{K} S_i \bigcup_{i=1}^{m} T^i \right)$, $u_p(Z) - t_{d_p}(Z) > \beta_m$.

Let M be such that at the end of step $K+M$, the construction is completed, that is $P_{K+M+1} = D_{K+M+1} = \emptyset$.

5.2 Main Result

At the end of step 1, each donor in $F_1 \setminus \{d_p : p \in S_1 \cap P_1^{u;u}\}$ donates her kidney to someone in $P_1^o(F_1)$ with probability 1. Each donor in D_{S_1} donates her kidney to someone in S_1 with probability 1. The next step continues with the remaining patients and donors. But there is a change in the decomposition of the patients: Note that each donor in $F_1 \setminus \{d_p : p \in S_1 \cap P_1^{u;u}\}$ donates her kidney in the current step with probability 1. Thus, in the next step, they are fully demanded or overdemanded donors of the underdemanded patients $\{p \in P_1^u : d_p \in (F_1 \cap D_1^{u;u}) \setminus \{d_p : p \in S_1 \cap P_1^{u;u}\}\}$. Thus, each such patient switches from being a member of $P_1^{u;u}$ to being a member of $P_2^{u;f,o}$.

At the end of step K, there is a matching such that the u-t difference for each remaining patient is positive. Thus, to maximize the leximin ordering among the remaining patients, we now have to consider all the patients including the overdemanded patients. (In the previous steps, since efficiency implies that the u-t difference for each overdemanded patient is positive, we ignored them.)

At the end of step K+1, for each $H \subseteq D_{K+1}^u$, the patients in $\{p : d_p \in H \cap D_{K+1}^{u;f,o}\}$ are overdemanded. For each T such that $\{d_p : p \in T \cap P_{K+1}^{u;u}\} \subseteq H$, the patients in $\{p : d_p \in H \cap D_{K+1}^{u;u}\}$ can receive at most

$$\left|D_T \cap D_{K+1}^o\right| - \left|T \cap P_{K+1}^{u;1}\right| + \left|H \cap D_{K+1}^{u;u}\right| - \left|T \cap P_{K+1}^{u;u}\right|$$

donors. Thus, together with the patients in $\{p : d_p \in H \cap D_{K+1}^{u;f,o}\}$, they can receive at most

$$\begin{aligned}
& \left|D_T \cap D_{K+1}^o\right| - \left|T \cap P_{K+1}^{u;1}\right| + \left|H \cap D_{K+1}^{u;u}\right| - \left|T \cap P_{K+1}^{u;u}\right| + \left|\{p : d_p \in H \cap D_{K+1}^{u;f,o}\}\right| \\
= & \left|D_T \cap D_{K+1}^o\right| - \left|T \cap P_{K+1}^{u;1}\right| + \left|H \cap D_{K+1}^{u;u}\right| - \left|T \cap P_{K+1}^{u;u}\right| + \left|H \cap D_{K+1}^{u;f,o}\right| \\
= & \left|D_T \cap D_{K+1}^o\right| - |T| + |H|
\end{aligned}$$

donors. Also, efficiency implies that their donors are matched to at least $\left|P_{K+1}^o(H)\right|$ patients. Thus, the upper bound for the lowest u-t difference for the patients in $\{p : d_p \in H\}$ is

$$\frac{\left|D_T \cap D_{K+1}^o\right| - \left|P_{K+1}^o(H)\right| - |T| + |H|}{|H|}.$$

Since H and T are arbitrarily chosen, to determine the upperbound for the lowest u-t difference, we need to consider each such pair of sets such that this upperbound as specified above is the minimum. Thus, $g_1(T^1, H^1)$ is the upper bound for the value of the first coordinate of the leximin ordering for the remaining patients. As we show in the Appendix, there is actually a matching such that the first coordinate of the leximin

ordering for the remaining patients is equal to $g_1(T^1, H^1)$.

Let $Z^* \in Z^{K+M}$ denote a random matching constructed above.

Theorem 1 *The random matching Z^* is egalitarian.*

The proof of this result highly relies on an elegant result from graph theory: Gale's Theorem. This result is relegated to the Appendix.

6 Concluding Remarks

Roth, Sönmez, and Ünver [35] proposed efficient kidney exchange mechanisms that integrates the KPD and LE. Roth, Sönmez, and Ünver [36] later suggested an alternative mechanism which involves only two-ways KPD's and no LE's, and assumes that each patient is indifferent between all compatible kidneys. In addition to this latter assumption, we also adopt the assumption that each patient prefers each compatible live-donor kidney to each deceased-donor kidney; and allow multiple-ways KPD's and as well as LE's as in the mechanism proposed by Roth, Sönmez, and Ünver [35]. Our contribution is to construct a stochastic kidney exchange mechanism that is efficient and egalitarian. Although we consider only the kidney exchange problem, the same mechanism applies to the assignment problems with private endowments where the endowment of each agent is ranked at the bottom of his preference ordering, and there is an outside option that is always feasible.

7 Appendix

7.1 Directed Graphs and Gale's Theorem

A **directed graph**, or **digraph** is a pair $G = (V, A)$, consisting of a set of **vertices** V and a set of ordered pairs of vertices, A, called **arcs**. We say that $a = (u, v)$ *leaves* u and *enters* v. For each vertex v, we denote

$$\delta_G^{in}(v) \equiv \delta_A^{in}(v) \equiv \delta^{in}(v) \equiv \text{set of arcs of } G \text{ entering } v,$$
$$\delta_G^{out}(v) \equiv \delta_A^{out}(v) \equiv \delta^{out}(v) \equiv \text{set of arcs of } G \text{ leaving } v.$$

For each $U \subseteq V$, an arc $a = (x, y)$ is said to *leave* U if $u \in U$ and $v \notin U$. It is said to *enter* U if $u \notin U$ and $v \in U$. We denote

$$\delta_G^{in}(U) \equiv \delta_A^{in}(U) \equiv \delta^{in}(U) \equiv \text{set of arcs of } G \text{ entering } U,$$

$$\delta_G^{out}(U) \equiv \ \delta_A^{out}(U) \equiv \delta^{out}(U) \equiv \text{set of arcs of } G \text{ leaving } U.$$

Let $s, t \in V$. A function $f : A \to \mathbb{R}$ is called a **flow from s to t**, or **an s − t flow**, if

$$\begin{aligned} &(i)\ f(a) \geq 0 \qquad \text{for each } a \in A, \\ &(ii)\ f(\delta^{out}(v)) = f(\delta^{in}(v)) \qquad \text{for each } v \in V \setminus \{s, t\}. \end{aligned}$$

Let $k : A \to \mathbb{R}_+$ be a function which associates each arc $a = (x, y)$ of G a nonnegative real number $k(x, y)$ called the **capacity of the arc**. We say that f is *under* k (or *subject to* k) if

$$f(a) \leq k(a) \qquad \text{for each } a \in A.$$

It will be convenient to make an observation on general functions $f : A \to \mathbb{R}$. Let $\mathcal{P}(V)$ denote the collection of all subsets of V. For each $f : A \to \mathbb{R}$, the **excess function** is the function $excess_f : \mathcal{P}(V) \to \mathbb{R}$ defined by

$$excess_f(U) \equiv f(\delta^{in}(U)) - f(\delta^{out}(U)).$$

Let $b \in \mathbb{R}^V$. A function $f : A \to \mathbb{R}$ is called a **b−transshipment** if $excess_f = b$.

Theorem 2 *(Gale's Theorem) Let $G = (V, A)$ be a digraph and let $k : A \to \mathbb{R}$ and $b : V \to \mathbb{R}$ with $b(V) \equiv \sum_{v \in V} b(v) = 0$. Then, there exists a b−transshipment f satisfying $0 \leq f \leq k$ if and only if*

$$\textit{for each } U \subseteq V : b(U) \leq k(\delta^{in}(U)) \equiv \sum_{a \in \delta^{in}(U)} k(a). \tag{1}$$

7.2 Proof of Theorem 1

Lemma 5 *Consider the first step of the egalitarian mechanism.*[8] *Suppose the sets $Y_1, Y_2 \subseteq P^{u;u}$, $Z_1, Z_2 \subseteq P^{u;f,o}$, $K_1, K_2 \subseteq D^{u;u}$, and $L_1, L_2 \subseteq D^{u;f,o}$ are such that*

$$f_1(Y_1 \cup Z_1, K_1 \cup L_1) = f_1(Y_2 \cup Z_2, K_2 \cup L_2) = \lambda_1.$$

Then,

$$f_1(Y_1 \cup Z_1 \cup Y_2 \cup Z_2, K_1 \cup L_1 \cup K_2 \cup L_2) = \lambda_1$$

as well.

[8] The result directly applies to steps 2,...,K as well.

Proof. For $i = 1, 2$, define

$$n_i = |Y_i \cup Z_i|\,,\ d_i = |D_{Y_i \cup Z_i} \cap D^o|\,,$$

$$m_i = |P(K_i \cup L_i)| - |(K_i \cup L_i) \setminus \{d_p : p \in Y_i\}|\,.$$

Also, define

$$n_3 = \left|\bigcap_{i=1,2} (Y_i \cup Z_i)\right|,\ n_4 = \left|\bigcup_{i=1,2} (Y_i \cup Z_i)\right|,$$

$$d_3 = \left|D_{(Y_1 \cup Z_1) \cap (Y_2 \cup Z_2)} \cap D^o\right|,\ d_4 = \left|D_{(Y_1 \cup Z_1) \cup (Y_2 \cup Z_2)} \cap D^o\right|,$$

and

$$m_3 = \left|P\left(\bigcap_{i=1,2} (K_i \cup L_i)\right)\right| - \left|\left(\bigcap_{i=1,2} (K_i \cup L_i)\right) \setminus \{d_p : p \in Y_1 \cap Y_2\}\right|,$$

$$m_4 = \left|P\left(\bigcup_{i=1,2} (K_i \cup L_i)\right)\right| - \left|\left(\bigcup_{i=1,2} (K_i \cup L_i)\right) \setminus \{d_p : p \in Y_1 \cup Y_2\}\right|.$$

By definition, we have

$$n_1 + n_2 = n_3 + n_4,\ \text{and}\ |Z_1| + |Z_2| = |Z_3| + |Z_4|\,.$$

Also,

$$d_1 + d_2 \geq d_3 + d_4.$$

This is because, not only the compatible kidneys of the patients in $\bigcap_{i=1,2} (Y_i \cup Z_i)$ are counted twice, but also two patients, one in $Y_1 \cup Z_1$, the other in $Y_2 \cup Z_2$, may reveal the same donor as compatible.

By definition of $P(\cdot)$, the patients in $P_1\left(\bigcap_{i=1,2} (K_i \cup L_i)\right)$ are the only double counted patients in $\bigcup_{i=1,2} P_1(K_i \cup L_i)$. Moreover, a patient who is neither in $P_1(K_1 \cup L_1)$ nor in $P_1(K_2 \cup L_2)$, may be in $P_1\left(\bigcup_{i=1,2} (K_i \cup L_i)\right)$. Thus,

$$\sum_{i=1,2} |P_1(K_i \cup L_i)| \leq \left|P_1\left(\bigcup_{i=1,2} (K_i \cup L_i)\right)\right| + \left|P_1\left(\bigcap_{i=1,2} (K_i \cup L_i)\right)\right|.$$

For each $i = 1, 2$, $d \in \left(\bigcap_{i=1,2} (K_i \cup L_i)\right) \setminus \{d_p : p \in Y_1 \cap Y_2\}$ and $p_d \in Y_i$ implies $d \in ((K_{-i} \cup L_{-i}) \setminus \{d_p : p \in Y_{-i}\})$, but $d \notin ((K_i \cup L_i) \setminus \{d_p : p \in Y_i\})$. Thus, the only double

counted donors in the set

$$\bigcup_{i=1,2} ((K_i \cup L_i) \setminus \{d_p : p \in Y_i\})$$

are the donors in

$$\left(\left(\bigcap_{i=1,2} (K_i \cup L_i)\right) \setminus \{d_p : p \in Y_1 \cap Y_2\}\right) \setminus (Y_1 \cup Y_2).$$

Thus,

$$\begin{aligned} &\sum_{i=1,2} |(K_i \cup L_i) \setminus \{d_p : p \in Y_i\}| \\ = \quad & \left|\left(\bigcap_{i=1,2} (K_i \cup L_i)\right) \setminus \{d_p : p \in Y_1 \cap Y_2\}\right| + \left|\left(\bigcup_{i=1,2} (K_i \cup L_i)\right) \setminus \{d_p : p \in Y_1 \cup Y_2\}\right|. \end{aligned}$$

Thus,

$$m_1 + m_2 \leq m_3 + m_4.$$

By definition of λ_1,

$$\lambda_1 = \frac{d_1 - m_1 - |Z_1|}{n_1} = \frac{d_2 - m_2 - |Z_2|}{n_2} \leq \frac{d_3 - m_3 - |Z_1 \cap Z_2|}{n_3},$$

and thus,

$$\begin{aligned} &\lambda_1 n_1 = d_1 - m_1 - |Z_1|, \\ &\lambda_1 n_2 = d_2 - m_2 - |Z_2|, \\ &\lambda_1 n_3 \leq d_3 - m_3 - |Z_1 \cap Z_2|. \end{aligned}$$

Adding the first two lines and subtracting the third line

$$\lambda_1 \underbrace{(n_1 + n_2 - n_3)}_{=n_4} \geq \underbrace{(d_1 + d_2 - d_3)}_{\geq d_4} - \underbrace{(m_1 + m_2 - m_3)}_{\leq m_4} - \underbrace{(|Z_1| + |Z_2| - |Z_1 \cap Z_2|)}_{=|Z_1 \cup Z_2|}$$

and thus,

$$\lambda_1 \geq \frac{d_4 - m_4 - |Z_1 \cup Z_2|}{n_4} = f_1(Y_1 \cup Z_1 \cup Y_2 \cup Z_2, K_1 \cup L_1 \cup K_2 \cup L_2).$$

Since λ_1 is the minimum value of f_1 among all possible sets as defined in the solution,

$$f_1(Y_1 \cup Z_1 \cup Y_2 \cup Z_2, K_1 \cup L_1 \cup K_2 \cup L_2) = \lambda_1.$$

■

Lemma 6 *Consider Step K+1 of the egalitarian mechanism.*[9] *Suppose the sets* $Y_1, Y_2 \subseteq P^{u;u}_{K+1} \cup P^{u;1}_{K+1}$, *and* $L_1, L_2 \subseteq D^u_{K+1}$ *are such that*

$$g_1(Y_1, L_1) = g_1(Y_2, L_2) = \beta_1.$$

Then,

$$g_1(Y_1 \cup Y_2, L_1 \cup L_2) = \beta_1$$

as well.

Proof. For $i = 1, 2$, define

$$n_i = |Y_i|, \; d_i = \left|D_{Y_i} \cap D^o_{K+1}\right|,$$

$$m_i = \left|P^o_{K+1}(L_i)\right|.$$

Also, define

$$n_3 = |Y_i \cup Y_2|, \; n_4 = |Y_1 \cup Y_2|,$$

$$d_3 = \left|D_{Y_1 \cap Y_2} \cap D^o_{K+1}\right|, \; d_4 = \left|D_{Y_1 \cup Y_2} \cap D^o_{K+1}\right|,$$

and

$$m_3 = \left|P^o_{K+1}(\{d_p : p \in Y_1 \cap Y_2\})\right|,$$
$$m_4 = \left|P^o_{K+1}(\{d_p : p \in Y_1 \cup Y_2\})\right|.$$

By definition, we have $n_1 + n_2 = n_3 + n_4$. Also,

$$d_1 + d_2 \geq d_3 + d_4.$$

This is because, not only the compatible kidneys of the patients in $\bigcap_{i=1,2} Y_i$ are counted twice, but also two patients, one in Y_1, the other in Y_2, may reveal the same donor as compatible.

By definition of $P(\cdot)$, the patients in $P^o_{K+1}(\{d_p : p \in Y_1 \cap Y_2\})$ are the only double counted patients in $\bigcup_{i=1,2} P^o_{K+1}(\{d_p : p \in Y_i\})$. Moreover, a patient who is not in the latter set may be in $P^o_{K+1}(\{d_p : p \in Y_1 \cup Y_2\})$. Thus,

$$m_1 + m_2 \leq m_3 + m_4.$$

[9] The result directly applies to steps K+2,...,K+M as well.

By applying the same technics as in the previous Lemma, we obtain

$$g_1(Y_1 \cup Y_2, L_1 \cup L_2) = \beta_1.$$

■

Thus, in the mechanism, the largest sets minimizing f_k and g_m in the sense of inclusion are well defined for each step $k \in \{1, \ldots, K\}$ and $K + m$ for $m \in \{1, \ldots, M\}$.

First, we need to show that $Z^* \in \mathcal{Z}^e$.

Lemma 7 *The matrix constructed through the egalitarian mechanism is an efficient random matching, that is, $Z^* \in \mathcal{Z}^e$.*

Proof. We prove by induction.

Step 1 : We claim that there exists a random matching $Z \in \mathcal{Z}^e$ such that, for each $p \in P$, $u_p(Z) - t_{d_p}(Z) \geq \lambda_1$. We construct the following digraph $G = (V, A)$:

$$V = \Big((P \cup D) \setminus (P^f \cup D^f)\Big) \cup \{t\}.$$

$$A = \left(\bigcup_{p \in P^o \cup P^u} \{(p, d) : d \in D_p\}\right) \cup \{(d, p_d) : d \in D^{u:u}\} \cup \{(d, t) : d \in D^o \cup D^{u;f,o}\}.$$

Define the capacity function $k : A \to \mathbb{R}_+$ as follows:

$$k(a) = \begin{cases} \infty & \text{if } a \in \left(\bigcup_{p \in P^o} \{(p, d) : d \in D_p\}\right) \cup \left(\bigcup_{p \in P^u} \{(p, d) : d \in D_p\}\right) \\ 1 & \text{if } a \in \{(d, p_d) : d \in D^{u;u}\} \cup \{(d, t) : d \in D^{u;f,o}\} \cup \{(d, t) : d \in D^o\} \end{cases}.$$

Define $b : V \to \mathbb{R}$ as follows:

$$b(v) = \begin{cases} -1 & \text{if } v \in P^o \\ 0 & \text{if } v \in D^u \cup D^o \\ -\lambda_1 & \text{if } v \in P^{u;u} \\ -1 - \lambda_1 & \text{if } v \in P^{u;f,o} \\ |P^o| + \lambda_1 |P^{u;u}| + (1 + \lambda_1) \left|P^{u;f,o}\right| & \text{if } v = t \end{cases}.$$

Let $f : A \to \mathbb{R}_+$ be a flow. Then, for

$$a = (p', d') \in \left(\bigcup_{p \in P^o} \{(p, d) : d \in D_p\}\right) \cup \left(\bigcup_{p \in P^u} \{(p, d) : d \in D_p\}\right),$$

$f(a)$ is the probability that patient p' receives kidney transplantation from donor d'. Similarly, for $a = (d'', p'') \in \{(d, p_d) : d \in D^{u;u}\}$, $f(a)$ is the probability that donor d'' of patient p'' donates her kidney to someone in the set P^o. The function b is specified to capture the efficiency, that is, the Gallai-Edmond Decomposition. For $p \in P^o$, since, by efficiency, patient p receives a live donor transplantation with probability 1, $b(p) = -1$. Also, for each $p \in P^{u;u}$, the difference between the flow leaving vertex p and the flow entering vertex p is the difference between the probability that patient p receives a live donor kidney transplantation and the probability that his donor donates her kidney someone in the pool. Our claim is that it is possible to stochastically match the patients to the donors such that for each underdemanded patient, the u-t difference is at least λ_1. Thus, a function $f : A \to \mathbb{R}$ such that $excess_f = b$, means that there is an ex ante efficient random matching Z such that, for each $p \in P$, $u_p(Z) - t_{d_p}(Z) \geq \lambda_1$. Thus, given the digraph $G = (V, A)$, and the functions $k : A \to \mathbb{R}_+$, and $b : V \to \mathbb{R}$, constructed above, we need to show that there exists a $b-$*transshipment* f satisfying $0 \leq f \leq k$. By Gale's Theorem, there exists such a $b-$*transshipment* f if and only if

$$\text{for each } U \subseteq V : b(U) \leq k(\delta^{in}(U)).$$

First, note that the inequality is satisfied for $U = \{t\}$. Suppose not. Then,

$$|P^o| + \lambda_1 |P^{u;u}| + (1 + \lambda_1) \left|P^{u;f,o}\right| = b(t) > k(\delta^{in}(t)) = |D^o| + \left|D^{u;f,o}\right|,$$

which implies

$$\lambda_1 > \frac{|D^o| + \left|D^{u;f,o}\right| - |P^o| - \left|P^{u;f,o}\right|}{|P^u|}. \tag{2}$$

Consider $P^{u;u} \cup P^{u;f,o}$, and $F' = D^{u;u} \cup D^{u;f,o}$. By the Gallai-Edmond Decomposition, $D_{P^u} \cap D^o = D^o$. Then, by definition of λ_1,

$$\frac{|D^o| - \left(|P^o| + \left|P^{u;f,o}\right| - \left|\left(D^{u;u} \cup D^{u;f,o}\right) \setminus \{d_p : p \in P^{u;u}\}\right|\right)}{|P^{u;u}| + |P^{u;f,o}|} \geq \lambda_1$$

which is equivalent to

$$\frac{|D^o| - \left(|P^o| + \left|P^{u;f,o}\right| - \left|D^{u;f,o}\right|\right)}{|P^u|} \geq \lambda_1,$$

and this contradicts with (2).

Let $S' = R' \cup T'$ with $R' \subseteq P^{u;u}$ and $T' \subseteq P^{u;f,o}$ and consider a set $U \subseteq V$ such that $\{t\} \cup S' \subseteq U$. If for some $d \in D_{P^u \setminus S'} \cap D^o$, $d \in U$, then, since $k(\delta^{in}(U)) = \infty$, the inequality

(1) is trivially satisfied. Thus, we need to check inequality (1) only for U such that $D^o \setminus D_{P^u \setminus S'} \subseteq U$. Similarly, if $D' \subseteq U$ for some $D' \subseteq D^u$, we need to check it only for U such that $P^o \setminus P(D^u \setminus D') \subseteq U$.

We claim that it is enough to check inequality (1) for $D' \subseteq D^u$ such that $D' \cap D^{u;u} \subseteq \{d_p : p \in R'\}$. Let $D' \subseteq U$ such that $D'' \subseteq D'$ where $D'' \subseteq D^{u;u} \setminus \{d_p : p \in R'\}$. As argued above, $P^o \setminus P(D^u \setminus D') \subseteq U$. Now, let us consider $U \setminus D''$. Since $P^o \setminus P(D^u \setminus D') \supseteq P^o \setminus P(D^u \setminus (D' \setminus D''))$, and for each $p \in P^o$, $b(p) = -1$, this implies that $b(U) \leq b(U \setminus D'')$. Also, the construction of the capacity function k implies that $k(\delta^{in}(U)) = k(\delta^{in}(U \setminus D''))$. Thus, if inequality (1) is satisfied for $U \setminus D''$, then it is satisfied for U as well. Thus, it is enough to check inequality (1) for U where $D' \subseteq U$ implies $D' \cap D^{u;u} \subseteq \{d_p : p \in R'\}$.

Now, consider $U = \{t\} \cup S' \cup \left(D^o \setminus D_{P^u \setminus S'}\right) \cup F' \subseteq V$ such that $F' \cap D^{u;u} \subseteq \{d_p : p \in R'\}$. Let $R = P^{u;u} \setminus R'$, $T = P^{u;f,o} \setminus T'$, and $F = D^u \setminus F'$. Suppose $b(U) > k(\delta^{in}(U))$. Thus,

$$\begin{aligned} & \left(\begin{array}{c} -\left(|P^o| - |P(F)|\right) + (-\lambda_1)\left(|P^{u;u}| - |R|\right) + \\ (-1-\lambda_1)\left(\left|P^{u;f,o}\right| - |T|\right) + |P^o| + \lambda_1 \left|P^{u;u}\right| + (1+\lambda_1)\left|P^{u;f,o}\right| \end{array} \right) \\ \geq \quad & |D_S| + |(F \cap D^{u;u}) \setminus \{d_p : p \in R\}| + \left|F \cap D^{u;f,o}\right|, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \lambda_1 \quad > \quad & \frac{|D_S| + |(F \cap D^{u;u}) \setminus \{d_p : p \in R\}| - |P(F)| - |T| + \left|F \cap D^{u;f,o}\right|}{|S|} \\ = \quad & \frac{|D_S| + |\{d_p : p \in R\}| - |P(F)| - |T| + |F|}{|S|} \\ = \quad & \frac{|D_S| + |R| - |P(F)| - |T| + |F|}{|S|} \\ = \quad & \frac{|D_S| - |P(F)| - |S| + |F|}{|S|}. \end{aligned}$$

This contradicts the definition of λ_1. Thus, $b(U) \leq k(\delta^{in}(U))$. Since U is arbitrarily chosen, this condition holds for each U. Then, by Gale's Theorem, there exists a $b-$*transshipment* f satisfying $0 \leq f \leq k$. Thus, there exists an ex ante efficient random matching $Z \in \mathcal{Z}^e$ such that, for each $p \in P$, $u_p(Z) - t_{d_p}(Z) \geq \lambda_1$. Let Z^1 be the set of all such ex ante efficient random matchings.

<u>Step k :</u> Let the sets S_{k-1}, and F_{k-1} be the largest sets in the sense of inclusion such that

$$\lambda_{k-1} = f_{k-1}(S_{k-1}, F_{k-1}).$$

The donors in F_{k-1} are matched only to the patients in $P^o_{k-1}(F_{k-1})$, and the donors

in $\left(D_{S_{k-1}} \setminus \bigcup_{n=1}^{k-2} D_{S_n}\right) \cap D^o$ are matched only to the patients in S_{k-1}. The donors in $F_{k-1} \setminus \{d_p : p \in S_{k-1}\}$ and $\left(D_{S_{k-1}} \setminus \bigcup_{n=1}^{k-2} D_{S_n}\right) \cap D^o$ are matched with probability 1. The patients and donors in $S_{k-1} \cup F_{k-1}$ leave. Then, we construct the digraph with the remaining patients and donors, as in the previous step. By Gale's Theorem, there exists $b-$*transshipment* f satisfying $0 \leq f \leq k$. Thus, there exists an ex ante efficient random matching $Z \in Z^{k-1}$ such that, for each remaining patient p, $u_p(Z) - t_{d_p}(Z) \geq \lambda_k$. Let Z^k be the set of all such ex ante efficient random matchings.

<u>Step $K+1$:</u> Let the sets S_K, and F_K be the largest sets in the sense of inclusion such that

$$\lambda_K = f_K(S_K, F_K).$$

The patients and donors in $S_K \cup F_K$ leave. If, among the remaining patients, there does exist an underdemanded patient p such that his donor has left at an earlier stage, then for each $Z \in Z^K$, $u_p(Z) - t_{d_p}(Z) \leq 0$. This contradicts the definition of λ_K. Thus, the donor of an underdemanded patient p is among the remaining donors. We construct the digraph $G = (V, A)$ where V is the set of remaining patients and donors together with the vertex t, that is, $V = \{t\} \cup D^u_{K+1} \cup D^o_{K+1} \cup P^o_{K+1} \cup P^u_{K+1}$ and

$$A = \left(\bigcup_{p \in P^o_{K+1} \cup P^u_{K+1}} \{(p,d) : d \in D_p\}\right) \cup \{(d, p_d) : d \in D^{u;u}_{K+1}\} \cup \{(d,t) : d \in D^o_{K+1} \cup D^{u;f,o}_{K+1}\}.$$

Define the capacity function $k : A \to \mathbb{R}_+$ as follows:

$$k(a) = \begin{cases} \infty & \text{if } a \in \left(\bigcup_{p \in P^o_{K+1}} \{(p,d) : d \in D_p\}\right) \cup \left(\bigcup_{p \in P^u_{K+1}} \{(p,d) : d \in D_p\}\right) \\ 1 & \text{if } a \in \{(d,p_d) : d \in D^{u;u}_{K+1}\} \cup \{(d,t) : d \in D^{u;f,o}_{K+1}\} \cup \{(d,t) : d \in D^o_{K+1}\} \end{cases}.$$

Define $b : V \to \mathbb{R}$ as follows:

$$b(v) = \begin{cases} -1 & \text{if } v \in P^o_{K+1} \cup P^{u;1}_{K+1} \\ 0 & \text{if } v \in P^{u;u}_{K+1} \cup D^o_{K+1} \\ -\beta_1 & \text{if } v \in D^{u;u}_{K+1} \\ 1 - \beta_1 & \text{if } v \in D^{u;f,o}_{K+1} \\ \left|P^o_{K+1}\right| + \left|P^{u;1}_{K+1}\right| + \beta_1 \left|D^{u;u}_{K+1}\right| - (1-\beta_1)\left|P^{u;f,o}_{K+1}\right| & \text{if } v = t \end{cases}.$$

Let $T' \subseteq P^u_{K+1}$. By the same argument used in Step 1, it is enough to check the condition in Gale's Theorem for $U \subseteq V$ such that $U \cap (D^{u;u}_{K+1} \setminus \{d_p : p \in T'\}) = \emptyset$. The definition of β_1

implies that for each $U \subseteq V$, $b(U) \leq k(\delta^{in}(U))$. Then, Gale's Theorem implies that there exists a $b-$*transshipment* f satisfying $0 \leq f \leq k$. Thus, there exists an ex ante efficient random matching $Z \in Z^K$ such that, for each remaining patient p, $u_p(Z) - t_{d_p}(Z) \geq \beta_1$. Let Z^{K+1} be the set of all such ex ante efficient random matchings.

<u>Step $K+m$:</u> Let the sets T^{m-1} and H^{m-1} be the largest sets in the sense of inclusion such that

$$\beta_{m-1} = g_{m-1}(T^{m-1}, H^{m-1}).$$

The donors in $\left(D_{T^{m-1}} \setminus \left(\bigcup_{n=1}^{K} D_{S_n}\right) \setminus \left(\bigcup_{j=1}^{m-2} D_{T^j}\right)\right) \cap D^o$ are matched to the patients in T^{m-1}, and the donors in $\{d_p : p \in T^{m-1}\} \cup H^{m-1}$ are matched to the patients in $P^o_{K+m-1}(\{d_p : p \in T^{m-1}\} \cup H^{m-1})$. These patients and donors leave and a digraph is constructed with the remaining patients and donors, as in the previous step. By Gale's Theorem, there exists a $b-$*transshipment* f satisfying $0 \leq f \leq k$. Thus, there exists an ex ante efficient random matching $Z \in Z^{K+m-1}$ such that, for each remaining patient p, $u_p(Z) - t_{d_p}(Z) \geq \beta_m$. Let Z^{K+m} be the set of all such ex ante efficient random matchings.

Thus, at the end of step $K+M$ Z^{K+M} is non-empty and contains only efficient random matchings. ■

Lemma 8 *For each* $k \in \{1, \ldots, K-1\}$, $\lambda_k < \lambda_{k+1}$. *For each* $m \in \{1, \ldots, M-1\}$, $\beta_m < \beta_{m+1}$.

Proof. Let the sets S_k, F_k; and S_{k+1}, F_{k+1} be the largest sets in the sense of inclusion such that

$$\lambda_k = f_k(S_k, F_k),$$

$$\lambda_{k+1} = f_{k+1}(S_{k+1}, F_{k+1}).$$

Suppose

$$\lambda_{k+1} \leq \lambda_k.$$

Thus,

$$\frac{\left|\left(D_{S_{k+1}} \setminus \left(\bigcup_{n=1}^{k} D_{S_n}\right)\right) \cap D^o\right| - \left(\left|P^o_{k+1}(F_{k+1})\right| + |S_{k+1}| - |F_{k+1}|\right)}{|S_{k+1}|}$$

$$\leq \frac{\left|\left(D_{S_k} \setminus \left(\bigcup_{n=1}^{k-1} D_{S_n}\right)\right) \cap D^o\right| - \left(|P^o_k(F_k)| + |S_k| - |F_k|\right)}{|S_k|}. \tag{3}$$

Now, consider $S_k \cup S_{k+1}$ and $F_k \cup F_{k+1}$. Note that

$$\left| \left(D_{S_k \cup S_{k+1}} \setminus \left(\bigcup_{n=1}^{k-1} D_{S_n} \right) \right) \cap D^o \right|$$

$$= \left| \left(D_{S_k} \setminus \left(\bigcup_{n=1}^{k-1} D_{S_n} \right) \right) \cap D^o \right| + \left| \left(D_{S_{k+1}} \setminus \left(\bigcup_{n=1}^{k} D_{S_n} \right) \right) \cap D^o \right|. \tag{4}$$

Also, that F_k, and F_{k+1} are mutually exclusive implies

$$|P_k^o(F_k \cup F_{k+1})| \geq \left|P_{k+1}^o(F_{k+1})\right| + |P_k^o(F_k)|. \tag{5}$$

Then, combining (3), (4) and (5), we obtain

$$f_k(S_k, F_k) \geq f_k(S_k \cup S_{k+1}, F_k \cup F_{k+1}).$$

This contradicts with the definition of S_k and F_k, that they are the largest sets in the sense of inclusion such that

$$\lambda_k = f_k(S_k, F_k).$$

Also, by construction $\beta_1 > 0$, and $\lambda_K \leq 0$. By using the same inequalities/equalities as above, we see that, for each $m \in \{1, \ldots, M-1\}$, $\beta_m < \beta_{m+1}$. ■

For each patient p in S_1, $u_p(Z^*) - t_{d_p}(Z^*)$ is the lowest under Z^*, for each patient p in S_2, $u_p(Z^*) - t_{d_p}(Z^*)$ is the lowest among the remaining patients under Z^*, and so on. The only thing that remains to show is that, for each $Z \in \mathcal{Z}^e$, if $p \in S_1$ such that $u_p(Z) - t_{d_p}(Z) > u_p(Z^*) - t_{d_p}(Z^*)$, then there is another patient p' in S_1 such that $u_{p'}(Z) - t_{d_{p'}}(Z) < u_{p'}(Z^*) - t_{d_{p'}}(Z^*)$. Note that $S_1 \subseteq P^u$, $F_1 \subseteq D^u$ such that $\{d_p : p \in S_1\} \cap P^{u;u} \subseteq F_1$, and

$$f_1(S_1, F_1) = \frac{|D_{S_1} \cap D^o| - (|P^o(F_1)| + |S_1| - |F_1|)}{|S_1|}.$$

By compatibility, the patients in S_1 can be matched to at most $|D_{S_1} \cap D^o|$ patients. Also, by efficiency, the patients in $P^o(F_1)$ are matched to the donors in F_1. At an efficient random matching, the least possible number of patients who are matched to $\{d_p : p \in S_1 \cap P^{u;u}\}$ is $|P^o(F_1)| - (|F_1| - |S_1 \cap P^{u;u}|)$; and this possible only if each donor in $F_1 \setminus (\{d_p : p \in S_1\} \cap D^{u;u})$ is matched to a patient in $P^o(F_1)$ with probability 1. After this matching of the donors in $F_1 \setminus (\{d_p : p \in S_1\} \cap D^{u;u})$, there are $|P(F_1)| - (|F_1| - |S_1 \cap P^{u;u}|)$ remaining patients in $P^o(F_1)$; and by efficiency each such patient is matched to the donors

in $\{d_p : p \in S_1 \cap P^{u;u}\}$ with probability one. Also, by efficiency, the donor of each patient $p \in S_1 \cap P^{u;f,o}$ is matched with probability 1. Thus, for each $Z \in \mathcal{Z}^e$,

$$\begin{aligned}
& \sum_{p \in S_1} \left(u_p(Z) - t_{d_p}(Z)\right) \\
\leq\ & |D_{S_1} \cap D^o| - (|P(F_1)| - (|F_1| - |S_1 \cap P^{u;u}|)) - \left|S_1 \cap P^{u;f,o}\right| \\
=\ & |D_{S_1} \cap D^o| - |P(F_1)| + |F_1| - \left(|S_1 \cap P^{u;u}| + \left|S_1 \cap P^{u;f,o}\right|\right) \\
=\ & |D_{S_1} \cap D^o| - |P(F_1)| + |F_1| - |S_1| \,.
\end{aligned}$$

We have already shown that there exists an ex ante efficient random matching Z^* such that this upper bound is reached in an egalitarian way, thus, in a way such that for each patient $p \in S_1$, $u_p(Z^*) - t_{d_p}(Z^*) = \lambda_1$. Similarly, for the patients in S_2, the upper bound is reached in an egalitarian way, and so on. This completes the proof.

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